Lecture 3: $U(1)$-bundles

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27th September 2011
• Remember that a $U(1)$-bundle is a space $L$ with a free action of $U(1)$ (i.e. no fixed points) so that the space of orbits is a manifold $M$.

• We’ll write the projection $\pi : L \to M$ and we call $L_p := \pi^{-1}(p)$ a fibre (where $p$ is a point in $M$).

• A section is a smooth map $\sigma : M \to L$ such that $\pi \circ \sigma = \text{id}$ (i.e. a smoothly varying choice of point in each fibre) and a local section is a section defined only over some open subset of $M$.

• Vertical distribution $V$ is the subbundle of $TL$ consisting of vectors tangent to fibres, i.e. $V_x = T_xL_p$ ($\pi(x) = p$).
Connections

However there’s no canonical subspace of horizontal vectors (i.e. which projects 1-1 onto the tangent space of \( M \) via \( \pi \)).

- A connection is a choice of horizontal subspace \( H_x \subset T_x L \) at each point \( x \in L \). We require \( \pi_* : H_x \rightarrow T_{\pi(x)} M \) to be an isomorphism.
- Horizontal lift of a vector field \( X \) on \( M \) is then the unique vector field \( \tilde{X} \) on \( L \) with \( \pi_* \tilde{X} = X \) and \( \tilde{X} \subset H \).
- Moreover we require that if \( g \in U(1) \) then \( g_* \tilde{X}_x = \tilde{X}_{xg} \) (so \( gH_x = H_{xg} \)).
- Write \( \alpha \) for the \( V \)-valued 1-form \( \alpha : TL \rightarrow V \) such that \( \alpha(v) = v \) for all \( v \in V \) and \( \alpha(H) = 0 \).
- Now we can understand what it means to take the vertical component of a vector, that is, the deviation of a local section \( \sigma \) from being horizontal:

\[
\nabla_X \sigma := \alpha_{\sigma(q)}(\sigma_* X) = \sigma_* X - \tilde{X}
\]

for a vector field \( X \) on \( M \) and a point \( q \in M \).
A gauge transformation is a bundle automorphism: a diffeomorphism $\Phi$ of $L$ such that

- $\Phi$ preserves the bundle structure ($\pi \circ \Phi = \pi$)
- and $\Phi$ is $U(1)$-equivariant, i.e. $\phi(xg) = \phi(x)g$ for all $x \in L, g \in U(1)$.

These form an infinite-dimensional Lie group of gauge transformations (Ex: You can see heuristically it’s infinite-dimensional by asking yourself what its Lie algebra is). If $u$ is a gauge transformation then it acts in the obvious way on sections ($\sigma \mapsto u\sigma$) and on connections: $(u\nabla)_{\chi}\sigma := u_{*}\nabla_{\chi}(u^{-1}\sigma)$ (Ex: Check this does what you think it does to the horizontal spaces).
Example

Take $L = U(1) \times M$ and let the horizontal space at $(e^{i\theta}, q)$ be $0 \oplus T_q M$. This is the trivial connection on the trivial $U(1)$-bundle.

- If $\sigma$ is a section then it can be considered as a map $e^{i\theta} : M \to U(1)$ and $\nabla_X \sigma = id\theta(X)$. We therefore usually write $\nabla = d$.

- More generally we can take $\nabla = d + iA$ for any 1-form $A$, so $\nabla_X e^{i\theta} = id\theta(X) + iA(X)$. This corresponds to the horizontal distribution given by $H = \text{gr}(-iA)$ where $\text{gr}$ denotes the graph of $-A$ considered as a linear map $\mathbb{R}^3 \to \mathbb{R}$.

- If $u : (e^{i\theta}, q) \mapsto (e^{i(\psi+\theta)}, q)$ is a gauge transformation then $(u\nabla)_X = d + iA - id\psi$. To see this, note that the image $ce^{i\psi}$ of a horizontal (constant) section $c$ should be horizontal for the new connection.

- In particular, a connection $d + iA$ is gauge equivalent to the trivial connection if and only if $A$ is exact.

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1This equation may seem more familiar if you multiply the right-hand side by $e^{i\theta}$.
The difference of two connections

The trick with the 1-form is quite general. Suppose that we have two connections, $H$ and $H'$. Pick a point $x \in L$ and consider how we might write $H'_x$ in terms of $H_x$. Since both project 1-1 to $T_{\pi(x)}M$ we can write $H'_x$ as the graph of a linear map $H_x \to V_x = i\mathbb{R}$. Since the connections are $U(1)$-invariant we see that this descends to a 1-form $A$ on $M$. In particular

$$(\nabla'_X - \nabla_X)\sigma = iA(X)$$

Technically, of course, $A$ takes values not in $\mathbb{R}$ but in the $\mathbb{R}$-bundle over $M$ whose fibre at $q = \pi(p) = \pi(p')$ is $V_p \cong V_{p'}$ but this turns out to be the trivial bundle in a canonical way if $L$ and $M$ are oriented.
We have seen that connections on the trivial bundle over $M$ are of the form $d + iA$ for a 1-form $A$ on $M$ and that these are gauge equivalent to the trivial connection $d$ if and only if $A$ is exact (if and only if $A$ is closed, by the Poincaré lemma). Therefore we see that the curvature 2-form $F = dA$ is a measure of how nontrivial a connection is. Now suppose we have a nontrivial bundle (so there’s no canonical $d$); how do we make sense of curvature?
Notice that any vector field $X$ on $M$ has a horizontal lift $\tilde{X}$ to $L$ with respect to a given connection $\nabla$: it’s just the unique vector field such that $\tilde{X}_p \in H_p$ and $\pi_* \tilde{X} = X$. A 2-form eats two vectors and outputs a number, so let’s try

$$F(X, Y) = \alpha([\tilde{X}, \tilde{Y}])$$

(remember $\alpha$ is the projection with kernel $H$).
Lemma

This defines a 2-form.

Proof.

We need to show that $F$ is $\mathcal{C}^\infty$-bilinear, i.e.

$$F(f\tilde{X}, g\tilde{Y}) = fgF(X, Y)$$

for any two functions $f, g \in \mathcal{C}^\infty(M)$. But

$$[f\tilde{X}, g\tilde{Y}] = f\tilde{X}(g\tilde{Y}) - g\tilde{Y}(f\tilde{X})$$

$$= fg[\tilde{X}, \tilde{Y}] + f\tilde{X}(g)\tilde{Y} - g\tilde{Y}(f)\tilde{X}$$

Since the final two terms are horizontal they are killed by $\alpha$. \hfill \Box

Ex: Check that when $L$ is the trivial bundle and $\nabla = d + iA$ this gives $F = idA$ (it may help to pass to coordinates).
Return to Maxwell’s equations

The $F$ we have here is precisely the magnetic field $\beta$ we had before.

**Lemma**

The curvature 2-form of a $U(1)$-bundle satisfies the intrinsic Maxwell equation $dF = 0$, i.e. $F$ is closed.

**Proof.**

In a local chart we have seen that $F = idA$ and therefore $dF = 0$.

Next we will seek connections which satisfy the extrinsic Maxwell equation $d \star F = \mu_0 J$. 