Lecture 23: The Atiyah-Bott formula

Jonathan Evans

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We are now in a position to explain the Atiyah-Bott formulae for the Poincaré polynomial of the moduli space of stable bundles. Let E be a Hermitian complex vector bundle on a Riemann surface. Consider a slope vector $\underline{\mu} = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \ldots, \mu_r, \ldots, \mu_r)$ where each slope μ_i has multiplicity n_i and $\mu_i = k_i/n_i$. Let F_μ be the bundle over M whose fibre at a point is the partial flag manifold $GL(n, \mathbb{C})/B_\mu$. Here B_μ is the subgroup of matrices preserving a flag in E (i.e. a sequence of subspaces $E_0 \subset \cdots \subset E_r = E$) where the ranks of the E_i are equal to the multiplicities of the slope vector. A section of F_μ is then a filtration of E and we consider the subspace $\mathcal{F}_\mu$ of sections where the subspaces E_i/E_{i-1} trace out a bundle of degree k_i (this restricts to a component of the space of sections). We say these are filtrations of type μ.

Canonicity of the HN filtration implies that there is a map

$$F\colon \mathcal{A}_\mu \to \mathcal{F}_\mu$$

taking a connection to the HN filtration of the corresponding holomorphic vector bundle.

Theorem (Atiyah-Bott, Section 15)

This map a) extends to Sobolev completions, b) is continuous.

We may or may not prove b), which involves hard work in algebraic geometry. If we fix $E_\mu \in \mathcal{F}_\mu$ and denote by $\mathcal{B}_\mu = F^{-1}(E_\mu) \subset \mathcal{A}_\mu$ and by $\mathrm{Aut}(E_\mu)$ the subgroup of $\mathcal{G}_\mathbb{C}$ consisting of automorphisms which preserve the filtration then we see $\mathcal{F}_\mu = \mathcal{G}_\mathbb{C}/\mathrm{Aut}(E_\mu)$ (because $GL(n,\mathbb{C})$ acts transitively on the fibre of F_μ) and $\mathcal{A}_\mu$ is the associated bundle

$$\mathcal{B}_\mu \to \mathcal{A}_\mu = \mathcal{G}_\mathbb{C} \times_{\mathrm{Aut}(E_\mu)} \mathcal{B}_\mu \to \mathcal{F}_\mu = \mathcal{G}_\mathbb{C} \times_{\mathrm{Aut}(E_\mu)} \{E_\mu\}$$

This means that the Borel space $\mathcal{A}_\mu \times_{\mathcal{G}_\mathbb{C}} E\mathcal{G}_\mathbb{C}$ is

$$\left(\mathcal{G}_\mathbb{C} \times_{\mathrm{Aut}(E_\mu)} \mathcal{B}_\mu\right) \times_{\mathcal{G}_\mathbb{C}} E\mathcal{G}_\mathbb{C} = \mathcal{B}_\mu \times_{\mathrm{Aut}(E_\mu)} E\mathcal{G}_\mathbb{C}$$

which is just homotopy equivalent to the Borel space $\mathcal{B}_\mu \times_{\mathrm{Aut}(E_\mu)} E\mathrm{Aut}(E_\mu)$. Moreover we can cut down further. Within $\mathcal{B}_\mu$ there is a locus of split bundles. Namely, use the Hermitian metric on E to define a splitting $E_i \cong D_1 \oplus \cdots \oplus D_i$ and restrict attention to $\mathcal{B}_\mu^0 \subset \mathcal{B}_\mu$ consisting of connections such that the D_i are semistable holomorphic subbundles (and $\mathrm{Aut}(E_\mu^0)$ automorphisms preserving the splitting). Then

$$\mathrm{Aut}(E_\mu^0) \cong \prod_{i=1}^{r} \mathrm{Aut}(D_i)$$

where $\mathrm{Aut}(D_i)$ denotes $GL(n_i, \mathbb{C})$-automorphisms and

$$\mathcal{B}_\mu^0 \cong \prod_{i=1}^{r} \mathcal{A}_{ss}(D_i)$$

where $\mathcal{A}_{ss}$ denotes connections inducing a semistable holomorphic vector bundle structure on D_i.

