# Alan Turing and the Riemann hypothesis

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## Introduction to $\zeta(s)$ and the Riemann hypothesis

- The Riemann $\zeta$-function is defined for a complex variable $s$ with real part $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- Central to the study of prime numbers because of the identity

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \left(1 + p^{-s} + p^{-2s} + \ldots\right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

- As discovered by Riemann (c. 1859), it has analytic continuation to $\mathbb{C}$, except for a simple pole at $s = 1$, and satisfies a functional equation:

$$\text{If } \gamma(s) := \pi^{-s/2}\Gamma(\tfrac{s}{2}) \text{ and } \Lambda(s) := \gamma(s)\zeta(s) \text{ then}$$

$$\Lambda(s) = \Lambda(1 - s).$$

Crucial question turns out to be where $\zeta(s)$ (or $\Lambda(s)$) vanishes:

**Theorem** (de la Vallée Poussin-Hadamard, 1896)**.**

*All zeros of* $\Lambda(s)$ *have real part in* $(0, 1)$.

**Corollary** (Prime number theorem)**.**

$$\pi(x) := \#\{\textit{primes } p \le x\} \sim \int_2^x \frac{dt}{\log t}.$$

**Conjecture** (Riemann hypothesis)**.**

*All zeros of* $\Lambda(s)$ *have real part* $\frac{1}{2}$.

If true, the Riemann hypothesis implies that the remainder term in the prime number theorem is small (of size about the square root of the main term).

## Vertical distribution of zeros

- Let $N(t)$ be the number of zeros of $\Lambda(s)$ with imaginary part $\Im(s) \in [0, t]$.

- $N(t)$ is about $\theta(t)/\pi + 1$, where $\theta(t)$ is the phase of $\gamma(\frac{1}{2} + it)$, i.e. the continuous function such that

$$\theta(0) = 0 \quad \text{and} \quad \gamma(\tfrac{1}{2} + it) = |\gamma(\tfrac{1}{2} + it)| e^{i\theta(t)}.$$

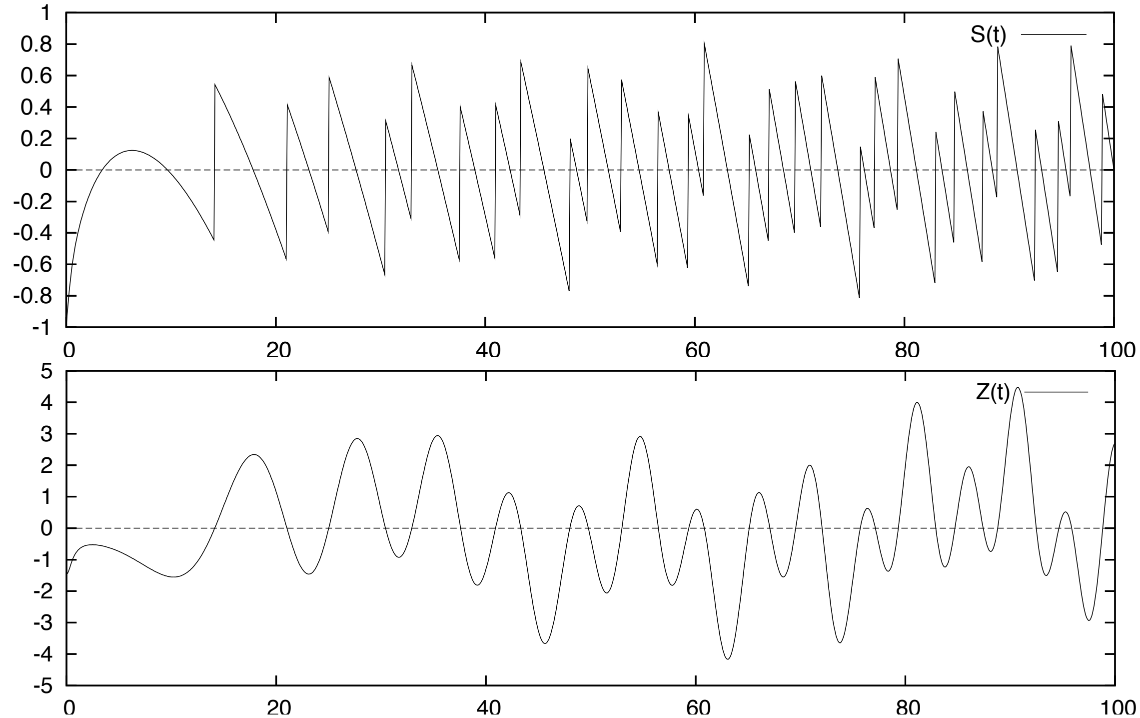
- Asymptotically, for large $t > 0$,

$$\frac{\theta(t)}{\pi} + 1 \approx \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{7}{8}.$$

In particular, $\Lambda(s)$ has many zeros.

