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Variation of Scattering Poles for Conformal Metrics

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Abstract. We estimate the size of the first variation of a scattering pole for $SL(2,\mathbb{Z})\setminus\mathbb{H}$ for certain conformal perturbations of the hyperbolic metric.

1. Introduction

The determinant of the Laplace operator $\det'\Delta$ is a global spectral invariant. It has been used in the study of sets of isospectral metrics on a Riemann surface, see [**14**], via the Polyakov formula. This formula gives the variation of $\det'\Delta$ along a conformal family of metrics [**15**]. For further applications of determinants of conformal operators and Polyakov's formula to isospectrality in 4 dimensions, see [**1**], [**2**], [**3**], [**4**]. The natural generalization of $\det'\Delta$ on surfaces with cusps, see [**11**] includes the eigenvalues and the scattering poles. These are the poles of $\det'\Phi(s)$ on $\Re s < 1/2$, where $\Phi(s)$ is the scattering matrix. Throughout this work we write the eigenfunction equation as $\Delta f + s(1-s)f = 0$. For $\det'\Delta$ we have at least formally

$$-\log\det{}'\Delta = \zeta'(0) = -\sum_{\eta\neq 1}\log|1-\eta| + c_1 \tag{1.1}$$

for a constant c_1 and where the summation is over the resonance set, which includes the eigenvalues, the scattering poles and $1/2$ with appropriate multiplicities. In order to study the variation of $\det'\Delta$ it is useful to study the variation of the individual terms in (1.1) and, more precisely, the size of the quotient $\dot{s}/s_0$, where s_0 is a scattering pole. In this work we study the variation of scattering poles arising from conformal perturbations of the hyperbolic metric. Another motivation to study the size of the variation of scattering poles is the problem of the location of scattering poles. The generalization of Weyl's law of counting eigenvalues for surfaces with cusps is

$$N(T) + \frac{1}{2}N_p(T) \sim \frac{\mathrm{Area}(\Gamma\setminus\mathbf{H})}{4\pi}T^2, \tag{1.2}$$

where $N(T)$ counts the eigenvalues less than or equal to T^2 and $N_p(T)$ counts the number of scattering poles in a ball of radius T. One is also interested in

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studying the number of scattering poles in other regions, in particular vertical strips $\sigma < \Re s < 1/2$. If the surface is hyperbolic, there exists a σ such that all the scattering poles are in the vertical strip $\sigma < \Re s < 1/2$, see [**20**]. There are counterexamples if it is not hyperbolic, see [**7**]. Our original motivation was to investigate whether in metrics conformal to the hyperbolic metric on $SL(2,\mathbb{Z})\setminus\mathbb{H}$ the scattering poles remain in a vertical strip or not.

The scattering matrix for $SL(2,\mathbb{Z})\setminus\mathbb{H}$ is

$$\phi(s) = \frac{\pi^{2s-1}\Gamma(1-s)\zeta(2-2s)}{\Gamma(s)\zeta(2s)}. \tag{1.3}$$

Throughout this work we assume the Riemann hypothesis (RH) on the nontrivial zeros ρ of $\zeta(s)$. It implies that the poles of $\phi(s)$ are at $s_0 = \rho/2 = 1/4 + i\gamma/2$, where $\gamma \in \mathbb{R}$. We also assume that the zeros are simple. Numerical evidence by Odlyzko [**12**] suggests that this is indeed true. Montgomery [**10**] proved that the pair correlation conjecture implies that almost all zeros are simple and RH implies that at least $2/3$ of the zeros are simple. Unconditionally Conrey [**5**] has proved that at least $2/5$ of the zeros are simple, while under RH and the generalized Lindelöf hypothesis at least $19/27$ of the zeros are simple [**6**]. The following weak form of the Mertens hypothesis implies that all the zeros of the zeta function are simple, [**22**, Th. 14.29, p. 376]. Let $\mu(n)$ be the Möbius function and $M(x) = \sum_{n\leq x}\mu(n)$. Then

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx = O(\log X). \tag{1.4}$$

We will also assume (1.4).

