Lecture 4: Harmonic forms

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29th September 2010

- Last lecture we introduced $U(1)$-bundles $L \to M$ and connections ∇ on them.
- We saw that two connections on the same bundle differ by a 1-form, $\nabla' = \nabla + iA$.
- We finished by defining the curvature F_∇ of a connection which is a 2-form on M.

If we take a different connection $\nabla' = \nabla + A$ on the same bundle then the curvature changes to $F' = F + idA$. This has the following interesting consequence:

Lemma

The de Rham cohomology class of the 2-form $-iF$ does not depend on the choice of connection. It is therefore a topological invariant of the bundle. We define

$$c_1(L) = \frac{1}{2\pi i}[F]$$

and call it the first Chern class (the $1/2\pi$ normalisation turns out to give us an integral cohomology class).

Aim

In what follows M will be a compact oriented Riemannian n-manifold.

Lemma

- *(Existence) If L is a $U(1)$-bundle over M and ω is a 2-form on M such that $[\omega] = c_1(L)$ then there exists a connection such that $\frac{1}{2\pi i}F = \omega$.*
- *(Uniqueness) The space of gauge equivalence classes of such connections is a torus modelled on $H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$.*

Theorem ('Hodge-Maxwell[a]' theorem)

[a] i.e. Hodge theorem in the language of Maxwell theory

Let j be an exact $n-k+1$-form. In each cohomology class there is a unique closed k-form β such that $d \star \beta = j$.

We first dispose of the Lemma.

Proof.

Existence: Let ∇' be a connection on L with curvature F' (so that by definition of $c_1(L)$, $[F'] = 2\pi i[\omega]$). Then $F' - 2\pi i\omega$ is nullhomologous and hence there exists A' such that $idA' = F' - 2\pi i\omega$. Now $\nabla = \nabla' + A'$ has the right curvature.

Uniqueness: Exercise! For those who know some algebraic geometry, when M is a smooth projective variety with a Fubini-Study metric the union of these tori over all possible Chern classes is precisely the Picard group of M (where ω is the harmonic representative of the Chern class which we will see at the end of this lecture). The single torus corresponding to the trivial bundle ($c_1 = 0$) is called the Jacobian. $\square$

The theorem above will follow from the following result (the Hodge theorem). This result, which is a nontrivial exercise in linear elliptic PDE, will serve as a good introduction to the rest of the course, which is a nontrivial exercise in nonlinear elliptic PDE.

Theorem (Hodge theorem)

Any k-form α has a unique decomposition

$$\alpha = \alpha_H + d^*\beta + d\gamma$$

where $d^* = (-1)^{nk+n+1} \star d \star$ and $\Delta\alpha_H := (dd^* + d^*d)\alpha_H = 0$. Uniqueness means that the three terms are unique, not that β and γ are themselves unique.

We note some important facts. First remember that by definition, for k-forms α, β,

$$\alpha \wedge \star\beta = g(\alpha, \beta)\mathrm{vol}$$

and therefore

$$\alpha \wedge \star\beta = \beta \wedge \star\alpha$$

Ex: Moreover $\star\star\alpha = (-1)^{k(n-k)}\alpha$. If we define $(\alpha, \beta) = \int \alpha \wedge \star\beta$ then this is an L^2-inner product on Ω^k (not complete!). We can define the adjoint d^* to d by

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

Ex: $d^* = (-1)^{nk+n+1} \star d \star$. The operator $\Delta = dd^* + d^*d$ (Laplace-Beltrami operator) is self-adjoint and in the case of the trivial connection on the trivial bundle over Euclidean $\mathbb{R}^4$ it agrees with the usual Laplacian. We call forms for which $\Delta\alpha = 0$ *harmonic forms*.

Harmonic forms

Lemma

A form α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$.

Proof.

$$(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha) \geq 0$$

with equality if and only if $d\alpha = 0$ and $d^*\alpha = 0$. □

Corollary

There is a unique harmonic form in each cohomology class.

Proof.

If $\alpha = \alpha_H + d^*\beta + d\gamma$ is closed then $dd^*\beta = 0$. Therefore $(d^*\beta, d^*\beta) = (dd^*\beta, \beta) = 0$ and so $\alpha = \alpha_H + d\gamma$. In particular α_H is a harmonic form in the same cohomology class and if α is harmonic then $d\gamma = 0$ by uniqueness in the Hodge theorem. □

Another version of the theorem:

Theorem

There is an orthogonal decomposition

$$\Omega^k(M) = \mathrm{im}(\Delta) \oplus \ker(\Delta)$$

Moreover $\ker(\Delta)$ *is finite-dimensional.*

From this we recover the Hodge decomposition by observing that $\alpha = \alpha_H + \Delta\kappa = \alpha_H + d^*(d\kappa) + d(d^*\kappa)$. Uniqueness is easy:

$$\alpha_H + d^*\beta + d\gamma = \alpha'_H + d^*\beta' + d\gamma'$$

implies $dd^*(\beta - \beta') = 0$ and $d^*d(\gamma - \gamma') = 0$. As before this means $d^*\beta = d^*\beta'$ and $d\gamma = d\gamma$ which finally implies $\alpha_H = \alpha'_H$.

