Lecture 21: Equivariant cohomology III

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Last time we introduced Morse-Bott functions $f\colon M \to \mathbb{R}$, whose critical points come in submanifolds on whose normal bundles the Hessian is a nondegenerate quadratic form. We observed that in a good case (perfect) the Poincaré polynomial of the ambient manifold M is

$$P(M) = \sum_{N \in \mathrm{Crit}(f)} P(N) q^{\lambda_N}$$

where $\mathrm{Crit}(f)$ is the set of critical manifolds and λ_N is the rank of the subbundle $\nu_-(N)$ of negative eigenvectors for the Hessian in $\nu(N)$. We gave a purely local criterion for a homology class $\sigma \in H_*(N;\mathbb{Z})$ to contribute to the homology of M: σ is *self-completing* if $\partial(\nu_-(N)|_\sigma)$ is nullhomologous in $\partial\nu_-(N)$, that is the image of σ under

$$H_*(N) \stackrel{\text{Thom}\cong}{\longrightarrow} H_{*+\lambda_N}(\nu_-(N), \partial\nu_-(N)) \stackrel{\text{connecting}}{\longrightarrow} H_{*+\lambda_N-1}(\partial\nu_-(N))$$

vanishes. We finished with the example of $\sigma \in H_0(\mathbb{CP}^1)$ contributing to $H_2(\mathbb{CP}^2)$.

This example is not as silly as it seems and indeed for the Yang-Mills functional, all the critical manifolds will be (equivariantly) self-completing in this way (which is what allows us to compute the cohomology of the moduli spaces!). Let me now say some words about the equivariant case. Equivariant cohomology can be computed by taking a G-invariant function $f\colon M \to \mathbb{R}$, considering the function $F(m,e) = f(m)$ on $M \times EG$ (which is still G-invariant) and then descending to f_G on the Borel space $M_G = M \times_G EG$. Suppose that f is Morse (in particular its critical points are fixed points of the G-action). Then the critical manifolds of f_G are copies of BG, i.e. if p is critical then

$$\{p\} \times_G EG \cong BG$$

is a critical submanifold of M_G.

More generally if p is a critical point whose G-stabiliser is H then there is a copy of G/H in the critical set and let's assume for a second that this is a whole connected component of the critical set and that f is Morse-Bott. Then upstairs the critical manifold looks like $(G/H) \times_G EG$. But remember that $G \times_G EG = EG$ so $(G/H) \times_G EG = EG/H = BH$ (since EG is a contractible space with a free H-action). What is the normal bundle of this submanifold of the Borel space? Well since H preserves the point p let's pick an H-invariant metric on M (which is fine as long as H is compact, by averaging) and observe that H preserves the normal bundle and Hessian of f. In particular it preserves ν_- and ν_+ and we get a representation $H \to \mathrm{Aut}(\nu_-(G/H)_p)$. It's not hard to see that the normal bundle of $BH = (G/H) \times_G EG$ in $M \times_G EG$ is precisely the bundle over BH associated (to the universal H-bundle) with this representation of H on $\nu_-(G/H \subset M)$. The most that such an orbit could contribute to the equivariant cohomology is therefore $q^{\lambda(N)}P(BH)$ where $N = G/H$ considered as a Morse-Bott critical submanifold of f.

Example

Let's illustrate this with the example of S^1 *acting by rotation around the z-axis of the unit sphere* $S^2 \subset \mathbb{R}^3$*. Take* z *as the* S^1*-invariant Morse function. This has two critical points (both fixed points) and so the Borel space has two critical manifolds, both copies of* $BS^1 = \mathbb{CP}^\infty$*. For the minimum,* $\nu_- = 0$ *as there are no downward flowlines on* M*. For the maximum, the normal bundle of* $\mathbb{CP}^\infty$ *is the associated bundle for the representation* $S^1 \to \mathrm{Aut}(T_N S^2)$ *where* N *is the North pole. This is (minus) the standard representation and hence the bundle is (minus) the tautological bundle over* $\mathbb{CP}^\infty$*. The unit normal bundle is therefore* $S^1 \to S^\infty \to \mathbb{CP}^\infty$ *and we see that any homology class* $[\mathbb{CP}^n]$ *in* $H_{2n}(\mathbb{CP}^\infty; \mathbb{Z})$ *lifts to a* S^{2n+1}*-sphere in* S^∞ *which is nullhomologous (in fact contractible) and hence the Morse function is equivariantly self-completing (and hence perfect).*

We will now find a criterion for when this happens, just in terms of the representation $H \to \mathrm{Aut}(\nu_-(G/H)_p)$. First, what is $\partial\nu_-(BH)$? Well for one thing it's a $\lambda(N) - 1$-sphere bundle over BH. There is a classifying map $BH \to BSO(\lambda(N))$ (which is just B applied to the representation $H \to SO(k+1) = \mathrm{Aut}(\nu_-(N)_p)$) along which one can pull back the Euler class so one has an Euler class in $H^*(BH)$.

Lemma

The map $H_*(BH) \cong H_*(\nu_-) \to H_*(\nu_-, \partial\nu_-)$ is surjective when the Euler class of this sphere bundle is not a zero divisor in $H^*(BH)$.

