Lecture 16: Narasimhan-Seshadri IV

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It remains to show that

Proposition

If the infimum of the Yang-Mills functional on a complexified L^2_2-gauge orbit $\mathfrak{Orb}(\mathcal{E})$ is achieved by some L^2_1-connection ∇ then ∇ is smooth, has constant central curvature and is the unique such connection up to (unitary) gauge transformations.

The idea of the proof is vary ∇ in a well-chosen direction ∇_t inside its gauge orbit and show that unless $\mathcal{YM}(\nabla) = 0$ we can make $\mathcal{YM}(\nabla_t)$ strictly smaller. As usual a "well-chosen direction" means the solution to a well-chosen PDE...

Proof.

Consider the operator $\nabla^*\nabla$ acting on L^2_2 self-adjoint sections of $\mathrm{End}(E)$. This is linear and elliptic because to a first approximation it's just a Laplacian. Its kernel consists of constant scalar matrices, because any other element of the kernel would satisfy

$$0 = \langle \nabla^*\nabla\sigma, \sigma\rangle = \langle \nabla\sigma, \nabla\sigma\rangle$$

i.e. $\nabla\sigma = 0$ so in particular $0 = \nabla^{0,1}\sigma = \bar{\partial}_{\mathcal{E}}\sigma$ and σ is a holomorphic section whose eigenspaces decompose $\mathcal{E}$ holomorphically. By the analogue of the Hodge theorem (where we need to take a Laplacian with nonsmooth coefficients since ∇ is only in the L^2_1-completion of $\mathcal{A}$) in this setting there is a self-adjoint section $h \in L^2_2$ such that

$$i\nabla^*\nabla h = 2\pi i\mu + \star F_\nabla$$

since $2\pi i\mu(\mathcal{E})\mathrm{id} + \star F_\nabla$ is orthogonal to the constant scalars. □

Proof, continued:

Now for small t, $1 + th =: g_t$ is a complexified gauge transformation (because $GL(n, \mathbb{C})$ is open in the space of all matrices). Let $\nabla_t = g_t \nabla$ so by the formula for the complexified gauge action

$$g_t \nabla = \nabla - (\nabla^{0,1} g_t) g_t^{-1} + ((\nabla^{0,1} g_t) g_t^{-1})^\dagger$$

so the curvature of ∇_t changes to

$$\begin{aligned} F_{\nabla_t} &= F_\nabla - \nabla^{1,0}((\nabla^{0,1} g_t) g_t^{-1}) + \nabla^{0,1}(g_t^{-1}(\nabla^{1,0} g_t)) \\ &\quad - (\nabla^{0,1} g) g^{-2} (\nabla^{1,0} g) + g^{-1}(\nabla^{1,0} g)(\nabla^{0,1} g) g^{-1} \\ &= F_\nabla - t(\nabla^{1,0}\nabla^{0,1} - \nabla^{0,1}\nabla^{1,0}) h + \epsilon(t, h) \end{aligned}$$

where the error term has L^2-norm bounded above by $C||h||_{L^2_2} t^2$ for small t. □

Proof, concluded:

Since $\nabla^*\nabla = i\star(\nabla^{1,0}\nabla^{0,1} - \nabla^{0,1}\nabla^{1,0})$ we get

$$\mathcal{YM}(\nabla_t) = \mathcal{YM}(\nabla)(1-t) + \mathcal{O}(t^2)$$

Since ∇ infimises $\mathcal{YM}$ over its $\mathcal{G}_{\mathbb{C}}$-orbit (and since $\nabla_t \in \mathcal{G}_{\mathbb{C}} \cdot \nabla$) we see that $\mathcal{YM}(\nabla) = 0$. □

It remains to see that a) ∇ is smooth; b) ∇ is unique up to gauge transformations. Smoothness follows from the fact that it is the solution to an elliptic equation (Yang-Mills). Strictly speaking we need to be careful because the Yang-Mills equations are not actually elliptic (in particular their solution spaces contain infinite-dimensional gauge orbits). When we prove Uhlenbeck's theorem we will come to understand the sense in which the equations are elliptic, and from that we will get regularity for their solutions from general elliptic theory.

