Lecture 12: Holomorphic bundles III (Harder-Narasimhan)

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Our next goal is to prove:

Theorem (Harder-Narasimhan)

Any holomorphic vector bundle admits a canonical Harder-Narasimhan filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = E$$

with $D_i = F_i / F_{i-1}$ *semi-stable and* $\mu(D_1) > \mu(D_2) > \cdots > \mu(D_r)$.

Exercise

Any semistable vector bundle E *admits a filtration*

$$0 = E_0 \subset \cdots \subset E_k = E$$

with $C_i := E_i / E_{i-1}$ *stable and* $\mu(C_i) = \mu(E)$.

We first look for F_{r-1}. We only need to look if $F_r = E$ is not semi-stable. We say $F \subset E$ is *destabilising* if for every $F \subsetneq F' \subset E$ $\mu(F') < \mu(F)$. Equivalently for any subbundle $0 \neq Q \subset E/F$, we have $\mu(Q) < \mu(F)$.

Proposition

If E is not semistable then there is a unique $F \subset E$ which is semistable and destabilising.

The proof of the theorem is then just to take F_1 to be the unique semistable destabilising subbundle of E and, inductively, F_k to be the preimage of the unique semistable destabilising subbundle of E/F_{k-1} under the quotient map $E \to E/F_{k-1}$. To see that this implies $\mu(D_i = F_i/F_{i-1})$ is decreasing, consider the SES

$$0 \to F_i/F_{i-1} \to E/F_{i-1} \to E/F_i \to 0$$

Since F_i is destabilising in E/F_{i-1}, take $Q = F_{i+1}/F_i \subset E/F_i$ and observe that $\mu(Q) < \mu(F_i/F_{i-1})$.

It remains to prove the Proposition. We need more Lemmata:

Lemma

If $F_1 \subset E$ is semistable and $F_2 \subset E$ is destabilising and $F_1 \not\subset F_2$ then $\mu(F_1) < \mu(F_2)$.

Proof.

By assumption the map $F_1 \to E/F_2$ is non-zero. As before we get a factorisation

$$\begin{array}{ccccc} F_1 & \longrightarrow & F_1' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ E/F_2 & \longleftarrow & F_1'' & \longleftarrow & 0 \end{array}$$

Since F_1 is semistable $\mu(F_1) \leq \mu(F_1')$. Since F_2 is destabilising $\mu(F_1'') < \mu(F_2)$. Since $\deg(F_1') < \deg(F_1'')$ while they have the same rank, $\mu(F_1') \leq \mu(F_1'')$. In total we get the desired inequality. □

Lemma (Uniqueness)

If $F_1, F_2 \subset E$ are semistable and destabilising then $F_1 = F_2$

Proof.

If $F_1 \not\subset F_2$ then $\mu(F_2) > \mu(F_1)$ by the previous Lemma. Then, also by the previous Lemma, $F_2 \subset F_1$. By symmetry we're done. □

Proof of Proposition.

We now define $m := \sup_{0 \neq F \subset E} \mu(F)$ which is $> \mu(E)$ because E is not semistable. There are F with $\mu(F) = m$ (Ex: why?) and among these we pick one, F_0, with maximal rank. If $0 \neq F' \subset F_0$ then $\mu(F') \leq \mu(F_0)$ so F_0 is semistable. If $F_0 \subsetneq F' \subset E$ then $\mathrm{rank}(F') > \mathrm{rank}(F_0)$ so $\mu(F') < \mu(F_0)$ by maximality of $\mathrm{rank}(F_0)$ and hence F_0 is destabilising. □

Now that we're familiar with stable bundles, we can state the Narasimhan-Seshadri theorem. Throughout, $G = U(n), \mathfrak{g} = \mathfrak{u}(n)$.

