Handout 8: Matrix Inverses

If $\underline{\underline{A}}$ is a square $(n \times n)$ matrix whose determinant is not zero, then there exists an **inverse** $\underline{\underline{A}}^{-1}$ satisfying:

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}} \qquad \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}.$$

Properties

Uniqueness If the inverse exists, it is unique (a matrix can only have one inverse).

Inverse of a product $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$.

Determinant of an inverse $|\underline{\underline{A}}^{-1}| = \frac{1}{|\underline{\underline{A}}|}$.

Inverse of an inverse $(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$.

Inverse of a small matrix For a 2×2 matrix, if

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and } |\underline{\underline{A}}| \neq 0 \quad \text{then} \quad \underline{\underline{A}}^{-1} = \frac{1}{|\underline{\underline{A}}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Gauss-Jordan elimination

If we write an augmented matrix containing $\left(\underline{\underline{A}} \,|\, \underline{\underline{I}}\right)$ we can use Gauss-Jordan elimination:

1. Carry out **Gaussian elimination** to get the matrix on the left to upper-triangular form. Check that there are no zeros on the digaonal (otherwise there is no inverse and we say $\underline{\underline{A}}$ is **singular**)
2. Working from bottom to top and right to left, use row operations to create zeros above the diagonal as well, until the matrix on the left is diagonal.
3. Divide each row by a constant so that the matrix on the left becomes the identity matrix.

Once this is complete, so we have $\left(\underline{\underline{I}} \,|\, \underline{\underline{B}}\right)$, the matrix on the right is our inverse: $\underline{\underline{B}} = \underline{\underline{A}}^{-1}$.

Cofactor method

There is a formal way to define the inverse. We find all the **cofactors** of our matrix (look back to determinants if you've forgotten) and put them in the *matrix of cofactors*. Then:

$$\underline{\underline{A}}^{-1} = \frac{1}{|\underline{\underline{A}}|}\left(\textit{matrix of cofactors}\right)^{\top} = \frac{1}{|\underline{\underline{A}}|}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}.$$

When to use the matrix inverse

- If we have a single system to solve, usually use Gaussian elimination to find the solution.
- If we have many systems with the same matrix: $\underline{\underline{A}}\,\underline{x}_1 = \underline{b}_1$, $\underline{\underline{A}}\,\underline{x}_2 = \underline{b}_2$, $\underline{\underline{A}}\,\underline{x}_3 = \underline{b}_3 \ldots$ then it may be more efficient to find $\underline{\underline{A}}^{-1}$.
- To find $\underline{\underline{A}}^{-1}$, the cofactor method is quicker for small matrices but we can't do anything like row sums to check during the calculation.
- In real life (for large matrices) it's usually more efficient to use the Gauss-Jordan method (with pivoting)
- At the end of either of these, we should check $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$ to be sure we have the right answer.