Lecture 10: Holomorphic bundles I (Existence)

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20th October 2011

Last time we introduced holomorphic vector bundles $\mathcal{E}$ over complex manifolds and we showed there is an operator

$$\bar{\partial}_{\mathcal{E}} \colon \Omega^k(M;\mathcal{E}) \to \Omega^{k+1}(M;\mathcal{E})$$

which vanishes on holomorphic sections ($k=0$) and obeys the Leibniz rule

$$\bar{\partial}_{\mathcal{E}}(f\sigma) = (\bar{\partial} f)\sigma + f\bar{\partial}_{\mathcal{E}}\sigma$$

We observed that if we pick a Hermitian metric on $\mathcal{E}$ then we can recover $\bar{\partial}_{\mathcal{E}}$ as the $(0,1)$-part of a unitary connection ∇. The aim of today's lecture is to see that when M is a Riemann surface, any unitary connection on a Hermitian complex vector bundle E induces a holomorphic structure $\mathcal{E}$ with

$$\bar{\partial}_{\mathcal{E}} = \nabla^{0,1}.$$

Proposition

If P is a principal $U(n)$-bundle over a Riemann surface M with associated bundle E and ∇ is a $U(n)$-connection then E inherits the structure of a holomorphic vector bundle over M such that

$$\nabla^{0,1} = \overline{\partial}$$

Proof.

It's easy to define complex charts on E: just pick local trivialisations, use the fibre coordinate vertically and pull back complex coordinates from M horizontally. The fact that M is a complex manifold means that these will glue to give the structure of a complex manifold globally and the projection will be holomorphic by construction. The main difficulty is to pick the trivialisation so as to ensure $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$. A trivialisation is the same as a choice of local sections $\sigma_1, \ldots, \sigma_n$ which form a unitary basis at each point. Notice that in the complex structure we have described these sections will trace out complex submanifolds and hence end up as holomorphic local sections... □

Proof.

...but holomorphic sections will obey $\bar{\partial}_{\mathcal{E}}\sigma = 0$, so to ensure $\nabla^{0,1} = \bar{\partial}_{\mathcal{E}}$ we'll have to find a basis of local sections $\sigma = \{\sigma_i\}_{i=1}^n$ for which $\nabla^{0,1}\sigma_i = 0$. To get us started, let's just pick a basis of local sections σ and record their $\nabla^{0,1}$-covariant derivatives as an n-by-n matrix θ of $(0,1)$-forms (in terms of the basis σ!).

$$\nabla^{0,1}_X \sigma = \theta(X)\sigma$$

Replace σ by $f\sigma$ for some matrix-valued function f and (by the Leibniz rule for $\nabla^{0,1}$) we get

$$\nabla^{0,1}(f\sigma) = (\bar{\partial}f + f\theta)\sigma$$

and it is sufficient to solve $f^{-1}\bar{\partial}f + \theta = 0$. Consider the operator

$$P\colon L^2_2 \to L^2_1$$

given by $P(f) = f^{-1}\bar{\partial}f$. Since Sobolev theory works best on compact manifolds we assume for now that L^2_2, etc are spaces of functions on S^2. We will see how to remedy this assumption shortly. □

Proof.

The operator P is not linear but its differential at $f = 1$ is the linear elliptic operator $\bar{\partial}$:

$$P(1+\epsilon) = (1-\epsilon+\mathcal{O}(\epsilon^2))\bar{\partial}(1+\epsilon) = \bar{\partial}\epsilon + \mathcal{O}(\epsilon^2)$$

I won't be specific about what I mean by elliptic, but I will tell you what I use when I need it. Now we see that $\bar{\partial}\colon L^2_2 \to L^2_1$ is surjective. Let ρ_1, ρ_2 be a partition of unity for the cover of S^2 by upper and lower hemispheres and let $f_i = f\rho_i$. Then by Cauchy's integral formula

$$f_i(\xi) = \frac{1}{2\pi i}\int \bar{\partial} f_i \frac{dz \wedge d\bar{z}}{z-\xi}$$

so we can recover a function from its $\bar{\partial}$-derivative. By Liouville's theorem, the kernel of $\bar{\partial}$ is just the space of constant matrices (each entry has to be an entire function). □

Proof.