We want to show that the orthogonal complement of the space of harmonic forms $\mathcal{H} := \ker(\Delta)$ is precisely the image of Δ. The hope is to make rigorous the following argument by passing to Hilbert space completions:

Nonsense

"The space of forms admits a direct sum decomposition into $\mathrm{im}(\Delta) \oplus \mathrm{im}(\Delta)^{\perp}$*. If* κ *is in* $\mathrm{im}(\Delta)^{\perp}$ *then* $0 = \langle \kappa, \Delta\omega \rangle = \langle \Delta\kappa, \omega \rangle$ *for all* ω *because* Δ *is self-adjoint. Therefore* $\Delta\kappa = 0$ *so* $\mathrm{im}(\Delta)^{\perp} = \mathcal{H}$ *and the Hodge decomposition follows."*

The difficulty is that direct sum decompositions like this only work well in the context of closed subspace of Hilbert spaces. All we can extricate from the above argument is the statement that harmonic forms are orthogonal to $\mathrm{im}(\Delta)$. We still need to solve the equation

$$\Delta\omega = \kappa$$

for any $\kappa \in \mathrm{im}(\Delta)^{\perp}$.

Sobolev spaces

So we are forced to introduce some Hilbert spaces. If the following goes a little fast, check out Gilbarg and Trudinger (henceforth GiT) Chapters 7 and 8 and replace $p = q = 2$ everywhere. In what follows Ω will denote a nice bounded open set in $\mathbb{R}^n$ and dx a measure coming from a Riemannian metric. We will blur the discussion: sometimes we talk about forms on manifolds, sometimes about functions on open sets of $\mathbb{R}^n$. The translation comes from taking local charts and is unproblematic.

Definition

By L^2 we will denote the space of k-forms whose coefficients are L^2-integrable measurable functions (measurable function means that if we were to take a local chart these functions would be Lebesgue measurable and we're considering them up to the equivalence of being equal almost everywhere (on a set of full measure)). This is a Hilbert space and the L^2-inner product restricts to the subspace of smooth forms as the inner product $(,)$ from earlier.

Definition

A function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has weak derivative $v =$ "$\partial_{i_1} \cdots \partial_{i_j} u$" (shorthand $D_{\underline{i}} u$, $j = |\underline{i}|$) if

$$\int_\Omega \phi v dx = (-1)^j \int_\Omega u D_{\underline{i}} \phi dx$$

for any $\mathcal{C}^j$-smooth function $\phi : \Omega \to \mathbb{R}$) (if u really were differentiable you could integrate by parts!).

This definition is ideally suited to our problem. We have expressions like "$0 = \langle \Delta\omega, \kappa \rangle$ for all smooth ω" and we want to understand them as "$\Delta\kappa = 0$" where κ may not be smooth (it will belong to some Hilbert space completion of the space of smooth functions). Weak derivatives are our friends.

Definition

The Sobolev space L^2_k is the space of p-forms whose coefficients are L^2-integrable measurable functions which possess all weak derivatives up to order k and these weak derivatives are L^2-integrable. This is a Hilbert space with norm

$$||u||_k := \sqrt{\sum_{|\underline{i}|\leq k} \int_M |D_{\underline{i}} u|^2}$$

The L^2_k notation follows Donaldson and Kronheimer. Many people would write $W^{2,k}$ or H^k for these spaces.

Proposition (GiT, Theorem 7.9)

For any k the space of smooth p-forms is dense in L^2_k.

We have the following nice lemmas.

Lemma (Sobolev, (GiT, Corollary 7.11))

If $k - \frac{n}{2} > r$ then $L^2_k \subset \mathcal{C}^r$ where $\mathcal{C}^r$ denotes the space of $\mathcal{C}^r$-smooth p-forms.

Lemma (Rellich, (GiT, Section 7.10)

For all k, $L^2_{k+1} \hookrightarrow L^2_k$ is a compact operator (i.e. the image of a bounded set has compact closure).

Given $\kappa \in L^2_{k+2}$ we can still define $\Delta\kappa \in L^2_k$.

Proposition (Elliptic regularity)

Suppose we are given $u \in L^2_1$ for which there exists $v \in L^2_k$ satisfying

$$\langle v, \omega \rangle = \langle u, \Delta\omega \rangle$$

for all $\omega \in L^2_2$. Then $u \in L^2_{k+2}$ and

$$||u||_{k+2} \leq C_k(||v||_k + ||u||_0).$$

These theorems we will treat as magical black boxes. Either you've studied enough PDE to prove them by yourself or you haven't and you now have motivation to do so. Next time we will prove the Hodge theorem assuming these results.