Lecture 2: Magnetostatics

Jonathan Evans

22nd September 2011

To avoid talking about special relativity I'll specialise to the case of a time-independent magnetic field $\mathbf{B}$ with no electric field ($\mathbf{E} = 0$) and (time-independent) current density $\mathbf{J}$. This satisfies two of Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1}$$

We will reformulate these equations to get rid of things like $\times$ which depends on our working with $\mathbb{R}^3$. First, what the hell is $\mathbf{B}$? The force it exerts on a particle of charge q and velocity $\mathbf{v}$ is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

Let's replace $\mathbf{B}$ by the 2-form

$$\beta = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2$$

This eats the vector $q\mathbf{v}$ and outputs

$$\beta(qv, \cdot) = \sum_{i=1}^{3} (q\mathbf{v} \times \mathbf{B})_i dx_i$$

which is the 1-form dual to the force vector exerted on the particle, i.e.

$$\beta(q\mathbf{v}, \mathbf{X}) = \mathbf{F} \cdot \mathbf{X} \text{ for all vector fields } \mathbf{X}$$

In general a magnetic field is precisely this, a 2-form which eats a vector $q\mathbf{v}$ and outputs the dual to the force vector exerted.

Maxwell's equation $\nabla \cdot \mathbf{B}$ translates into $d\beta = 0$. Let's replace $\mathbf{J}$ by its dual 1-form (say ψ) and see how the other Maxwell equation translates. We need one more operation: define $\star$ on forms by sending dx_1 to $dx_2 \wedge dx_3$, $dx_1 \wedge dx_2$ to dx_3 and more generally

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

to the $n-k$-form $dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}$ such that

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}} = \mathrm{vol} := dx_1 \wedge \cdots \wedge dx_n$$

In particular

$$\star\beta = B_1 dx_1 + B_2 dx_2 + B_3 dx_3$$

so note that $d \star \beta$ is the $n-1$-form dual to $\nabla \times \mathbf{B}$.

$$d\beta = 0 \qquad\qquad d \star \beta = \mu_0(\star\psi) \tag{2}$$

Now these equations make sense on any Riemannian manifold (M, g) provided we can make sense of duality between 1-forms and vector fields and of the Hodge star $\star$. Indeed, define the dual 1-form to a vector field X to be χ, such that

$$\chi(Y) = g(X, Y)$$

and define the Hodge star by

$$\chi \wedge \star\chi = |\chi|^2 \mathrm{vol}_g$$

Note that $\star^2 = \pm 1$.

Remark

In fact the current is usually thought of as the $n-1$-form $j = \star\psi$ which takes as its input a "hyperplane" $v_1 \wedge \cdots \wedge v_{n-1}$ and outputs the flux through an infinitesimal hyperplane $j(v_1, \ldots, v_{n-1})$.

Observe that by Maxwell's equations the 2-form β is closed.

Lemma (Poincaré's lemma)

Let κ be a k-form on a contractible n-manifold (like $\mathbb{R}^n$). If $d\kappa = 0$ then there exists an $k-1$-form α (the potential or antiderivative) such that $d\alpha = \kappa$.

Proof.

Bookwork exercise! $\rightsquigarrow$☺ (a bookworm) □

Therefore $\beta = d\alpha$ for a potential 1-form α. However, there is ambiguity in the choice of α, since

$$d(\alpha + df) = d\alpha = \beta$$

for any function f. In fact this is the only ambiguity on $\mathbb{R}^n$, since the difference of two potentials defining the same fields is an closed 1-form and hence there exists such an f by Poincaré's lemma. This ambiguity is called gauge freedom and changing potential using a function f is called a gauge transformation.

The gauge principle

We now seek to understand this formalism a little better. We begin with a beautiful observation of Weyl. Quantum mechanics uses a complex-valued function ψ to describe physics, the *wave-function*, whose squared norm is supposed to represent probability density and whose phase is something unobservable. By unobservable, I mean that you only ever measure expectation values when you make observations and the expectation value of an observable $\hat{A}$ (a $\mathbb{C}$-linear operator on the space of functions) is

$$\int \psi^* \hat{A} \psi \mathrm{vol}$$

Therefore you should be able to multiply ψ by $e^{i\theta}$ (θ constant) and obtain the same physics. However, it seems unnatural to make global changes of phase: physics is supposed to be local. We ask ourselves: what if we allow θ to depend on position in space-time? (For magnetostatics let's just allow θ to vary in space).