We state Corrolary 2 in [**16**] as

THEOREM 1.1. *For a compactly supported conformal perturbation of the metric $g_\epsilon = e^{\epsilon f} g_0$, where $f \in C_c^\infty(\Gamma\setminus\mathbb{H})$ the first variation of the scattering pole at s_0 with multiplicity one for a surface with one cusp is*

$$\dot{s} = \frac{s_0(1-s_0)}{2s_0-1}\mathrm{Res}_{s=s_0}\Phi(s)\int_{\Gamma\setminus\mathbb{H}} f(z)E(z,1-s_0)^2\, d\mu. \tag{1.5}$$

The first variation of a scattering pole of multiplicity one is defined through the Taylor series

$$s(\epsilon) = s_0 + \epsilon\dot{s} + \cdots$$

We first look at metrics which are asymptotically hyperbolic.

THEOREM 1.2. *For a conformal perturbation of the metric $g_\epsilon = e^{\epsilon f}$, where f is an even Maaß cusp eigenform and for any scattering pole s_0 we have*

$$\dot{s} = G(s_0)\frac{1}{\zeta'(2s_0)}L(f,1/2)L(f,3/2-2s_0), \tag{1.6}$$

where $L(f,s)$ is the L-series of f, $G(s_0)$ is a nonzero product of Gamma factors, powers of π and values of $\zeta(s)$ in the domain of absolute convergence. If $L(f,1/2)\neq 0$ then all the scattering poles move at the same time and

$$|s_0|^{-\epsilon} \ll \left|\frac{\dot{s}}{s_0}\right| = o(|s_0|^{1+\epsilon}). \tag{1.7}$$

REMARK 1.3. We take f to be even, because it is clear from (1.5) that if f is odd the integral in (1.5) is 0. The exact formula for $G(s_0)$ is given in the proof. The proof shows that only the upper bound depends on (1.4). The first part of the theorem gives a spectral interpretation of the central value of a Maaß cusp form. Vanderkam [**23**] has recently proved that $1/3-\epsilon$ of the values $L(f,1/2)$ are nonzero for f even Maaß cusp form.

For a compactly supported conformal perturbation of $SL(2,\mathbb{Z})\setminus\mathbb{H}$ we have the following theorem.

THEOREM 1.4. *Let $f\in C_c^\infty(SL(2,\mathbb{Z})\setminus\mathbb{H})$. Under the same assumptions as in Theorem 1.2 for any scattering poles s_0 and for the conformal perturbation of the metric $g_\epsilon=e^{\epsilon f}g_0$ we have*

$$\dot{s}=o(|s_0|^{5/2}).$$

From (1.6) it is clear that $\dot{s}$ is influenced by the size of $1/\zeta'(\rho)$. Gonek (in preparation) has recently conjectured that

$$\frac{1}{\zeta'(\rho)}\ll|\rho|^\epsilon \tag{1.8}$$

for all $\epsilon>0$. This gives the following corollary.

COROLLARY 1. Under the same assumptions as Theorem 1.2 and (1.8) we get the improved upper bound

$$\left|\frac{\dot{s}}{s_0}\right|\ll|s_0|^\epsilon. \tag{1.9}$$

This corollary together with Theorem 1.2 says that up to arbitrarily small powers of the scattering pole, the variation $\dot{s}$ has the same size as the scattering pole s_0, when $L(f,1/2)\neq 0$.

2. Proof of Theorems 1.2 and 1.4

By symmetry of the zeros we can assume that $\gamma>0$. For the f in Theorem 1.4 we consider the expansion of $f(z)\in C_c^\infty(\Gamma\setminus\mathbb{H})$ in automorphic eigenfunctions. Let $\phi_0,\phi_1,\dots$ be an orthonormal basis of eigenfunctions with eigenvalues $1/4+t_j^2$. We can assume that they are real-valued and that $\phi_0=1/\sqrt{\mathrm{Area}(\Gamma\setminus\mathbb{H})}$. Then

$$f(z)=\sum_{j=0}^{\infty}(f,\phi_j)\phi_j(z)+\frac{1}{4\pi}\int_{-\infty}^{\infty}(f(z),E(z,1/2+it))E(z,1/2+it)\,dt \tag{2.1}$$

in the L^2 sense. For Theorem 1.2 f is equal to an ϕ_j for some even Maaß cusp form.