Theorem (Narasimhan-Seshadri, Donaldson)

An indecomposable Hermitian holomorphic vector bundle $\mathcal{E}$ on a Riemann surface (M, g) is stable if and only if there is a compatible unitary connection on $\mathcal{E}$ with constant central curvature

$$\star F_\nabla = -2\pi i \mu(\mathcal{E}).$$

Remark

- *Indecomposable means we can't decompose it as a direct sum of holomorphic subbundles. We'll see in a couple of lectures' time how to cope with decomposable bundles.*
- $\mu(\mathcal{E})$ *denotes the slope of the holomorphic vector bundle, namely* $c_1(\det(\mathcal{E}))/\mathrm{rank}(\mathcal{E})$. *Stable means that any holomorphic subbundle has strictly smaller slope.*
- *We need a Hermitian metric on the bundle to make sense of unitarity.*
- *What about the curvature statement?*

- Since $F_\nabla \in \Omega^2(M; \mathrm{ad}(P))$ and $\dim(M) = 2$, $\star F_\nabla \in \Omega^0(M; \mathrm{ad}(P))$. Sections of $\mathrm{ad}(P)$ biject with G-equivariant maps $\sigma : P \to \mathfrak{g}$ (G acting by ad on the RHS).
- To say that a connection has *constant curvature*, i.e. for this G-equivariant map to be constant, we therefore need $\sigma(P) = X \in \ker(\mathrm{ad})$. These are precisely the central elements of the Lie algebra.
- Certainly a connection with constant central curvature satisfies $\nabla \star F_\nabla = 0$. We will see in a few lectures' time that these are precisely the minima of the Yang-Mills functional.
- A central element of $\mathfrak{u}(n)$ is diagonal (with diagonal entries being the eigenvalues). Since the bundle is indecomposable, it is clear we need these eigenvalues to be the same (to avoid an eigenspace decomposition). Since the integral $\int_M F_\nabla$ is $2\pi i$ times the first Chern class, we see that these eigenvalues are all $-2\pi i \mu(\mathcal{E})$.

Here's an alternative way of stating the theorem which is closer in spirit to the Kempf-Ness theorem (and indeed to the proof). Remember that we defined a complexification $\mathcal{G}_{\mathbb{C}}$ of the action of the gauge group $\mathcal{G}$ on the space of connections $\mathcal{A}$ whose orbits were isomorphism classes of holomorphic vector bundles.

Theorem

Every stable $\mathcal{G}_{\mathbb{C}}$-orbit contains a unique $\mathcal{G}$-orbit of solutions to $\mathcal{YM}^{-1}(0)$ where

$$\mathcal{YM}(\nabla) = \int_M ||F_\nabla||^2 d\mathrm{vol} - \mu(\mathcal{E})$$

.

Here stability means that the corresponding holomorphic vector bundle is stable in the algebro-geometric sense we described last lecture.

The proof goes something like the following. Let ∇_i be a sequence of connections in the $\mathcal{G}_{\mathbb{C}}$-orbit of ∇ (corresponding to $\mathcal{E}$) such that $\mathcal{YM}(\nabla_i) \to \inf_{\mathcal{G}_{\mathbb{C}}(\nabla)} \mathcal{YM}$. A theorem of Uhlenbeck (which we will prove in a couple of lectures' time) guarantees the existence of a limiting connection ∇_∞. If $\nabla_\infty \in \mathcal{G}_{\mathbb{C}}(\nabla)$ then a quick variational calculation will ensure that ∇_∞ has constant central curvature. If not, we will use ∇_∞ to construct a subbundle contradicting stability of $\mathcal{E}$. This last step requires an inductive argument, but we notice that in the case $\mathrm{rank}(\mathcal{E}) = 1$ (i.e. $U(1)$-bundles) the stability condition is empty (all line bundles are stable) and the theorem reduces to the Hodge-Maxwell theorem (which we've already proved). Therefore we will assume the theorem is true for all bundles of rank $\leq k$ and try to prove it for rank $k+1$.