Lecture 13: Narasimhan-Seshadri theorem I

Jonathan Evans

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Today we begin the proof of...

Theorem (Narasimhan-Seshadri, Donaldson)

An indecomposable Hermitian holomorphic vector bundle $\mathcal{E}$ on a Riemann surface (M, g) is stable if and only if there is a compatible unitary connection on $\mathcal{E}$ with constant central curvature

$$\star F_\nabla = -2\pi i \mu(\mathcal{E}).$$

...or equivalently...

Theorem

Every stable $\mathcal{G}_\mathbb{C}$-orbit on $\mathcal{A}$ contains a unique $\mathcal{G}$-orbit of solutions to $\mathcal{YM}^{-1}(0)$ where

$$\mathcal{YM}(\nabla) = \int_M ||F_\nabla||^2 d\mathrm{vol} - \mu(\mathcal{E})$$

.

The proof goes something like the following. Let ∇_i be a sequence of connections in the $\mathcal{G}_{\mathbb{C}}$-orbit of ∇ (corresponding to $\mathcal{E}$) such that $\mathcal{YM}(\nabla_i) \to \inf_{\mathcal{G}_{\mathbb{C}}(\nabla)} \mathcal{YM}$. A theorem of Uhlenbeck (which we will prove in a couple of lectures' time) guarantees the existence of a limiting connection ∇_∞. If $\nabla_\infty \in \mathcal{G}_{\mathbb{C}}(\nabla)$ then a quick variational calculation will ensure that ∇_∞ has constant central curvature. If not, we will use ∇_∞ to construct a subbundle contradicting stability of $\mathcal{E}$. This last step requires an inductive argument, but we notice that in the case $\mathrm{rank}(\mathcal{E}) = 1$ (i.e. $U(1)$-bundles) the stability condition is empty (all line bundles are stable) and the theorem reduces to the Hodge-Maxwell theorem (which we've already proved). Therefore we will assume the theorem is true for all bundles of rank $\leq k$ and try to prove it for rank $k+1$.

The first technical caveat is that we do not use the Yang-Mills functional, but rather a modification with the same minima (to simplify the proof). Define the norm

$$\nu(M) = \mathrm{Tr}(M^\dagger M) = \sum_{i=1}^{n} |\lambda_i| = \max_{\{e_i\}} \sum_{i=1}^{n} |\langle Me_i, e_i\rangle|$$

on Hermitian n-by-n matrices. Here $\dagger$ is the adjoint, λ_i are the eigenvalues and the maximum is taken over all orthonormal bases $\{e_i\}$ of $\mathbb{C}^n$. For a section $s \in \Omega^0(M; \mathrm{ad}(P))$ define the norm

$$N(s) = \sqrt{\int_M \nu(s)^2 \mathrm{vol}}$$

Now we use the modified functional

$$\mathcal{YM}(\nabla) = N\left(\frac{\star F_\nabla}{2\pi i} + \mu\right)$$

Observe that the zeros of this functional are precisely the constant central curvature Yang-Mills connections.

Lemma (Exercise!)

If $M = \begin{pmatrix} A & -B^\dagger \\ B & C \end{pmatrix}$ *then* $\nu(M) \geq |\mathrm{Tr}(A)| + |\mathrm{Tr}(C)|$.

We can now prove the converse direction of the theorem:

Proposition

If an indecomposable holomorphic vector bundle $\mathcal{E}$ *admits a compatible connection* ∇ *with* $\mathcal{YM}(\nabla) = 0$ *then* $\mathcal{E}$ *is stable.*

Suppose that $\mathcal{E}$ is not stable and let $\mathcal{M} \subset \mathcal{E}$ be a subbundle with $\mu(\mathcal{M}) \geq \mu(\mathcal{E})$ and quotient

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$$

We will show that:

Lemma

If $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$ is an exact sequence of holomorphic bundles with $\mu(\mathcal{M}) \geq \mu(\mathcal{E})$ then for any compatible connection ∇ on $\mathcal{E}$,

$$\mathcal{YM}(\nabla) \geq \mathrm{rank}(\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{E})) + \mathrm{rank}(\mathcal{N})(\mu(\mathcal{E}) - \mu(\mathcal{N}))$$

with equality if and only if the extension splits.

The Proposition obviously follows from this: the existence of a subbundle like $\mathcal{M}$ implies this inequality and it must be strict since the sequence cannot split (since $\mathcal{E}$ is indecomposable). Hence $\mathcal{YM}(\nabla)$ cannot be zero since the RHS is ≥ 0. Morally, since "curvature decreases in subbundles", having a subbundle contradicting stability (i.e. with large slope) means that the curvature must be large!