This clearly never happens in a finite-dimensional, nonequivariant setting: the Euler class is nilpotent because cohomology is supported in finitely many degrees. However equivariant cohomology is a module over $H^*(BG)$ and in particular can be very infinite! Here is an example of the Lemma in action: in the example of S^1 acting on S^2 from earlier, the Euler class was $c_1 \in H^2(BU(1))$. Since $H^2(BU(1)) \cong \mathbb{Z}[c_1]$, the class c_1 is not a zero-divisor.

Proof.

Despite being over an infinite-dimensional base manifold, one can prove (using finite-dimensional approximations) that the sphere bundle admits a cohomology Gysin sequence

$$\cdots \leftarrow H^{r-\lambda(N)+1}(BH) \stackrel{\check{a}}{\leftarrow} H^r(\partial\nu_-) \stackrel{\check{b}}{\leftarrow} H^r(BH) \stackrel{\check{c}}{\leftarrow} H^{r-\lambda(N)}(BH) \leftarrow \cdots$$

which is the dual of the homology LES of the pair $(\nu_-, \partial\nu_-)$ under the duality $H^* \cong (H_*)\check{}$ (take field coefficients)

$$\cdots \to H_{r-\lambda(N)+1}(BH) \stackrel{a}{\to} H_r(\partial\nu_-) \stackrel{b}{\to} H_r(BH) \stackrel{c}{\to} H_{r-\lambda(N)}(BH) \to \cdots$$

(where we have implicitly used the Thom isomorphism and the fact that $\nu_- \simeq BH$). The map $\check{c}$ is cup product with the Euler class of the sphere bundle and but since it is the dual of c, it is injective precisely when c is surjective. □

See Dold's "Lectures on algebraic topology", p.326, for the Gysin sequence and this duality in the finite-dimensional setting.

Now we will talk about stratifications. To each critical point p of f you look at the stable submanifold $S_p^{\pm}$ (of points q which flow down to p in the sense that $\phi_t(q) \to p$ as $t \to \pm\infty$). Clearly a generic point q will live in S_m^+ because the other stable and unstable manifolds have some codimension. Since every point is in the stable manifold of some critical point

$$M = \bigcup_p S_p^+$$

is a stratification.

Moreover the stratification has the property that

$$\bar{S}_p^+ \subset \bigcup_{\mathrm{ind}(q) \geq \mathrm{ind}(p)} S_q^+$$

Here is why this is true, without the gory details of proving compactness and transversality in Morse theory. If $x_i \in S_p^+$ then there is a Morse flowline γ_i from x_i to p. As $i \to \infty$ this flowline breaks into a union of flowlines joining critical points and for a generic Morse function there are no flowlines from a critical point of lower index to one of higher index.

All we actually need to do the Mayer-Vietoris calculations is a stratification $M = \bigcup_{i \in I} M_i$ where the indexing set I is partially ordered and:

- $\bar{M}_i \subset \bigcup_{j \geq i} M_j$,
- for any finite subset $J \subset I$ there is a finite number of minimal elements in the complement $I \setminus J$ (in particular, when $J = \emptyset$, there are a finite number of minimal elements in I),
- for each q there are finitely many indices i such that $\mathrm{codim}(M_i) < q$.

Now we have a natural stratification on the space $\mathcal{A}$. Identifying ∇ with $\mathcal{E}_\nabla$ we can ask what the Harder-Narasimhan filtration of $\mathcal{E}_\nabla$ looks like. If $D_1, \ldots, D_r$ are the semistable quotients and $\mu_i = \mu(D_i)$ are their decreasing slopes then we write μ for the vector

$$(\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \ldots, \mu_r, \ldots, \mu_r)$$

where each slope μ_i is repeated $\dim(D_i)$-times. Now write $\mathcal{A}_\mu$ for the space of connections whose associated holomorphic vector bundle gives the vector μ (we say ∇ has type μ). In particular, if $\mu = (\mu(\mathcal{E}), \ldots, \mu(\mathcal{E}))$ then $\mathcal{E}$ is semistable (its Harder-Narasimhan decomposition is only two steps long: $0 \subset \mathcal{E}$). We don't know much about semistable as opposed to stable bundles so the following observation is useful:

Exercise

If $\gcd(\dim(\mathcal{E}), \deg(\mathcal{E})) = 1$ *then* $\mathcal{E}$ *is semistable if and only if it is stable.*

We want a partial ordering on the vectors μ such that

$$\bar{\mathcal{A}}_\mu \subset \bigcup_{\lambda \geq \mu} \mathcal{A}_\lambda$$

This is not hard to define. Given a vector μ, draw a piecewise linear path P_μ in $\mathbb{R}^2$ starting at the origin and moving to the right such that the segment in the strip $i \leq x \leq i+1$ has slope equal to the ith component of μ. For example a semistable bundle will give a straight line of slope $\mu(\mathcal{E})$ joining the origin to $(\dim(\mathcal{E}), \deg(\mathcal{E}))$. This piecewise linear path is part of the boundary of a convex polygon since the slopes are decreasing.

Theorem (Shatz, Atiyah-Bott)

If we define $\lambda \geq \mu$ to mean that P_λ lies entirely above P_μ then

$$\bar{\mathcal{A}}_\mu \subset \bigcup_{\lambda \geq \mu} \mathcal{A}_\lambda.$$