2.1. Rankin-Selberg convolutions of Maaß cusp forms with $E(z,1-s_0)$.

To apply (1.5) we first analyze the integrals

$$\int_{\Gamma\setminus\mathbb{H}}\phi_j(z)E(z,1-s_0)^2\,d\mu,\quad j=1,2,\dots. \tag{2.2}$$

If $\phi_j(z)$ is odd, i.e., $\phi_j(-\bar{z})=-\phi_j(z)$, the integral (2.2) is zero, since the Eisenstein series $E(z,s)$ is even, i.e., $E(-\bar{z},s)=E(z,s)$. So we can assume that $\phi_j(z)$ is also even and has the expansion

$$\phi_j(z) = \rho_j(1)y^{1/2}\sum_{n=1}^{\infty}\lambda_j(n)K_{it_j}(2\pi ny)\cos(2\pi nx), \tag{2.3}$$

where $\lambda_j(1) = 1$. We also used the fact that for $SL(2,\mathbb{Z})$ all the L^2-eigenfunctions are cusp forms with the exception of the constant eigenfunction ϕ_0. We assume that $\phi_j(z)$ is also a Hecke eigenform. This means that the coefficients $\lambda_j(n)$ satisfy the multiplicative relations

$$\lambda_j(mn) = \lambda_j(m)\lambda_j(n), \quad (m,n) = 1,$$

$$\lambda_j(p^k) = \lambda_j(p^{k-1})\lambda_j(p) - \lambda_j(p^{k-2}), \quad p \text{ prime}.$$

The L-series of $\phi_j(z)/\rho_j(1)$ is

$$L(\phi_j, s) = \sum_{n=1}^{\infty}\frac{\lambda_j(n)}{n^s} = \prod_p (1-\lambda_j(p)p^{-s}+p^{-2s})^{-1}. \tag{2.4}$$

We set

$$I_j(s) = \int_{\Gamma\backslash\mathbb{H}} \phi_j(z)E(z,1-s_0)E(z,s)\,d\mu \tag{2.5}$$

so that the integral in (2.2) is $I_j(1-s_0)$. The integral $I_j(s)$ is essentially the Rankin-Selberg convolution of $\phi_j(z)$ with $E(z,1-s_0)$. Since we take the Rankin-Selberg convolution with an Eisenstein series, which is a Hecke eigenform, we expect that the Rankin-Selberg convolution factors as product of L-series. This is seen as follows. The Fourier expansion of the Eisenstein series is

(2.6)

$$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2y^{1/2}}{\xi(2s)}\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)K_{s-1/2}(2\pi ny)\cos(2\pi nx),$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, $\phi(s) = \xi(2s-1)/\xi(2s)$ and $\sigma_\nu(n) = \sum_{d|n} d^\nu$. For $\Re s \gg 0$ we unfold the integral in (2.5) to get

$$I_j(s) = \int_0^\infty\int_0^1 \phi_j(z)E(z,1-s_0)y^s\,\frac{dx\,dy}{y^2}$$

and use (2.6) and (2.3) to get

$$\begin{aligned} I_j(s) &= \frac{\rho_j(1)}{\xi(2-2s_0)}\sum_{n=1}^{\infty}\lambda_j(n)n^{1/2-s_0}\sigma_{2s_0-1}(n)\int_0^\infty K_{1/2-s_0}(2\pi ny)K_{it_j}(2\pi ny)y^s\,\frac{dy}{y} \\ &= \frac{\rho_j(1)}{\xi(2-2s_0)}\sum_{n=1}^{\infty}\frac{\lambda_j(n)n^{1/2-s_0}\sigma_{2s_0-1}(n)}{(2\pi n)^s}\int_0^\infty K_{1/2-s_0}(y)K_{it_j}(y)y^s\,\frac{dy}{y} \end{aligned}$$

The last integral can be evaluated as in [**8**, p. 716, 6.576.4]. Therefore,

$$\begin{aligned} I_j(s) &= \frac{\rho_j(1)\pi^{-s}}{8\xi(2-2s_0)\Gamma(s)}\Gamma\left(\frac{s+1/2-s_0+it_j}{2}\right)\Gamma\left(\frac{s-1/2+s_0+it_j}{2}\right) \\ &\cdot\Gamma\left(\frac{s+1/2-s_0-it_j}{2}\right)\Gamma\left(\frac{s-1/2+s_0-it_j}{2}\right)R(s), \end{aligned} \tag{2.7}$$

where we set

$$R(s)=\sum_{n=1}^{\infty}\frac{\lambda_j(n)n^{1/2-s_0}\sigma_{2s_0-1}(n)}{n^s}. \tag{2.8}$$