However $\mathrm{Aut}(E_\mu^0) \subset \mathrm{Aut}(E_\mu)$ is a homotopy equivalence. To see this, notice that the elements of the left-hand side are block-diagonal whereas the elements of the right-hand side are block-upper-triangular. Now one can multiply the off-diagonal chunks by $t \in [0,1]$ without affecting invertibility and deformation retract one onto the other. Similarly (by rescaling the second fundamental form) one can see that $\mathcal{B}_\mu$ retracts onto $\mathcal{B}_\mu^0$. Therefore one sees that the Borel space

$$\left(\mathcal{B}_\mu^0\right)_{\mathrm{Aut}(E_\mu^0)}$$

is homotopy equivalent to $(\mathcal{B}_\mu)_{\mathrm{Aut}(E_\mu)}$ which we already know is equivalent to $(\mathcal{A}_\mu)_{\mathcal{G}_\mathbb{C}}$.

Hence

Theorem

$$H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}_\mu;\mathbb{Q}) \cong \bigotimes_{i=1}^{r} H^*_{\mathrm{Aut}(D_i)}(\mathcal{A}_{ss}(D_i);\mathbb{Q})$$

where we have used field coefficients to invoke Künneth's theorem. This will be a useful fact once we have established the following theorem

Theorem

The stratification $\mathcal{A}_\mu$ of $\mathcal{A}$ is $\mathcal{G}$-equivariantly perfect, so

$$H^*(B\mathcal{G}) = H^*(B\mathcal{G}_{\mathbb{C}}) = H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}) = \sum_\mu H^{*-d_\mu}_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}_\mu)$$

Here d_μ is the codimension of the stratum which we computed last time from Riemann-Roch:

$$d_\mu = 2\dim_{\mathbb{C}}(H^1(\mathrm{End}_1(\mathcal{E}))) = 2\sum_{i>j}((n_i k_j - n_j k_i) + n_i n_j(g-1))$$

This theorem is proved by showing that every stratum is self-completing! We will postpone the proof until we have derived the Atiyah-Bott formula in the simplest case, $n = 2$, $k = 1$ (when semistable is the same as stable). In this case the Atiyah-Bott formula becomes quite manageable. In general you should think of it as more of a recursive procedure... Now we have

$$P(B\mathcal{G};\mathbb{Q}) = \frac{\prod_{k=1}^{n}(1+q^{2k-1})^{2g}}{(1-q^{2n})\prod_{k=1}^{n-1}(1-q^{2k})^2} = \frac{(1+q)^{2g}(1+q^3)^{2g}}{(1-q^4)(1-q^2)^2}$$

and we have the following possibilities for slopes: $\mu = \left(\frac{1}{2}, \frac{1}{2}\right)$ which corresponds to the semistable case (where the HN filtration is just $0 \subset E$) and $\mu = (r+1, -r)$ which corresponds to the HN filtration $0 \subset L \subset E$ where $c_1(L) = r+1$ (so the quotient L' has $c_1 = -r$).

We write $\mathcal{A}_r$ for the stratum with $\mu = (r+1, -r)$. Now by the last theorem we proved,

$$\begin{aligned} H^*_{\mathcal{G}_\mathbb{C}}(\mathcal{A}_r) &= H^*_{\mathrm{Aut}(L)}(\mathcal{A}(L)) \otimes H^*_{\mathrm{Aut}(L')}(\mathcal{A}(L')) \\ &= H^*(B\mathrm{Aut}(L)) \otimes H^*(B\mathrm{Aut}(L')) \end{aligned}$$

because any holomorphic line bundle is stable. Now $\mathrm{Aut}(L)$ and $\mathrm{Aut}(L')$ are the (complexified) gauge groups of $U(1)$-bundles and we computed $P(B\mathcal{G})$ for $U(1)$-bundles last time:

$$P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}_r) = \left(\frac{(1+q)^{2g}}{1-q^2} \right)^2$$

and the codimension formula gives

$$d_\mu = 4r + 2g$$

The equivariant perfection of our stratification (which we have yet to prove) implies

$$P(B\mathcal{G};\mathbb{Q}) = \frac{(1+q)^{2g}(1+q^3)^{2g}}{(1-q^4)(1-q^2)^2}$$
$$= P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}_{ss}) + \sum_{r=0}^{\infty} q^{4r+2g}\left(\frac{(1+q)^{2g}}{1-q^2}\right)^2$$

which rearranges to give

$$P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}_{ss}) = (1+q)^{2g}\frac{(1+q^3)^{2g} - q^{2g}(1+q)^{2g}}{(1-q^2)^2(1-q^4)}$$

