Lecture 9: Moment map in Yang-Mills theory

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The moment map

When M is a Riemann surface, the space of connections $\mathcal{A}$ has a canonical symplectic form. Since it's just an affine space modelled on $\Omega^1(M; \mathrm{ad}(P))$ its tangent space at any point is canonically isomorphic to $\Omega^1(M; \mathrm{ad}(P))$ and so we can use the 2-form

$$\int_M \alpha \wedge \beta, \ \alpha, \beta \in \Omega^1(M; \mathrm{ad}(P))$$

(where the integral makes sense because M is 2-dimensional). Now each element $\phi \in \mathrm{Lie}(\mathcal{G}) = \Omega^0(M; \mathrm{ad}(P))$ generates a vector field V on $\mathcal{A}$ by its infinitesimal action $V_\nabla = \nabla\phi$.

Lemma

The function $f : \nabla \mapsto -\int_M F_\nabla \wedge \phi$ is a Hamiltonian function on $\mathcal{A}$ generating V.

Proof.

The proof is just integration-by-parts.

$$Q(\nabla\phi, A) = \int_M \nabla\phi \wedge A$$
$$= -\int_M \phi \wedge \nabla A$$
$$(df)_\nabla(A) = \langle \nabla A, \phi \rangle$$

since $F_{\nabla+\epsilon A} = F_\nabla + \epsilon\nabla A + \mathcal{O}(\epsilon^2)$ □

More generally if a group $\mathcal{G}$ acts on a symplectic manifold $\mathcal{A}$ in such a way that its infinitesimal action is through Hamiltonian vector fields you can write a *moment map*

$$\mu : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*$$

encoding the Hamiltonian functions associated to infinitesimal actions of $\mathcal{G}$. In our case $\mathrm{Lie}(\mathcal{G})^* = \Omega^2(M; \mathrm{ad}(P))$ and the moment map is just

$$\nabla \mapsto -F_\nabla$$

The Yang-Mills functional is the L^2-norm of the moment map.

Complexifying the action of $\mathcal{G}$

We want to consider $\mathcal{A}$ as a space of holomorphic vector bundles. Let us recall what that means.

Definition

A holomorphic vector bundle $\mathcal{E}$ over a complex manifold (in our case a Riemann surface) is a complex vector bundle $\pi : E \to M$ where the total space is a complex manifold and the projection is holomorphic.

As usual on complexified 1-forms one writes $d = \partial + \bar{\partial}$ corresponding to the splitting $\Omega^1(M) \otimes \mathbb{C} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ (here $\Omega^{1,0}, \Omega^{0,1}$ are the $\pm i$-eigenspaces for J where $(J\alpha)(v) = \alpha(iv)$). For example if $z = x + iy$ is a local complex coordinate then $dz \in \Omega^{1,0}$ since in
$dz(iv) = (dx + idy)(-v_y, v_x) = -v_y + iv_x = i(dx + idy)(v_x, v_y)$.

On a holomorphic vector bundle $\mathcal{E}$ one can also write down an operator $\bar{\partial}_{\mathcal{E}}$ on forms with values in $\mathcal{E}$ ($\Omega^k(M;\mathcal{E})$). This is defined so that

$$\bar{\partial}_{\mathcal{E}}\sigma = 0$$

for holomorphic sections $\sigma \in \Omega^0(M;\mathcal{E})$ and so that the Leibniz rule

$$\bar{\partial}_{\mathcal{E}}(f\sigma) = (\bar{\partial}f)\sigma + f\bar{\partial}_{\mathcal{E}}\sigma$$

holds for $\sigma \in \Omega^k(M;\mathcal{E})$, $f \in \Omega^\ell(M)$.

It is easy to define an operator satisfying these conditions. In some local chart $U \subset M$ there is a holomorphic trivialisation of the bundle

$$\mathcal{E}|_U \overset{\phi}{\cong} U \times \mathbb{C}^n$$

and we define $\bar{\partial}_{\mathcal{E}}\sigma$ to be

$$(\bar{\partial}\sigma_1, \ldots, \bar{\partial}\sigma_n)$$

Changing local coordinates is accomplished by a holomorphic map $g : U \to GL(n, \mathbb{C})$, the new coordinates being

$$\mathcal{E} \ni e \mapsto (\mathrm{pr}_U \phi(e), g\,\mathrm{pr}_{\mathbb{C}^n}\phi(e))$$

Under such a change

$$\bar{\partial}_{\mathcal{E}}(g\sigma) = (\bar{\partial}g)\sigma + g\bar{\partial}_{\mathcal{E}}\sigma = g\bar{\partial}_{\mathcal{E}}\sigma$$

since g is holomorphic.

Suppose we have a holomorphic vector bundle with a fibrewise Hermitian metric. Then we can make sense of unitary frames and observe that there is an underlying principal $U(n)$-bundle P. A unitary connection on the associated unitary vector bundle E underlying $\mathcal{E}$ is then

$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E) = \Omega^{1,0}(M; E) \oplus \Omega^{0,1}(M; E)$$

so we can define $\nabla^{0,1}, \nabla^{1,0}$. We say ∇ is compatible with $\mathcal{E}$ if $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$. If $\bar{\partial}_{\mathcal{E}} = \bar{\partial} + \alpha$ in a local unitary trivialisation then define $\nabla^{1,0} = \partial - \alpha^{\dagger}$. Ex: $\nabla = \nabla^{1,0} + \bar{\partial}_{\mathcal{E}}$ is a unitary connection compatible with $\mathcal{E}$. The next proposition shows that this is reversible...

Proposition

If P is a principal $U(n)$-bundle over a Riemann surface M and ∇ is a $U(n)$-connection then $\mathrm{ad}(P)$ inherits the structure of a holomorphic vector bundle over M such that

$$\nabla^{0,1} = \overline{\partial}$$

Proof.

It's easy to define complex charts on E: just pick local trivialisations, use the fibre coordinate vertically and pull back complex coordinates from M horizontally. The fact that M is a complex manifold means that these will glue to give the structure of a complex manifold globally and the projection will be holomorphic by construction. The main difficulty is to pick the trivialisation so as to ensure $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$. A trivialisation is the same as a choice of local sections $\sigma_1, \ldots, \sigma_n$ which form a unitary basis at each point. Notice that in the complex structure we have described these sections will trace out complex submanifolds and hence end up as holomorphic local sections... □

...but holomorphic sections will obey $\bar{\partial}_{\mathcal{E}}\sigma = 0$, so to ensure $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$ we'll have to find a basis of local sections $\sigma = \{\sigma_i\}_{i=1}^n$ for which $\nabla^{0,1}\sigma_i = 0$. We'll do this next time!