As in [**9**] we have

$$R(s)=\frac{L(\phi_j,s-1/2+s_0)L(\phi_j,s+1/2-s_0)}{\zeta(2s)}. \tag{2.9}$$

To deduce equation (2.9) we notice that, since the coefficients $\lambda_j(n)n^{1/2-s_0}\sigma_{2s_0-1}(n)$ are multiplicative, $R(s)=\prod_p R_p(s)$, where

$$R_p(s)=\sum_{k=0}^{\infty}\lambda_j(p^k)p^{k(1/2-s_0)}\sigma_{2s_0-1}(p^k)p^{-ks}.$$

Since $\sigma_{2s_0-1}(p^k)=(1-p^{(2s_0-1)(k+1)})/(1-p^{2s_0-1})$,

$$\begin{aligned}
R_p(s) &= \frac{1}{1-p^{2s_0-1}}\left[\sum_{k=0}^{\infty}\lambda_j(p^k)p^{-k(s-1/2+s_0)}-p^{2s_0-1}\sum_{k=0}^{\infty}\lambda_j(p^k)p^{-k(s+1/2-s_0)}\right]\\
&= \frac{(1-p^{2s_0-1})^{-1}}{1-\lambda_j(p)p^{-s+1/2-s_0}+p^{-2s+1-2s_0}}-\frac{(1-p^{2s_0-1})^{-1}p^{2s_0-1}}{1-\lambda_j(p)p^{-s-1/2+s_0}+p^{-2s-1+2s_0}}\\
&= \frac{1-p^{-2s}}{(1-\lambda_j(p)p^{-s+1/2-s_0}+p^{-2s+1-2s_0})(1-\lambda_j(p)p^{-s-1/2+s_0}+p^{-2s-1+2s_0})}.
\end{aligned}$$

Since $\zeta(s)=\prod(1-p^{-s})^{-1}$, we get (2.9) using (2.4). Alternatively, we apply Lemma 1 in [**21**] that gives the Euler product of the Rankin-Selberg convolution of two Dirichlet series and notice that $\sum\sigma_a(n)n^{-s}=\zeta(s)\zeta(s-a)$, [**22**, 1.3.1, p. 8].

Using (2.7) and (2.9) we get

$$\begin{aligned}
I_j(1-s_0) &= \frac{\rho_j(1)|\Gamma(1/4+it_j/2)|^2\Gamma(1/2-i\gamma/2+it_j/2)\Gamma(1/2-i\gamma/2-it_j/2)}{8\Gamma(3/4-i\gamma/2)^2\zeta(3/2-i\gamma)^2}\\
&\quad\cdot L(\phi_j,1-i\gamma)L(\phi_j,1/2).
\end{aligned}$$

We notice that $1-i\gamma$ is on the edge of the critical strip of $L(\phi_j,s)$ and using the argument of de la Vallée Poussin and Hadamard proving that $\zeta(1+it)\neq 0$, see [**19**], we see that this value is nonzero.

We would like to estimate $\mathrm{Res}_{s=1/4+i\gamma/2}\phi(s)I_j(1-s_0)$.

We have

$$\mathrm{Res}_{s=1/4+i\gamma/2}\phi(s)=\frac{\pi^{-1/2+i\gamma}\zeta(3/2-i\gamma)\Gamma(3/4-i\gamma/2)}{2\zeta'(1/2+i\gamma)\Gamma(1/4+i\gamma/2)}. \tag{2.10}$$

We estimate the Gamma factors using the asymptotic formula

$$|\Gamma(x+iy)|\sim\sqrt{2\pi}e^{-\pi|y|/2}|y|^{x-1/2},\quad |y|\to\infty,\quad x,y\in\mathbb{R}, \tag{2.11}$$

see [**8**, p. 945, 8.328.1].

For fixed t_j and as $\gamma\to\infty$ we get, using (2.11),

$$\begin{aligned}
|\Gamma(1/2-i\gamma/2\pm it_j/2)| &\sim \sqrt{2\pi}e^{-\pi|\gamma\mp t_j|/4}\\
|\Gamma(3/4-i\gamma/2)| &\sim \sqrt{2\pi}e^{-\pi|\gamma|/4}|\gamma/2|^{1/4}\\
|\Gamma(1/4+i\gamma/2)| &\sim \sqrt{2\pi}e^{-\pi|\gamma|/4}|\gamma/2|^{-1/4}
\end{aligned}$$