To prove the Lemma, recall that on such an exact sequence of holomorphic bundles a unitary connection splits as

$$\begin{pmatrix} \nabla_{\mathcal{M}} & -\beta^\dagger \\ \beta & \nabla_{\mathcal{N}} \end{pmatrix}$$

Here $\nabla_{\mathcal{M}}\sigma = \mathrm{pr}_{\mathcal{M}}\nabla\sigma$ where $\mathrm{pr}_{\mathcal{M}}$ denotes orthogonal projection with respect to the Hermitian metric and $\beta = \nabla - \nabla_{\mathcal{M}} : \Omega^0(M; \mathcal{M}) \to \Omega^1(M; \mathcal{N})$ is a 1-form with values in $\mathcal{M}^* \otimes \mathcal{N}$ (called the 2nd fundamental form of $\mathcal{M}$). The curvature is

$$F_\nabla = \begin{pmatrix} F_{\nabla_{\mathcal{M}}} - \frac{1}{2}\beta^\dagger \wedge \beta & -\nabla_{\mathrm{Hom}(\mathcal{M}^*\otimes\mathcal{N})}\beta^\dagger \\ \nabla_{\mathrm{Hom}(\mathcal{M}^*\otimes\mathcal{N})}\beta & F_{\nabla_{\mathcal{N}}} - \frac{1}{2}\beta \wedge \beta^\dagger \end{pmatrix}$$

Using our inequality for the norm ν in terms of block matrices we see that

$$\nu\left(\frac{\star F_\nabla}{2\pi i}+\mu\right)\geq\left|\mathrm{Tr}\left(\frac{F_{\nabla_{\mathcal{M}}}-\frac{1}{2}\beta^\dagger\wedge\beta}{2\pi i}+\mu\right)\right|+\left|\mathrm{Tr}\left(\frac{F_{\nabla_{\mathcal{N}}}-\frac{1}{2}\beta^\dagger\wedge\beta}{2\pi i}+\mu\right)\right|$$

Remember also that $\langle\beta^\dagger\wedge\beta\rangle=-2\pi i|\beta|^2$. Therefore

$$\begin{aligned}\mathcal{YM}(\nabla)&=\sqrt{\int_M\nu\left(\frac{\star F_\nabla}{2\pi i}+\mu\right)^2}\\&\geq\int_M\nu\left(\frac{\star F_\nabla}{2\pi i}+\mu\right)\\&\geq\left|\int_M\mathrm{Tr}\left(\frac{\star F_{\nabla_{\mathcal{M}}}}{2\pi i}+\mu\right)-|\beta|^2\right|+\left|\int_M\mathrm{Tr}\left(\frac{\star F_{\nabla_{\mathcal{N}}}}{2\pi i}+\mu\right)-|\beta|^2\right|\end{aligned}$$

...and $\int_M \mathrm{Tr}\left(\frac{\star F_\nabla}{2\pi i}\right) = -\deg(\mathcal{M})$ so these terms give

$$\deg(\mathcal{M}) - \mathrm{rank}(\mathcal{M})\mu(\mathcal{E}) + |\beta|^2 + \deg(\mathcal{N}) - \mathrm{rank}(\mathcal{N})\mu(\mathcal{E}) + |\beta|^2$$

(why have I switched signs?) This in turn is bigger than

$$\mathrm{rank}(\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{E})) + \mathrm{rank}(\mathcal{N})(\mu(\mathcal{E}) - \mu(\mathcal{N}))$$

as claimed. Phew! The Lemma is proved.

The second technical caveat is that we need slightly more Sobolev theory than I gave you before because we're now in a nonlinear setting. Let me describe the setup today and next time we'll continue with the proof.

- We're using the L^2_1-completion of $\mathcal{A}$ (i.e. fix ∇ and identify $\mathcal{A}$ with $\Omega^1(M; \mathrm{ad}(P))$ then take the Sobolev completion of the vector space).
- We're using L^2_2-gauge transformations. These form a group because the product of two L^2_2 functions is again L^2_2. More generally this is true of L^2_k functions whenever $k > n/2$ (for us $n = 2$). The product of an L^2_2 function and an L^2_1 function is L^2_1 and hence the L^2_2-gauge group actually acts on the space of L^2_1-connections!
- Perhaps more importantly, $L^2_2 \subset \mathcal{C}^0$ and hence the gauge transformations make sense with respect to the topology of the bundle.

More important facts:

- ν is equivalent to the L^2-norm on $\Omega^0(M; \mathrm{ad}(P))$ and in particular extends to L^2-sections. $\mathcal{YM}$ extends to L^2_1-sections. To see this last fact, note that when curvature transforms under change of connection $\nabla + A$, we get $\nabla A + [A, A]$ and if $A \in L^2_1$ then both of these are in L^2 (it's clear for the derivative; the other follows from the embedding $L^2_1 \subset L^4$ and the fact that the product of two L^4-functions is L^2 by the Hölder inequality).
- $L^2_1 \subset L^4$ follows from the more general form of the Sobolev/Rellich theorems
$$L^p_k \subset L^q_\ell$$
which holds when $k \geq \ell$ and $k - n/p \geq \ell - n/q$ (and is a compact inclusion when the inequalities are strict).
- Also $L^p_k \subset \mathcal{C}^\ell$ if $k - n/p > \ell$ (also compact). These theorems all come with accompanying inequalities on norms, e.g. $|| \cdot ||_{\mathcal{C}^\ell} \leq C|| \cdot ||_{L^p_k}$.

Finally we note an improved version of elliptic regularity:

Theorem

If P is an elliptic operator of order d and $Pu = v$ weakly with $v \in L^2_k$ and $u \in \ker(P)^\perp$ then $u \in L^2_{k+d}$ and

$$||u||_{L^2_{k+d}} \leq C||v||_{L^2_k}$$

(The improved inequality doesn't have a term in $||u||_{L^2}$. Note that this cannot hold unless $u \in \ker(P)^\perp$).