Lecture 1: Introduction

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Essentially everything I will say is contained in Atiyah-Bott "The Yang-Mills equations on a Riemann surface". This mixes

- ∞-dimensional Morse theory,
- equivariant cohomology,
- holomorphic vector bundles,
- ideas from physics

so we will have to be a little sketchy! STOP me if you stop understanding.

- Exercises and notes: http://www.jde27.co.uk, see handout for further reading.
- Exercise classes: Tuesday 14:00-15:00 (here) starting optimistically next week
- Thursday lectures in HG E 3.

What we will cover

First few lectures: Maxwell's equations of magnetostatics

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2}$$

- Linear PDE:
  - the first equation satisfied by all static magnetic fields (intrinsic),
  - the second equation involves current density $\mathbf{J}$.
- We'll rewrite them in a new form to clarify their symmetries and to allow generalisation to an arbitrary background (not just $\mathbb{R}^3$).
- On the way we'll discover gauge theory.

Punchline

Space of solutions depends only on the topology of the background.

Major theme in modern geometry

Nice (elliptic) PDE gives a space of solutions whose topology reflects that of the background space.

Example (Donaldson theory (Yang-Mills in 4-D))

Cohomological invariants of the moduli space of Yang-Mills instantons

- *detect exotic 4-manifolds,*
- *prove the diagonalisability theorem,*
- *and the failure of h-cobordism in 4-D.*

In 2-D we hope to understand the full space of solutions. So what are the Yang-Mills equations?

- Maxwell's equations come from thinking about the group $U(1)$ of unit complex numbers.
- In physics one thinks of this as the possible phases of a quantum wavefunction ψ.
- Measurable quantities like $\bar{\psi}\psi$ do not depend on phase so one hopes that the equations of QM are invariant under local changes of phase $\psi(x) \mapsto e^{i\phi(x)}\psi(x)$.
- TRUE if you introduce a new field and absorb phase shifts into a potential for this field. This yields Maxwell theory.

This is called the gauge principle. We will cover it properly next lecture. It generalises to other groups e.g. $SU(2), SO(3)$ and the corresponding theory is called Yang-Mills theory.

Goal of course

The Atiyah-Bott formulae for the cohomology of the moduli space of Yang-Mills instantons.

What is a moduli space? And how did Atiyah and Bott work out its cohomology? Take a step back to Maxwell's equations.

- $\nabla \cdot \mathbf{B} = 0$ implies (Poincaré's lemma) there exists a *potential* $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$.
- However it is not uniquely determined:

$$\mathbf{A} + \nabla f$$

gives the same electromagnetic field! In fact any two potentials for the same field are related in this way, called a *gauge transformation*.

Why bother?

Underlying reason

There is a function $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ on the (∞-dimensional) space $\mathcal{A}$ of potentials which is invariant under gauge transformations and whose critical points $D\mathcal{F}(\phi, A) = 0$ are precisely the solutions to Maxwell's equations.

In Yang-Mills theory we have exactly the same picture. We are interested in the MODULI SPACE of Yang-Mills fields:

$$\mathcal{M} := \{D\mathcal{F} = 0\}/\mathcal{G}$$

where $\mathcal{G}$ is the group of gauge transformations. This means we're not interested in potentials, just fields (which are equivalence classes of potentials up to gauge transformations). $\mathcal{M}$ is a finite-dimensional manifold (maybe with some singularities).

Example

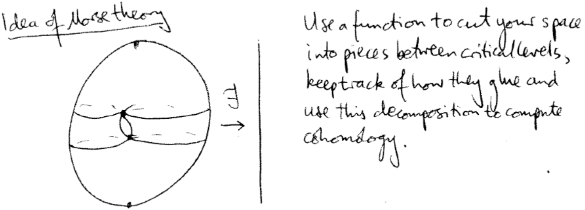
In Maxwell theory on a Riemannian manifold X the moduli space is very simple because the equations are linear.

$$\mathcal{M} \cong H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$$

When $U(1)$ is replaced by a nonabelian group the Yang-Mills equations become nonlinear and the moduli spaces develop interesting topology.

Genius idea (Atiyah-Bott)

Calculate the cohomology groups of $\mathcal{M}$ using Morse theory.



Genius idea (Continued)

Use $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ as a Morse function!

$\mathcal{A}$ is topologically very simple, just an ∞-dimensional affine space. Working backwards one should be able to get information about its critical locus! (Caveat: only interested in critical locus modulo $\mathcal{G}$ so use $\mathcal{G}$-equivariant cohomology). This MASTERPIECE will be our focus later when we have developed the tools to understand it.

Because Yang-Mills theory on a Riemann surface is closely related to algebraic geometry via the *Narasimhan-Seshadri theorem*. This will be the focus of the first half of the course.

Theorem (Narasimhan-Seshadri)

A stable holomorphic vector bundle on a Riemann surface (something entirely algebro-geometric) admits a Yang-Mills connection (unique up to gauge) and conversely if a holomorphic vector bundle admits such a connection it is stable.

Example

If X is a complex projective manifold then the Maxwell moduli space $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$ is identified with the Jacobian torus of holomorphic line bundles (which are all stable).

This has an intriguing geometric parallel:

Theorem (Calabi-Yau-Aubin)

A Kähler manifold with $c_1 \leq 0$ admits a unique Kähler-Einstein metric.

While it's hard (impossible?) to write down an explicit Ricci-flat metric on a quartic K3 surface, its existence follows from this theorem!

Understanding the precise nature of algebro-geometric stability to go on the left-hand-side to ensure existence of a "best metric" on the right is a current hot topic and all the ideas seem to be directly inspired by the Narasimhan-Seshadri theorem and its proof by Donaldson which we will cover in the course!

The moduli spaces of stable bundles on Riemann surfaces were long of interest to algebraic geometers, but their topology was somehow mysterious. Harder-Narasimhan gave formulae for the Betti numbers but the formulae were magic: they worked over a finite field and counted points, passing back to the topology of the locus of complex points using the Weil conjectures. Atiyah and Bott recovered these formulae using their totally different approach and the relationship between the two is frustratingly unclear!