Lecture 15: Narasimhan-Seshadri theorem III

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15th November 2011

Today we continue the proof of...

Theorem (Narasimhan-Seshadri, Donaldson)

An indecomposable Hermitian holomorphic vector bundle $\mathcal{E}$ on a Riemann surface (M, g) is stable if and only if there is a compatible unitary connection on $\mathcal{E}$ with constant central curvature

$$\star F_\nabla = -2\pi i \mu(\mathcal{E}).$$

...or equivalently...

Theorem

Every stable $\mathcal{G}_{\mathbb{C}}$-orbit on $\mathcal{A}$ contains a unique $\mathcal{G}$-orbit of solutions to $\mathcal{YM}^{-1}(0)$ where

$$\mathcal{YM}(\nabla) = N\left(\frac{\star F_\nabla}{2\pi i} + \mu\right)$$

and N is this funny norm we defined last time.

To summarise what we have already proved:

- If $\mathcal{E}$ is indecomposable and admits a constant central curvature connection then it is stable,
- Either the infimum of $\mathcal{YM}$ over the complexified gauge orbit of connections compatible with $\mathcal{E}$ is attained inside the orbit or there exists a subbundle $\mathcal{F} \subset \mathcal{E}$ with certain properties.

Today we will show a) the existence of this subbundle implie E is not stable and b) with the infimum is attained then it is attained by a constant central curvature connection. Alas for a) we will need to invoke some theory we haven't proved (Hodge theorem for Dolbeaut cohomology) but happily this theory runs along entirely parallel lines to the Hodge theory we developed in the first part of the course. If you're bothered by this discrepancy, consider its rectification an exercise.

Proposition

If $\mathcal{E}$ is a stable holomorphic vector bundle then there is no bundle $\mathcal{F} \not\cong \mathcal{E}$ with $\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{E})$, $\deg(\mathcal{F}) = \deg(\mathcal{E})$, $\inf_{\mathfrak{Orb}(\mathcal{F})} \mathcal{YM} \leq \int_{\mathfrak{Orb}(\mathcal{E})} \mathcal{YM}$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$.

We argue by contradiction. Recall that the map $\mathcal{E} \to \mathcal{F}$ factors:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow \kappa & & \\ 0 & \longleftarrow & \mathcal{Q} & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{P} & \longleftarrow & 0 \end{array}$$

where $\mathrm{rank}(\mathcal{L}) = \mathrm{rank}(\mathcal{P})$, $\det(\kappa)$ is not identically zero and $\mu(\mathcal{L}) \leq \mu(\mathcal{P})$. Stability of $\mathcal{E}$ implies $\mu(\mathcal{L}) < \mu(\mathcal{E}) = \mu(\mathcal{F})$ so $\mu(\mathcal{P}) > \mu(\mathcal{F})$.

Now by the Lemma we worked so hard to prove last time, this implies that for all compatible connections on $\mathcal{F}$ the functional $\mathcal{YM}$ is bounded from below

$$\inf_{\mathfrak{Orb}(\mathcal{F})} \mathcal{YM} \geq \mathrm{rank}(\mathcal{P})(\mu(\mathcal{P}) - \mu(\mathcal{F})) + \mathrm{rank}(\mathcal{Q})(\mu(\mathcal{F}) - \mu(\mathcal{Q}))$$

If we can show that $\inf_{\mathfrak{Orb}(\mathcal{E})} \mathcal{YM}$ is bounded above by

$$\mathrm{rank}(\mathcal{K})(\mu(\mathcal{E}) - \mu(\mathcal{K})) + \mathrm{rank}(\mathcal{L})(\mu(\mathcal{L}) - \mu(\mathcal{E}))$$

then we would get a contradiction to the assumed inequality in the statement of the Proposition (work through the inequalities!). I can feel another Lemma coming on...

Lemma

Suppose that the Narasimhan-Seshadri theorem holds for bundles of rank $\leq \ell$ *and let* $\mathcal{E}$ *be a stable holomorphic vector bundle of rank* $\ell + 1$. *If*

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{L} \to 0$$

is an exact sequence then there is a compatible connection ∇ *on* $\mathcal{E}$ *with*

$$\mathcal{YM}(\nabla) < \mathrm{rank}(\mathcal{K})(\mu(\mathcal{E}) - \mu(\mathcal{K})) + \mathrm{rank}(\mathcal{L})(\mu(\mathcal{L}) - \mu(\mathcal{E}))$$

To prove the lemma we use the Harder-Narasimhan filtration on $\mathcal{K}$

$$0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_r = \mathcal{K}$$

where the quotients $\mathcal{D}_i = \mathcal{K}_i/\mathcal{K}_{i-1}$ are semistable and have strictly decreasing slope.

