Lecture XII: Hamiltonian group actions

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In this lecture we will be concerned with groups G acting on a symplectic manifold (X, ω) in a way which preserves the symplectic form. In fact, we'll mostly be interested in groups which act via Hamiltonian diffeomorphisms, i.e. homomorphisms

$$G \to \mathrm{Ham}(X, \omega)$$

The groups we are interested in are compact Lie groups, mostly tori. There is a very beautiful, explicit finite-dimensional theory which will give us a better understanding of the geometry of blow-ups and of highly symmetric ("toric") symplectic manifolds. This finite-dimensional theory inspires many ideas in the infinite-dimensional setting which includes Yang-Mills theory and the existence theory of Kähler metrics (sadly I won't have time to discuss this).
Let's start with circles.

Circle actions and symplectic reduction

- Let H be a function on (X, ω) and consider the time-t flow ϕ_t of the Hamiltonian vector field X_H defined by $\iota_{X_H}\omega = dH$. If all orbits of X_H are 1-periodic then the map $\mathbb{R} \ni t \mapsto \phi_t \in \mathrm{Ham}(X, \omega)$ descends to a homomorphism $S^1 \to \mathrm{Ham}(X, \omega)$. Such a homomorphism is called a Hamiltonian circle action.
- Observe that $\mathfrak{L}_{X_H} H = \iota_{X_H} dH = \omega(X_H, X_H) = 0$ so the flow of X_H preserves the level sets of H. On each regular level set the circle action is free (fixed points are those points where $X_H = 0$, which happens iff $dH = 0$) and so the quotient $M_c := H^{-1}(c)/S^1$ is a manifold if c is a regular value of H. We call it the *reduced space at level c*.

Let $\iota_c : H^{-1}(c) \to X$ be inclusion of the level set. The 2-form $\iota_c^*\omega$ is degenerate precisely along the direction field generated by X_H, that is $TH^{-1}(c)^\omega = \langle X_H \rangle$ so we can define an S^1-equivariant symplectic vector bundle over $H^{-1}(c)$ whose fibre at x is $T_x H^{-1}(c)/\langle X_H \rangle$. When we perform the reduction this symplectic vector bundle descends to the quotient and is naturally identified with the tangent bundle of M_c.
If we write ω_c for the reduced symplectic form and $\pi_c : H^{-1}(c) \to M_c$ for the quotient map then $\pi_c^*\omega_c = \iota^*\omega$ by construction and we see that $d\omega_c = 0$.

Definition

The symplectic manifold (M_c, ω_c) is called the symplectic reduction of X along the circle action generated by H at level c.

Example: the squared distance function on the standard $\mathbb{C}^n$

$$H(z) = |z|^2 = \sum_{i=1}^{n} |z_i|^2$$

The level sets are $2n-1$-spheres. The Hamiltonian vector field generated by H is precisely the Hopf field, whose integral curves are the fibres of the Hopf fibration. To see this, recall that a fibre of the Hopf fibration is the circle of intersection between a complex line through the origin and a sphere of fixed radius. Since

$$\omega(X_H, \cdot) = dH(\cdot) = g(JX_H, \cdot)$$

where g and J are the standard Euclidean metric and complex structure on $\mathbb{C}^n$, we see that X_H is $J\nabla H$, but ∇H is radial and together

$$\langle \nabla H, J\nabla_H \rangle$$

span a complex line through 0.

The symplectic reduction of the level set $H^{-1}(c)$ is then a symplectic manifold diffeomorphic to the base of the Hopf fibration, i.e. $\mathbb{CP}^{n-1}$. It should come as no surprise that we obtain a multiple of the Fubini-Study form. Indeed, this is obvious since the Hamiltonian is $U(n)$-invariant. The value of c tells us the total volume of this symplectic manifold.

Integrable geodesic flow

Another great example comes when we have a Riemannian manifold (M, g) whose geodesics all have minimal period exactly 1. For example, S^n, $\mathbb{RP}^n$. Such a Riemannian metric is called a *Zoll metric* and there are many interesting examples (not just the obvious ones!). Then the squared length function on T^*M is a Hamiltonian which generates the geodesic flow. Although this is not a circle action (because (p, v) will only go all the way around a geodesic if $|v| \in \mathbb{Z}$) it is a free circle action on $H^{-1}(1)$, i.e. the unit cotangent bundle. We can still symplectically reduce on this level. For S^n we get a symplectic quadric $n-1$-fold this way.