Now why is this relevant to us? What does this equivariant cohomology of the semistable stratum have to do with the moduli space of stable vector bundles? The moduli space is

$$\mathcal{A}_{ss}/\mathcal{G}_\mathbb{C}$$

but the points have stabilisers...

Proposition

The stabiliser of a compatible connection on a stable bundle under the $\mathcal{G}_{\mathbb{C}}$-action consists of the constant central scalars, i.e. the global gauge transformations $U(1) \overset{\text{diag}}{\subset} U(n)$ acting by multiplication on P. Therefore the $\mathcal{G}_{\mathbb{C}}/\mathbb{C}^*$-equivariant cohomology of $\mathcal{A}_{ss}$ is the actual cohomology of $\mathcal{A}_{ss}/\mathcal{G}_{\mathbb{C}}$. Moreover the $\mathcal{G}_{\mathbb{C}}$-equivariant cohomology of $\mathcal{A}_{ss}$ is (rationally) $H^*(\mathcal{A}_{ss}/\mathcal{G}_{\mathbb{C}}) \otimes H^*(B\mathbb{C}^*)$. (Can replace $\mathcal{G}_{\mathbb{C}}$ by $\mathcal{G}$ and $\mathbb{C}^*$ by $U(1)$).

Proof.

A complex gauge transformation fixing a compatible connection gives a holomorphic automorphism of the bundle. If the automorphism is not a constant scalar this decomposes the bundle into proper holomorphic eigensubbundles $E = E_1 \oplus \cdots \oplus E_k$. Since E is stable, both E_1 and $E_2 \oplus \cdots \oplus E_k$ have both strictly smaller slope than E (they're subbundles) and strictly larger slope (they're quotients). We saw early on that the only constant gauge transformations are the central ones. Now to see the statement about equivariant cohomology it suffices to show that the fibration $U(1) \to \mathcal{G} \to \mathcal{G}/U(1)$ is rationally a product. But this is true because there is a homomorphism $\mathcal{G} \to U(1)$ which takes the determinant of a gauge transformation at some framed basepoint and the composition $U(1) \to \mathcal{G} \to U(1)$ is just a map of degree n. □

Since $P(BU(1)) = \frac{1}{1-q^2}$ this gives a formula for the Poincaré polynomial for the moduli space $\mathcal{M}_{ss}(2,1)$ of semistable (i.e. stable) bundles of rank 2 and degree 1 on a curve of genus g

$$\boxed{P(\mathcal{M}_{ss}(2,1)) = (1+q)^{2g}\frac{(1+q^3)^{2g} - q^{2g}(1+q)^{2g}}{(1-q^2)(1-q^4)}}$$

Exercise

Check in the case $g = 2$ that this is indeed the generating function for the Betti numbers of a compact finite-dimensional manifold. What is its dimension?

We now have three lectures and three things to prove:

- Equivariant perfection of the Yang-Mills functional,
- The statement that the compactification of the stratum $\mathcal{A}_\mu$ involves only higher strata,
- The claim that the map $\mathcal{A}_\mu \to \mathcal{F}_\mu$ sending a connection compatible with a holomorphic vector bundle whose HN filtration has type $\underline{\mu}$ to the filtration (considered as a section of a bundle of partial flag manifolds) is continuous.

Next lecture we will try and prove the first two. We will not prove the last due to lack of time, motivation and background in algebraic geometry. In the final lecture I will talk instead about how our stratification can be thought of as a Morse stratification for the Yang-Mills functional (conjectured by Atiyah-Bott, proved by Daskalopoulos).