To see that ∇ is the unique constant central curvature connection in its complexified gauge orbit (up to (non-complexified) gauge transformations), suppose that $g\nabla$ is another constant central curvature connection (for some $g \in \mathcal{G}_{\mathbb{C}}$). Since every complex matrix can be written in the form PU with U unitary and P positive definite Hermitian matrix, we can write $g = pu$ with $u \in \mathcal{G}$ and $g^\dagger = g$. Therefore WLOG $g = g^\dagger$.

Now $F_{g\nabla} = F_{\nabla} = -2\pi i\mu(\mathcal{E})$ implies that

$$\nabla^{0,1}\nabla^{1,0}g^2 = -((\nabla^{0,1}g^2)g^{-1})((\nabla^{0,1}g^2)g^{-1})^{\dagger}$$

and the trace of this implies that (for $\tau = \mathrm{Tr}(g^2)$)

$$\Delta\tau \leq 0$$

with equality if and only if $\nabla^{0,1}g^2 = 0$. The maximum principle for subharmonic functions implies that $\Delta\tau \equiv 0$ and $\nabla^{0,1}g^2 = \nabla^{1,0}g^2 = 0$. Unless g is a constant scalar its eigenspaces decompose $\mathcal{E}$ and hence $\nabla = g\nabla$. This proves uniqueness.

We have now seen a proof of the Narasimhan-Seshadri theorem modulo Uhlenbeck compactness.

Theorem

Let ∇_i be a sequence of L^2_1-connections on a principal $U(n)$-bundle over a Riemann surface. Suppose that $||F_{\nabla_i}||^2_L \leq C$. Then there exists a subsequence ∇_{i_j} and L^2_2-gauge transformations u_j such that $u_j\nabla_{i_j}$ converges weakly in L^2_1 to a limiting connection ∇_∞ and $||F_{\nabla_\infty}||_{L^2} \leq C$.

I will now outline a sketch proof of this theorem. We may or may not fill in all the details.

The first step is to prove a *gauge-fixing theorem*. Remember that in magnetostatics we have a (let's say compactly-supported) vector potential $\mathbf{A}$ and we can change it to $\mathbf{A} + \nabla f$ for any (compactly-supported) function f. The Maxwell equations then become $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$. Note that ∇^2 is the Laplacian and hence the equation looks elliptic except for the screwy first term. If only it were to vanish...but we can make it vanish! We only need to find an $\mathbf{A}' = \mathbf{A} + \nabla f$ which solves $\nabla \cdot \mathbf{A}' = 0$ but this means finding a function f such that $\nabla^2 f = -\nabla \cdot \mathbf{A}$. But this is Poisson's equation for f and so admits a unique compactly-supported solution (given explicitly by a formula involving an integral and the Green's function for ∇^2)!

The analogue of this local gauge fixing for 2D Yang-Mills is:

Theorem (Uhlenbeck)

Consider the trivial $U(n)$-bundle over the unit 2-disc. There exist $\kappa > 0$ and $c < \infty$ such that if $\nabla = d + A$ is a connection with $||F_\nabla||_{L^2} \leq \kappa$ then there is an L^2_2-gauge transformation u such that $u\nabla = d + A'$ satisfies

- $d^* A' = 0$,
- $||A'||_{L^2_1} \leq c||F_{\nabla'}||_{L^2}$.

so not only do we have a nice gauge-fixing condition but we have control over the L^2_1-norm of the gauge-fixed connection matrices.

Now it the Banach-Alaoglu theorem tells us that when a sequence in a Hilbert space is bounded it has a weakly-convergent subsequence; thus we see how Uhlenbeck's gauge-fixing theorem gives Uhlenbeck compactness for connections on the trivial bundle over the disc. To go to the global result requires a patching argument which we will not go into. The regularity properties of Yang-Mills connections which we used at the end of the proof of the Narasimhan-Seshadri theorem also follow relatively easily from this gauge fixing theorem because (just as in magnetostatics) a solution to Yang-Mills which is in Coulomb gauge satisfies an elliptic equation. Next time we will prove the gauge fixing theorem.