Symplectic cut

We now want to cut a symplectic manifold along the level set of a Hamiltonian circle action and collapse the circles. This gives a nice way to look at blow-up. Suppose (X, ω) admits a symplectic S^1-action generated by a Hamiltonian H and c is a regular value of H such that the circle acts freely on $H^{-1}(c)$. Form

$$(X \times \mathbb{C}, \omega \oplus dx \wedge dy)$$

and take the new Hamiltonian function $F = H - x^2 - y^2$. The action of $e^{i\theta} \in S^1$ is

$$(x, z) \mapsto (e^{i\theta}(x), ze^{-i\theta})$$

Now $F^{-1}(c) = \{(x, z) \in X \times \mathbb{C} : H(x) = |z|^2 + c\}$ which contains $H^{-1}(c) \times \{0\}$. Forming the quotient by S^1 we get

$$\tilde{X}_c = F^{-1}(c)/S^1 \supset H^{-1}(c)/S^1$$

Away from $H^{-1}(c)$, $\tilde{X}$ can be identified symplectically with $X \setminus H^{-1}(-\infty, c]$ via

$$[x, z] \mapsto e^{i \arg z}(x)$$

Return to blow-up

- As an example, we can take $X = \mathbb{C}^n$ with $H(z) = |z|^2$. The reduced manifold at any nonzero radius is $\mathbb{CP}^n$, the symplectic cut replaces the ball of some radius by the reduced manifold at that radius. This is a much more elegant way of looking at the construction of the symplectic form on the blow-up. Of course this can be implanted inside a symplectic ball of larger radius, regardless of whether that ball is contained in a Hamiltonian S^1-manifold.
- We also remark that there is a link with last lecture's fibre sum construction. If we have a symplectic $\mathbb{CP}^n = E \subset X^{2n+2}$ such that the normal bundle of E has first Chern class $-H$ then we can fibre sum with $\mathbb{CP}^{n+1}$ along $\mathbb{CP}^n \subset \mathbb{CP}^{n+1}$ (which has opposite normal bundle, first Chern class $+H$). This replaces E with a ball (the complement of $\mathbb{CP}^n \subset \mathbb{CP}^{n+1}$) so acts as a *symplectic blow-down*.
- There is also a more exciting blow-down operation which turns a neighbourhood of a symplectic -4-sphere into a neighbourhood of a Lagrangian $\mathbb{RP}^2$. This comes from fibre-summing with a quadric curve in $\mathbb{CP}^2$ whose complement is $T^*\mathbb{RP}^2$.

We can also symplectically cut the cotangent bundle of a Zoll manifold along the unit cotangent bundle. For S^n we thereby obtain the quadric n-fold with real part S^n and a quadric $n-1$-fold at infinity (the symplectically reduced boundary). For $\mathbb{RP}^2$ we get $\mathbb{CP}^2$ (the symplectically reduced boundary is a quadric curve) - compare with the last comment on the previous slide.

Averaging over symmetries

If we have an ω-compatible almost complex structure J on a Hamiltonian S^1-manifold (X, ω, H) we would like to turn it into an S^1-invariant almost complex structure by averaging. Unforunately, the space of compatible almost complex structures is not convex so we can't just average. Instead, we pass to the (convex) space of metrics by $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and average there to obtain an S^1-invariant metric. Unfortunately this may not be associated to an ω-compatible J. So we recall the following trick implicit in Lecture II. Given our new g, define A (uniquely) by

$$\omega(\cdot, \cdot) = g(A\cdot, \cdot)$$

Check that $A^T = -A$, so AA^T is a symmetric matrix which is positive definite in the sense that

$$g(AA^T u, u) = g(A^T u, A^T u) > 0$$

Therefore we can define $\sqrt{AA^T}$ and set $J = \sqrt{AA^T}^{-1} A$ and check this is an ω-compatible a.c.s. which is invariant by uniqueness.

Fixed submanifolds

As an application of the existence of an invariant compatible almost complex structure, let's prove that

Lemma

A component of a fixed submanifold F of a Hamiltonian S^1-action is a symplectic submanifold.

- Let ψ_t be the time t Hamiltonian flow and let p be a fixed point for all t. The derivatives $d\psi_t(p)$ form a family of matrices on T_pX. This is certainly a symplectic matrix. In fact if J is an invariant compatible a.c.s. then $d\psi_t(p)$ is a family of unitary matrices.
- If v is tangent to the fixed submanifold containing p then $d\psi_t(p)v = v$ so v is an eigenvector with eigenvalue 1.