We also have

$$\frac{1}{\zeta(3/2)} \le |\zeta(3/2 - i\gamma)| \le \zeta(3/2), \tag{2.12}$$

where the left inequality follows from

$$\frac{1}{\zeta(3/2 - i\gamma)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2-i\gamma}}.$$

Under (1.4) we have

$$\frac{1}{\zeta'(1/2 + i\gamma)} = o(|\gamma|), \tag{2.13}$$

which follows from

$$\sum_{\rho} \frac{1}{|\rho\zeta'(\rho)|^2} < \infty,$$

see [**22**, 14.29.4, p. 377]. In fact the series converges to approximately 0.029, see [**13**]. On the other hand Cauchy's integral formula gives for any t real

$$\zeta'(1/2 + it) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta(1/2 + it + re^{i\theta})}{r^2 e^{2i\theta}} re^{i\theta}\, d\theta = O\left(\frac{1}{r}\int_0^{2\pi} |\zeta(1/2 + it + re^{i\theta})|\, d\theta\right).$$

The Riemann hypothesis implies the Lindelöf hypothesis, according to which

$$\begin{aligned} \zeta(\sigma + it) &= O_\epsilon(t^\epsilon), \quad \sigma > 1/2, \\ \zeta(\sigma + it) &= O_\epsilon(t^{1/2-\sigma+\epsilon}), \quad \sigma \le 1/2. \end{aligned} \tag{2.14}$$

We choose $r = 1/\log t$. The above estimates imply that $\zeta(1/2+it+re^{i\theta}) = O_\epsilon(|t|^\epsilon)$, so that

$$\zeta'(1/2 + it) = O_\epsilon(t^\epsilon \log t)$$

from which follows, in particular, the existence of a constant K_ϵ so that

$$\frac{K_\epsilon}{\gamma^\epsilon} \le \frac{1}{|\zeta'(1/2 + i\gamma)|}. \tag{2.15}$$

As $\gamma \to \infty$

$$\mathrm{Res}_{s=1/4+i\gamma/2}\phi(s)I_j(1 - s_0) \ll \frac{|\rho_j(1)||\Gamma(1/4 + it_j/2)|^2 |L(\phi_j, 1/2)|\gamma^\epsilon}{|\zeta'(1/2 + i\gamma)|}, \tag{2.16}$$

which is $o(\gamma^{1+\epsilon})$, since $L(\phi_j, 1 - i\gamma) \ll \gamma^\epsilon$ and (2.13) holds. This proves the upper bound in Theorem 1.2. The lower bound follows from $L(\phi_j, 1 - i\gamma) \gg \gamma^{-\epsilon}$. The proof of the corollary is also obvious if we assume (1.8).

2.2. Incomplete Eisenstein series. Let $h(y) \in C^\infty(\mathbb{R}^+)$ be a function which decreases rapidly at 0 and ∞. This means that $h(y) = O_N(y^N)$ for $0 < y \le 1$ and $h(y) = O(y^{-N})$ for $y \gg 1$. Its Mellin transform is

$$H(s) = \int_0^\infty h(y) y^{-s} \frac{dy}{y} \tag{2.17}$$

and the Mellin inversion formula gives

$$h(y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} H(s) y^s\, ds \tag{2.18}$$

for any $\sigma \in \mathbb{R}$. The function $H(s)$ is entire and $H(\sigma+it)$ is in the Schwartz space in the t variable for any $\sigma \in \mathbb{R}$. We consider the incomplete Eisenstein series

$$F_h(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(\Im(\gamma z)) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H(s)E(z,s)\,ds. \tag{2.19}$$

The function $F_h(z)$ is smooth and rapidly decreasing in the cusp. Then

$$\int_{\Gamma\backslash\mathbb{H}} F_h(z)(E(z,1-s_0))^2\frac{dx\,dy}{y^2} = \frac{1}{2\pi i}\int_{\Re s=2}\int_0^\infty\int_0^1 H(s)y^s(E(z,1-s_0))^2\frac{dx\,dy}{y^2}\,ds. \tag{2.20}$$

Since s_0 is a scattering pole, the functional equation $\phi(s)\phi(1-s)=1$ gives that $\phi(1-s_0)=0$. By (2.6)

$$\int_0^1 (E(z,1-s_0))^2\,dx = y^{3/2-i\gamma} + \frac{2y}{(\xi(2-2s_0))^2}\sum_{n=1}^\infty \frac{\sigma_{-1+2s_0}(n)^2}{n^{i\gamma-1/2}}(K_{1/2-s_0}(2\pi n y))^2.$$