We also use the filtration on each semistable quotient

$$0 = \mathcal{D}_{i,0} \subset \mathcal{D}_{i,1} \subset \cdots \subset \mathcal{D}_{i,k_i} = \mathcal{D}$$

with stable quotients $\mathcal{C}_{i,j} = \mathcal{D}_{i,j}/\mathcal{D}_{i,j+1}$ of slope $\mu(\mathcal{C}_{i,j}) = \mu(\mathcal{D}_i) < \mu(\mathcal{K}_1) < \mu(\mathcal{E})$ where the last inequality follows from stability of $\mathcal{E}$. By the inductive hypothesis, there exist constant central curvature connections on these $\mathcal{C}_{i,j}$. For an extension of holomorphic bundles

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

we recall the decomposition of a compatible connection into $\nabla_{\mathcal{A}}, \nabla_{\mathcal{C}}$ and a second fundamental form β. Given connections on $\mathcal{A}$ and $\mathcal{C}$ one can extend them to compatible connections on $\mathcal{B}$ (Ex: Why?).

The complex gauge transformation

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

conjugates the $(0,1)$-part of the connection to

$$\begin{pmatrix} \nabla^{0,1}_{\mathcal{A}} & -t\beta^{\dagger} \\ 0 & \nabla^{0,1}_{\mathcal{C}} \end{pmatrix}$$

So one can multiply β by a nonzero constant t and the connection stays in the gauge orbit. As $t \to 0$ the connection converges in $\mathcal{C}^{\infty}$ to a compatible connection on the split bundle $\mathcal{A} \oplus \mathcal{C}$. Since $\mathcal{K}$ is an iterated extension involving the $\mathcal{C}_{i,j}$ we use this observation to find a sequence of connections ∇_t on $\mathcal{K}$ such that as $t \to 0$, $\nabla_t \to \nabla_0$ where ∇_0 is a compatible connection on $\bigoplus_{i,j} \mathcal{C}_{i,j}$ with central curvature

$$\star F_{\nabla_0} = -2\pi i \mathrm{diag}(\mu(\mathcal{C}_{i,j})) = -2\pi i \Lambda_{\mathcal{K}}$$

One finds something analogous for $\mathcal{L}$. The difference is that while $\mu(\mathcal{C}_{i,j}) < \mu(E)$, the diagonal entries for $\mathcal{L}$ will all be strictly bigger than $\mu(\mathcal{E})$.

As usual $\nabla_{t,\mathcal{K}}, \nabla_{t,\mathcal{L}}$ define an operator d_t on forms with values in $\mathcal{L}^* \otimes \mathcal{K}$. If we take the $(0,1)$-part of d_t this is a differential, i.e. $d_t^{0,1}$ squares to zero (since it takes a 0-form to a $(0,2)$-form and on a Riemann surface all 2-forms are of type $(1,1)$). The cohomology of this differential is a version of *Dolbeault cohomology* and it's not hard to prove a version of the Hodge theorem, i.e. that every Dolbeault cohomology class contains a "harmonic representative" (that is $d_t^{0,1}\beta = 0$, $d_t^{0,1*}\beta = 0$). However $d_t^{0,1*} = d_t^{1,0}$ so in fact a harmonic representative will be covariantly constant. We choose the d_t-harmonic representative β_t of the second fundamental form of the extension $\mathcal{K} \to \mathcal{E} \to \mathcal{L}$, so $\nabla_{\mathrm{Hom},t}\beta_t = 0$ and WLOG $||\beta_t||L^2 = 1$.

Now one can change the second fundamental form arbitrarily in its Dolbeault cohomology class by a complex gauge transformation. To see this, note that a splitting of the short exact sequence $\mathcal{K} \to \mathcal{E} \to \mathcal{L}$ is given by a map $\mathcal{L} \to \mathcal{E}$ and is equivalent to a choice of Hermitian metric. Changing Hermitian metric corresponds to acting by a complex gauge transformation ($GL_n(\mathbb{C})$ acts transitively on Hermitian metrics). Because $\mathcal{L} \to \mathcal{E} \to \mathcal{L}$ is the identity for a splitting, two splittings σ_1, σ_2 differ by a map $\kappa\colon \mathcal{L} \to \mathcal{K}$ (given by applying σ_1 and then projecting to $\mathcal{K}$ along σ_2). From $\nabla_{\mathcal{L},t}$ and ∇_t one gets a connection on $\mathcal{L}^* \otimes \mathcal{E}$ whose projection to $\mathcal{L}^* \otimes \mathcal{K}$ agrees with d_t. There are two things to note: the second fundamental form for a splitting γ is actually $\beta = \nabla^{0,1}_{\mathcal{L}^*\otimes\mathcal{E}}\gamma$ and hence β is $\nabla^{0,1}_{\mathcal{L}^*\otimes\mathcal{K}}$-(Dolbeault)-closed; the second fundamental form for a splitting $\gamma + \kappa$ where $\kappa\colon \mathcal{L} \to \mathcal{K}$ differs from β by $\nabla^{0,1}_{\mathcal{L}^*\otimes\mathcal{K}}\kappa$, which is an arbitrary Dolbeault-exact form!