- Conversely if v is a 1-eigenvector then the geodesic through (p, v) stays in the fixed submanifold. To see this, note that its initial conditions are S^1-invariant and hence uniqueness of geodesics with given initial conditions means that the geodesic is fixed pointwise by the circle action. Therefore T_pF is identified with the 1-eigenspace of $d\psi_t(p)$, which is J-invariant because the matrix is unitary, and hence symplectic.

Torus actions

We now generalise to the case of a torus action, or equivalently a collection of several circle actions which all commute with one another. The most interesting case is when an n-torus acts in a Hamiltonian way on a symplectic $2n$-manifold but let's start by looking at when the Hamiltonian flows of two functions commute.

Definition

The Poisson bracket of two functions F, G on a symplectic manifold (X, ω) is the function defined by

$$\{F, G\} = \omega(X_F, X_G)$$

Lemma

$$[X_F, X_G] = X_{\{F,G\}}$$

Proof.

Note that $[X, Y] = \mathfrak{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\phi_t)_*^{-1} Y$ where ϕ_t is the time t flow along X. Therefore

$$\begin{aligned} \iota_{[X,Y]}\omega &= \frac{d}{dt}\Big|_{t=0} \left(\iota_{(\phi_t)_*^{-1} Y}\omega\right) \\ &= \mathfrak{L}_X(\iota_Y\omega) \\ &= \iota_X d\iota_Y\omega + d\omega(X, Y) \end{aligned}$$

so if $X = X_F$, $Y = X_G$ then the first term vanishes and the second term is $d\{F, G\}$. □

Corollary

Two autonomous (i.e. time-independent) Hamiltonian flows commute if and only if the corresponding Hamiltonian functions Poisson commute.

This tells us that

- A Hamiltonian torus action is given by a collection of k linearly independent Poisson-commuting functions $H_1, \ldots, H_n$ each defining a circle action.
- Here linear independence refers to the 1-forms dH_i being linearly independent on a dense open subset.
- The orbits are isotropic (if $\{F, G\} = 0$ then $\omega(X_F, X_G) = 0$), in particular for an n-torus acting on a $2n$-manifold the generic orbits are Lagrangian (tori) and one can never have more than n linearly independent Poisson-commuting functions.

Definition

The map $(H_1, \ldots, H_n) : X \to \mathbb{R}^n$ is called the moment map of the Hamiltonian torus action.

Examples: $\mathbb{C}^n$, blow-up

Consider the functions $H_1 = |z_1|^2, \ldots, H_n = |z_n|^2$ on $\mathbb{C}^n = \{(z_1, \ldots, z_n)\}$. These generate circle actions (X_{H_i} acting by $e^{i\theta}$ on the i-th coordinate) which clearly commute. the image of $\mathbb{C}^n$ under this collection of maps is the positive orthant in $\mathbb{R}^n$. For example, the image of $\mathbb{C}^2$ is just $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. The origin goes to $(0, \ldots, 0)$. Over each real coordinate axis lives the complex line in the corresponding complex coordinate direction.

If we blow up a ball of radius r then the moment image gets truncated along the line $|z|^2 = |z_1|^2 + |z_2|^2 = H_1 + H_2 = 1$ and over this line lives the exceptional sphere.

Examples: $\mathbb{CP}^n$

Use homogeneous coordinates $[z_0 : \ldots : z_n] \in \mathbb{CP}^n$ and set

$$(H_1, \ldots, H_n) = \left(\frac{|z_1|^2}{|z|^2}, \ldots, \frac{|z_n|^2}{|z|^2} \right)$$

where $|z|^2 = \sum_{i=0}^n |z_i|^2$. Normalising away from $z_0 = 0$ so that $|z|^2 = 1$ things look like $\mathbb{C}^n$ with the same circle action as before. However we note that

$$0 \leq |z_0|^2 = 1 - \sum_{i=1}^n |z_i|^2$$

means that we restrict to the subset $\sum_{i=1}^n |z_i|^2 \leq 1$, i.e. the ball of radius 1 in $\mathbb{C}^n$. The moment image is the subset of the positive orthant below the hyperplane

$$K = \{H_1 + \ldots + H_n = 1\}$$

At infinity we compactify by adding in a $\mathbb{CP}^{n-1}$. The moment image of this is precisely the intersection of the hyperplane K with the positive orthant.