The term $y^{3/2-i\gamma}$ gives

$$\frac{1}{2\pi i}\int_{\Re s=2}\int_0^\infty H(s)y^{s-1/2-i\gamma}\,dy\,ds = \int_0^\infty h(y)y^{-1/2-i\gamma}\,dy.$$

This term is $O_N(\gamma^{-N})$ for all $N>0$ by the Riemann-Lebesgue Lemma. In the other terms the y-integration can be performed explicitly. We call the sum of these terms $I_2(\gamma)$ and we have

$$I_2(\gamma) = \frac{1}{\pi i(\xi(2-2s_0))^2}\int_{\Re s=2}\frac{H(s)}{(2\pi)^s}\sum_{n=1}^\infty\frac{(\sigma_{-1+2s_0}(n))^2}{n^{s-1/2+i\gamma}}\int_0^\infty y^{s-1}(K_{1/2-s_0}(y))^2\,dy\,ds. \tag{2.21}$$

As before, we evaluate the integral in y using [**8**, p. 716, 6.576.4] to get

$$\begin{aligned} I_2(\gamma) &= \frac{1}{\pi i(\xi(2-2s_0))^2}\int_{\Re s=2}\frac{H(s)}{8\Gamma(s)\pi^s}\Gamma\left(\frac{s}{2}+\frac14-\frac{i\gamma}{2}\right)\Gamma\left(\frac{s}{2}\right)^2 \\ &\quad\cdot\Gamma\left(\frac{s}{2}-\frac14+\frac{i\gamma}{2}\right)\sum_{n=1}^\infty\frac{(\sigma_{-1+2s_0}(n))^2}{n^{s-1/2+i\gamma}}\,ds.\end{aligned}$$

The series was first evaluated by Ramanujan, see [**22**, 1.3.3, p. 8]. We get

$$\sum_{n=1}^\infty\frac{(\sigma_{-1+2s_0}(n))^2}{n^{s-1/2+i\gamma}} = \frac{\zeta(s-1/2+i\gamma)\zeta(s+1/2-i\gamma)\zeta(s)^2}{\zeta(2s)}. \tag{2.22}$$

The contour of integration can be moved to the line $\Re s=1/2$. The poles in this region occur at $3/2-i\gamma$ and $s=1$. The pole of $\zeta(s+1/2-i\gamma)$ at $s=1/2+i\gamma$ cancels with the zero of $\zeta(s)$ and, similarly, the pole of $\Gamma(s/2-1/4+i\gamma/2)$ at $s=1/2-i\gamma$. Since $\zeta(s-1/2+i\gamma)$ has a zero of order 1 at 1, the pole of the integrand at 1 is indeed a first order pole. Notice that

$$\lim_{s\to1}(s-1)\zeta(s)^2\zeta(s-1/2+i\gamma) = \zeta'(1/2+i\gamma).$$

The residue at $s=1$ times the residue of the scattering matrix at $s=1/4+i\gamma$ give together

$$\frac{3}{4\pi}\int_0^\infty h(y)y^{-2}\,dy,$$

since $H(1) = \int h(y)y^{-2}\,dy$. This is $O(1)$. We remark that $H(1) = 0$ iff $F_h \perp 1$. The residue at $3/2 - i\gamma$ times the residue of the scattering matrix at $s = 1/4 + i\gamma$ give together

$$\frac{\pi^{-1/2+i\gamma} H(3/2-i\gamma)\sqrt{\pi}\Gamma(1-i\gamma)\Gamma(3/4-i\gamma/2)\zeta(2-2i\gamma)\zeta(3/2-i\gamma)}{8\Gamma(3/2-i\gamma)\Gamma(1/4+i\gamma/2)\zeta(3-2i\gamma)\zeta'(1/2+i\gamma)},$$

which is easily seen to be $O_N(\gamma^{-N})$, using (2.11), the Riemann-Lebesgue Lemma applied to $H(3/2-i\gamma)$ and the bounds (2.13) and (2.12). We denote the integrand in (2.21) by $G(s)$. For the integral on $\Re s = 1/2$ we again use (2.11) and (2.14) to get

$$\int_{\Re s=1/2} G(s) \ll e^{-\pi\gamma/2}\gamma^{2\epsilon}.$$

Multiplying with $1/\xi(2-2s_0)^2$ and $\mathrm{Res}_{s=s_0}\phi(s)$, we estimate by $o(\gamma^{1+2\epsilon})$.