All of this means that we can let $\nabla_{s,t}$ be the connection which splits relative to the extension $\mathcal{K} \to \mathcal{E} \to \mathcal{L}$ as $(\nabla_{t,\mathcal{K}}, \nabla_{t,\mathcal{L}}, s\beta_t)$. Now the curvature of $\nabla_{s,t}$ on $\mathcal{E}$ splits as

$$\begin{pmatrix} F_{\nabla_{t,\mathcal{K}}} - \frac{1}{2}s^2\beta_t^\dagger \wedge \beta_t & 0 \\ 0 & F_{\nabla_{t,\mathcal{L}}} + \frac{1}{2}s^2\beta_t^\dagger \wedge \beta_t \end{pmatrix}$$

since $\nabla_{\mathrm{Hom},t}\beta_t = 0$ by construction. As $s, t \to 0$, $\mathcal{YM}(\nabla_{s,t}) \to \mathrm{rank}(\mathcal{K})(\mu(\mathcal{E}) - \mu(\mathcal{K})) + \mathrm{rank}(\mathcal{L})(\mu(\mathcal{L}) - \mu(\mathcal{E})) =: J$. We will show that for small s, t, $\mathcal{YM}(\nabla_{s,t}) < J$.

One useful observation is that when $s = t = 0$ (so $\mathcal{YM}(\nabla_{0,0}) = N(\mu(\mathcal{E})\mathrm{id} - \Lambda_{\mathcal{K}} \oplus \Lambda_{\mathcal{L}})$) the eigenvalues of $\mu(\mathcal{E})\mathrm{id}_{\mathcal{K}} - \Lambda_{\mathcal{K}}$ are all positive (since $\mu(\mathcal{E}) > \mu(\mathcal{C}_{i,j})$) and hence the same will be true of $\mu(\mathcal{E})\mathrm{id}_{\mathcal{K}} - \star F_{t,\mathcal{K}}$ for small enough t. On such matrices $\nu = \mathrm{Tr}$. We can also find a bound on the $\mathcal{C}^0$-norm of β_t (uniform in t). This follows from the elliptic bounds

$$||\beta_t||_{L^2_{k+1}} \leq C_t(||d_t\beta_t||_{L^2_k} + ||\beta_t||_{L^2}) = C_t$$

for the elliptic operators d_t (where C_t are uniformly bounded from above in t because d_t converges to d_0) and the Sobolev lemma ($L^2_k \subset \mathcal{C}^0$ for large enough k). Now

$$\nu\left(\frac{\star F_{s,t}}{2\pi i} + \mu(\mathcal{E})\mathrm{id}\right) = J_1 - 2s^2|\beta_t|^2 + \epsilon(t)$$

where $\epsilon(t) \to 0$ as $t \to 0$. This is easy to see by taking a trace, which we can do because of the above observation and the fact that there is a uniform $\mathcal{C}^0$-bound on the β_t so for small s the ν-norm is still given by taking traces.

Therefore

$$\mathcal{YM}(\nabla_{s,t}) = \sqrt{\int_M \left(J_1 - 2s^2|\beta_t|^2 + \epsilon(t)\right)^2 d\mathrm{vol}}$$

Using the uniform bound on the $\mathcal{C}^0$-norm of β_t, we can pick s to be very small and ensure that

$$s^4 \int_M |\beta_t|^4 d\mathrm{vol} \ll s^2 \int_M |\beta_t|^2 d\mathrm{vol} = s^2$$

for all t. Finally we let $t \to 0$ and we see that the expression is strictly less than $\mathcal{YM}(\nabla_{0,0})$ for very small s, t.

So we have proved the Lemma and thereby the Proposition. So we know that if Case 2 of our dichotomy holds (i.e. the complexified gauge orbit corresponding to $\mathcal{E}$ is "not closed", i.e. the infimising sequence of compatible connections tends to a limiting connection in another orbit) then the bundle is NOT stable.