All this means that the linearisation of P at $1 \in L^2_2$ is surjective. Therefore by the implicit function theorem for Banach spaces we see that $P(f) = -\theta$ has a unique solution orthogonal to $\ker(\bar{\partial})$ provided θ has small L^2_1-norm. Unique means unique in a neighbourhood of $1 \in L^2_2$. Ellipticity will imply that if $\theta \in \mathcal{C}^\infty$ then $f \in \mathcal{C}^\infty$. Now we never wanted to work over a sphere. We wanted to work over a disc. To that end, let $\rho(|z|)$ be a cutoff function on S^2 such that

$$\rho(x) = \begin{cases} 1 & \text{if } x \leq \delta/2 \\ 1 - \frac{2x-\delta}{\delta} & \text{if } x \in [\delta/2, \delta] \\ 0 & \text{otherwise} \end{cases}$$

Note that $\rho \in L^2_1$ and

$$||\rho||_1 \leq 2\sqrt{\pi}$$

□

Proof.

Now if $\phi = \rho\theta$ then

$$\begin{aligned} ||\phi||_1^2 &= ||\rho\theta||^2 + ||\rho'\theta + \rho\theta'||^2 \\ &\leq ||\rho\theta||^2 + ||\rho'\theta||^2 + ||\rho\theta'||^2 + 2||\rho'\theta||||\rho\theta'|| \\ &\leq ||\rho\theta||^2 + ||\rho'\theta||^2 + ||\rho\theta'||^2 + 2||\rho'\theta||^2 + 2||\rho\theta'||^2 \\ &\leq 3||\rho\theta||^2 + 3||\rho'\theta||^2 + 3||\rho\theta'||^2 + 3||\rho\theta||^2 \\ &\leq 12 \sup |\theta|^2 + 3||\theta||_1^2 \end{aligned}$$

By suitably choosing σ to begin with we can assume that $\theta(0) = 0$. Then $\sup |\theta|^2$ can be made arbitrarily small by reducing δ. So can $||\theta||_1^2$. Therefore $P(f) = -\rho\theta$ has a solution for small δ and we can restrict to the disc of interest to find our local holomorphic frame. □

This gorgeous argument is due to Atiyah and Bott in their Yang-Mills equations over Riemann surfaces paper. It's a "linear" version of the Newlander-Nirenberg theorem (which is much harder and constructs systems of local complex coordinates under much weaker assumptions).

Corollary

We can think of unitary connections on a $U(n)$-bundle as giving the structure of a holomorphic vector bundle to the associated complex vector bundle.

The action of the gauge group on $\mathcal{A}$ now extends to an action of the *complexified gauge group* $\mathcal{G}_{\mathbb{C}}$ consisting of gauge transformations of the $GL(n, \mathbb{C})$-bundle associated to the representation $U(n) \to GL(n, \mathbb{C})$. Notice that our identification of a connection $\nabla \in \mathcal{A}$ with a holomorphic structure depended on a choice of Hermitian metric. The space of Hermitian metrics compatible with a given $GL(n, \mathbb{C})$ bundle admits a transitive $GL(n, \mathbb{C})$-action and $U(n)$ is the stabiliser of a given metric. Therefore we can act on $\mathcal{A}$ by $GL(n, \mathbb{C})$-gauge transformations and we get unitary connections which are compatible with the same holomorphic vector bundle (using a different choice of Hermitian metric).

In formulae

Let's remind ourselves that a gauge transformation $u \in \mathcal{G}$ is a G-equivariant diffeomorphism of P living over id and that

$$(u\nabla)_X\sigma = u\nabla_X(u^{-1}\sigma)$$

What is $u\nabla - \nabla$? Well we can now differentiate u, considered as a section of $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} G$ (not $\mathrm{ad}(P)$!). We get

$$(u\nabla)_X\sigma = \nabla_X\sigma + (u\nabla_X u^{-1})\sigma$$

so $a = u\nabla - \nabla = u\nabla u^{-1}$, which is a section of $\mathrm{ad}(P)$. Since $uu^{-1} = \mathrm{id}$, $u\nabla u^{-1} = -(\nabla u)u^{-1}$. In these terms, the complexified gauge action is

$$(g\nabla)^{0,1} = \nabla^{0,1} - (\nabla^{0,1}g)g^{-1}$$

We see that $\mathcal{A}/\mathcal{G}_{\mathbb{C}}$ is the "moduli space" of holomorphic vector bundles. By analogy with the Kempf-Ness theorem we expect there to be a notion of stability of holomorphic vector bundles such that the stable $\mathcal{G}_{\mathbb{C}}$-orbits contain a unique minimum of the Yang-Mills functional. This is the Narasimhan-Seshadri theorem. Next time we will define the relevant notion of stability, before moving on to the proof (à la Donaldson).