In Dirac's relativistic quantum theory of spin $1/2$ particles ψ is actually a *spinor* but it still carries an action of $U(1)$ which is "irrelevant" for the expectation values. ψ satisfies Dirac's equation

$$D\psi := -i\gamma^{\mu}\partial_{\mu}\psi + m\psi = 0$$

where the μ runs over space-time coordinates and the γ^{μ} are certain matrices. But, lo and behold, when we make an arbitrary phase change $\psi \mapsto e^{i\theta}\psi$ we get

$$D(e^{i\theta}\psi) = e^{i\theta}D\psi - i\gamma^{\mu}(\partial_{\mu}\theta)\psi$$

so the local change of phase screws up our favourite equations. However, we could get away with this if we absorb the error term into the equation, that is we simultaneously transform

$$\psi \mapsto e^{i\theta}\psi,\ \partial_{\mu} \mapsto \partial_{\mu} + \partial_{\mu}\theta$$

We have therefore introduced a new field (actually a potential!) $\partial_\mu \theta$ which interacts with the 'particles' ψ in our theory. Of course we want our physics to be locally phase invariant, so this interaction must somehow be trivial. But remember that the magnetostatic potential $\alpha = d\theta$ has no observable effect on particles since the magnetic field $\beta = d\alpha = dd\theta = 0$! So this is the perfect receptacle for our new potential. That is, we use 'gauge invariance' of magnetostatics (or EM) to eat up the 'local phase invariance' of quantum mechanics. Neat idea.

More generally, this is the idea behind a *gauge theory*: a formalism for localising some symmetry of a physical theory. Instead of thinking of $\arg(\psi)$ as a function, we think of it as a *section of a $U(1)$-bundle*.

- A $U(1)$-bundle is just a space L with a free $U(1)$-action (for any $x \in L$, $xg = x$ implies $g = 1 \in U(1)$). There is a quotient map $\pi : L \to L/U(1) =: M$ where M is a manifold (in our case $U(1) \times \mathbb{R}^3 \to \mathbb{R}^3$). π commutes with the $U(1)$-action.
- A section is then a map $\sigma : M \to L$ such that $\pi \circ \sigma = \mathrm{id}$. A *global* section may not exist but local sections always make sense.
- The ∂_μ are replaced by a differential operator called a connection (for differentiating local sections) and the "change of phase" is replaced by a "gauge transformation" which acts on local sections and on the connection simultaneously.

Example (Stupidest example)

$M \times U(1)$ admits a free $U(1)$-action $(x, e^{i\phi}) \mapsto (x, e^{i(\phi+\theta)})$ and the space of orbits is M. For any map $s : M \to U(1)$ the map $x \mapsto (x, e^{is(x)})$ is a section. This is called the product bundle.

Two bundles $\pi : L \to M$ and $\pi' : L' \to M$ over the same space M are called isomorphic if there is a diffeomorphism $\Psi : L \to L'$ such that Ψ commutes with the $U(1)$ actions. In particular $\pi' \circ \Psi = \pi$. A bundle is called trivial if it is isomorphic to the product bundle. Ex: A $U(1)$-bundle is trivial if and only if it admits a section.

Connections

On the total space (L) of a $U(1)$-bundle it is easy to pick out the *vertical vectors* (tangent to the fibres of π, or equivalently to the orbits of $U(1)$). However there's no canonical subspace of *horizontal* vectors (i.e. which projects 1-1 onto the tangent space of M via π). A connection is a choice of horizontal subspace $H_p \subset T_pL$ at each point $p \in L$. Moreover we require that if $g \in U(1)$ then $gH_p = H_{g(p)}$. Now we can understand what it means to take the vertical component of a vector (we can define a projection $\alpha_p : T_pL \to V_p$ whose kernel is H_p, where V_p is the space of vertical vectors at p). That is, we can understand the deviation of a local section σ from being horizontal:

$$\nabla_X \sigma := \alpha_{\sigma(q)}(d\sigma)(X)$$

for a vector field X on M and a point $q \in M$.