2.3. Approximating $f \in C_c^\infty(SL(2,\mathbb{Z})\setminus\mathbb{H})$. Let $g(z) = g_1(z) + g_2(z)$, where $g_1(z)$ is a finite linear combination of Maaß cusp forms and $g_2(z)$ is in the space of incomplete Eisenstein series and $\|f-g\|_\infty < \epsilon$. Since $f - g$ is rapidly decreasing in the cusp, we can find a $k \geq 0$ which is also rapidly decreasing in the cusp and such that

$$F_k(z) \geq |f(z) - g(z)|$$

and F_k has small L^1 norm. This imitates the argument in [**9**, Prop. 2.3]. For the argument to go through one needs to estimate

$$\int_{\Gamma\setminus\mathbb{H}} F_k(z)|E(z,1-s_0)|^2\,d\mu.$$

We follow the same process as above.

$$\int_0^1 E(z,1-s_0)E(1-\bar{s}_0)\,dx = y^{3/2} + \frac{2y}{|\xi(2-2s_0)|^2}\sum_{n=1}^\infty n^{1/2}|\sigma_{2s_0-1}(n)|^2|K_{1/2-s_0}(2\pi n y)|^2.$$

The term $y^{3/2}$ contributes

$$\frac{1}{2\pi i}\int_{\Re s=2}\int_0^\infty H(s)y^{2-1/2}\,dy\,ds = \int_0^\infty h(y)y^{-1/2}\,dy,$$

which is $O(1)$. In the other terms the y-integration can be performed explicitly. We call the sum of these terms $J_2(\gamma)$ and we have

(2.23)
$$J_2(\gamma) = \frac{1}{\pi i|\xi(2-2s_0)|^2}\int_{\Re s=2}\frac{H(s)}{(2\pi)^s}\sum_{n=1}^\infty\frac{|\sigma_{-1+2s_0}(n)|^2}{n^{s-1/2}}\int_0^\infty y^{s-1}|K_{1/2-s_0}(y)|^2\,dy\,ds.$$

We evaluate the integral in y using [**8**, p. 716, 6.576.4] to get

$$\begin{aligned} J_2(\gamma) &= \frac{1}{\pi i|\xi(2-2s_0)|^2}\int_{\Re s=2}\frac{H(s)}{8\Gamma(s)\pi^s}\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}\right)\\ &\cdot\Gamma\left(\frac{s}{2}+\frac{i\gamma}{2}\right)\Gamma\left(\frac{s}{2}-\frac{i\gamma}{2}\right)\sum_{n=1}^\infty\frac{|\sigma_{-1+2s_0}(n)|^2}{n^{s-1/2}}\,ds.\end{aligned}$$

The series can be evaluated using [**22**, 1.3.3, p. 8]. We get

$$\sum_{n=1}^\infty\frac{|\sigma_{-1+2s_0}(n)|^2}{n^{s-1/2}} = \frac{\zeta(s-1/2)\zeta(s+1/2)\zeta(s-i\gamma)\zeta(s+i\gamma)}{\zeta(2s)}.$$

The contour of integration can be moved to the line $\Re s = 1/2$. The only pole in this region occurs at $3/2$ arising from $\zeta(s-1/2)$. The pole at $s = 1/2$ from $\zeta(s+1/2)$ and $\Gamma(s/2-1/4)$ cancels with the double zero of $\zeta(s-i\gamma)\zeta(s+i\gamma)$. The poles at $s = 1 \pm i\gamma$ from $\zeta(s \mp i\gamma)$ cancel with the zeros of $\zeta(s-1/2)$ at the same points. The residue at $s = 3/2$ times $1/|\xi(2-2s_0)|^2$ can be estimated as $O(1)$. For the integral on $\Re s = 1/2$ we again use (2.11) and (2.14), multiply with $1/|\xi(2-2s_0)|^2$ and estimate by $O(\gamma^{-1/2})$. Finally we use the asymptotic growth for $\mathrm{Res}_{s=s_0}\phi(s)$ $(= o(\gamma^{3/2}))$ and the factor $s_0(1-s_0)/(2s_0-1)$ in (1.5) to complete the proof of Theorem 1.4.

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