Point

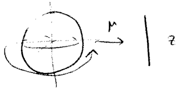
That is to say the moment image of $\mathbb{CP}^n$ is a simplex in $\mathbb{R}^n$ whose vertices are at the origin and at the unit points along the coordinate axes. The face spanned by all vertices except the origin is the moment image of the $\mathbb{CP}^{n-1}$ "at infinity" and is itself a simplex...

Exercise

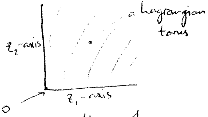
Show that the height function on $S^2 \subset \mathbb{R}^3$ generates the circle action of rotation around the z-axis. What is the moment image of $S^2 \times \cdots \times S^2$ under the moment map generating the torus action which rotates each sphere in this way?

It's clear that we can read a lot of geometry off the moment image. In $\mathbb{CP}^n$ we can see linear subspaces living over faces, we can see Lagrangian tori living over interior points. What kind of moment image do we expect in general?

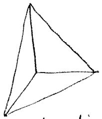
Pictures of moment maps & moment polytopes



Moment map for rotation about z-axis of the unit sphere

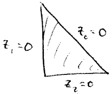


Moment image of $\mathbb{C}^2$

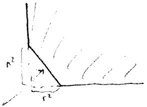


Moment image of $\mathbb{CP}^3$

4 Facets $\longleftrightarrow$ $\mathbb{CP}^2$ linear subsets where $x_0=0, x_1=0, x_2=0, x_3=0$



Moment image of $\mathbb{CP}^2$



Exceptional sphere projects to this line.

Blowing up a ball of radius r^2

Convexity theorem

Theorem (Atiyah)

The image of a compact symplectic manifold (X, ω) under the moment map $\mu = (H_1, \ldots, H_k) : X \to \mathbb{R}^k$ of a Hamiltonian torus action is the convex hull of the points $\mu(p)$ where p is a fixed point of the torus action. The fixed points form a finite collection of symplectic submanifolds over each of which the moment map is constant.

The proof of this theorem is not hard, it is a combination of observations we have already made and some Morse-Bott theory. However, I don't have time to prove it. I also want to point out a kind of converse.

Delzant's theorem

Definition

A polytope is a subset of $\mathbb{R}^n$ defined by a finite collection of linear inequalities $\langle u, x\rangle \geq c$. The vectors u are the normals of the supporting hyperplanes defining these inequalities. A polytope with vertices in $\mathbb{Z}^n \subset \mathbb{R}^n$ is called a Delzant polytope if at each vertex, the collection of normal vectors u to all the supporting hyperplanes through the vertex form a $\mathbb{Z}$-basis for the integer lattice in $\mathbb{R}^n$.

One can use a $\mathbb{Z}$-affine transformation of $\mathbb{R}^n$ to transform a Delzant polytope so that it has any given vertex at the origin and the germ of the polytope at that vertex is just the positive orthant. Given a Delzant polytope Δ we will construct a symplectic manifold with a Hamitonian torus action whose moment polytope is Δ.

The construction is easy enough. Let's do it in the case of $\mathbb{CP}^1$ and it'll be clear(ish) how to generalise. Take the vector space with basis e_i corresponding to the facets $v \geq 0$, $v \leq 1$ of the polytope (in this case 2-d, generated by the endpoints of a line). Consider the map sending the basis vector e_i to u_i, in this case:

$$\begin{pmatrix} 1 & -1 \end{pmatrix}$$

The kernel of this map is spanned by $(1,1)^T$. Consider $\mu : \mathbb{C}^2 \to \mathbb{R}^2$ the (shifted by c_i) moment map sending (z_1, z_2) to $(|z_1|^2, |z_2|^2 - 1)$ which generates the standard $(S^1)^2$-action on $\mathbb{C}^2$. Let $\psi = p \circ \mu$ where p is projection onto the subspace $\langle (1,1)^T \rangle$. This generates a subtorus (circle) action and we can divide out the zero-level by this to obtain a space W. In our case we just have the subset $|z_1|^2 + |z_2|^2 - 1 = 0$ or $|z_1|^2 = 1 - |z_2|^2$ divided out by a circle (i.e. the unit disc with the unit circle $|z_2| = 0$ identified to a point where S^1 fixes $z_2 = 0$ and rotates $|z_1| = 1$). This still has a residual moment map to $\mathbb{R}$ by projecting onto the complement of $\langle (1,1)^T \rangle$, namely $\frac{|z_1|^2 - |z_2|^2}{2} = |z_1|^2$.