- Define

$$S(t) := N(t) - \left(\frac{\theta(t)}{\pi} + 1\right).$$




- One often wants to compute $\Lambda(s)$ at arguments $s = \frac{1}{2} + it$.
- Since $|\gamma(\frac{1}{2} + it)|$ decreases exponentially for large $t$, we work instead with $Z(t) := \Lambda(\frac{1}{2} + it)/|\gamma(\frac{1}{2} + it)|$, which is real valued for $t \in \mathbb{R}$ and has the same zeros as $\Lambda(\frac{1}{2} + it)$.
- Riemann-Siegel formula:

$$Z(t) = 2 \sum_{n=1}^{\lfloor \sqrt{t/2\pi} \rfloor} n^{-1/2} \cos(\theta(t) - t \log n) + O(t^{-1/4}).$$

Alan Mathison Turing (1912–1954)

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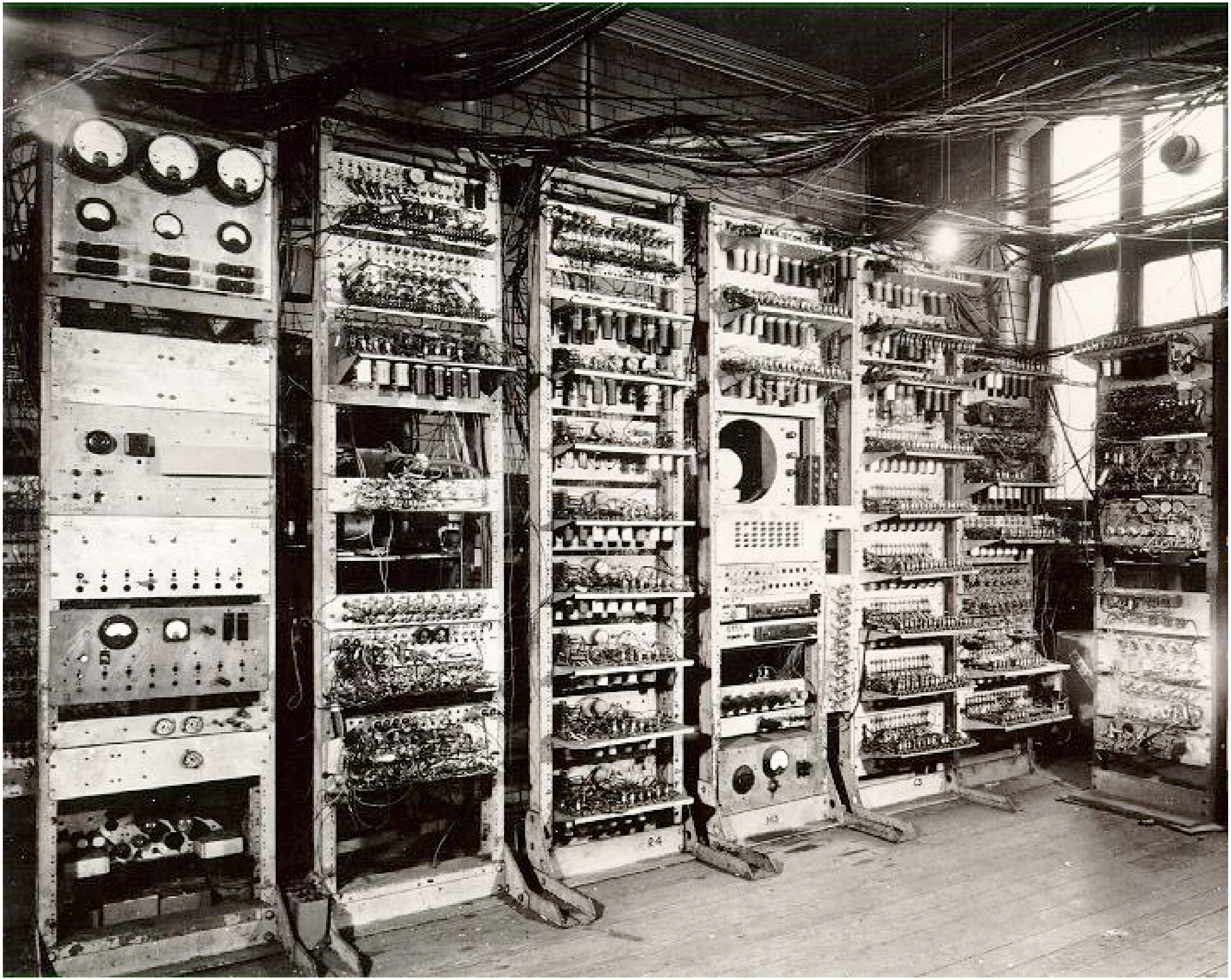


From Turing's 1953 paper:

"The calculations had been planned some time in advance, but had in fact to be carried out in great haste. If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all. As it was, the interval $2\pi.63^2 < t < 2\pi.64^2$ was investigated during that period, and very little more was accomplished."

"If definite rules are laid down as to how the computation is to be done one can predict bounds for the errors throughout. When the computations are done by hand there are serious practical difficulties about this. The computer will probably have his own ideas as to how certain steps should be done. [...] However, if the calculations are being done by an automatic computer one can feel sure that this kind of indiscipline does not occur."

## How to test the Riemann hypothesis

1. Locate zeros on the line $\Re(s) = \frac{1}{2}$ up to height $T$ by computing $Z(t)$ and noting its changes of sign.

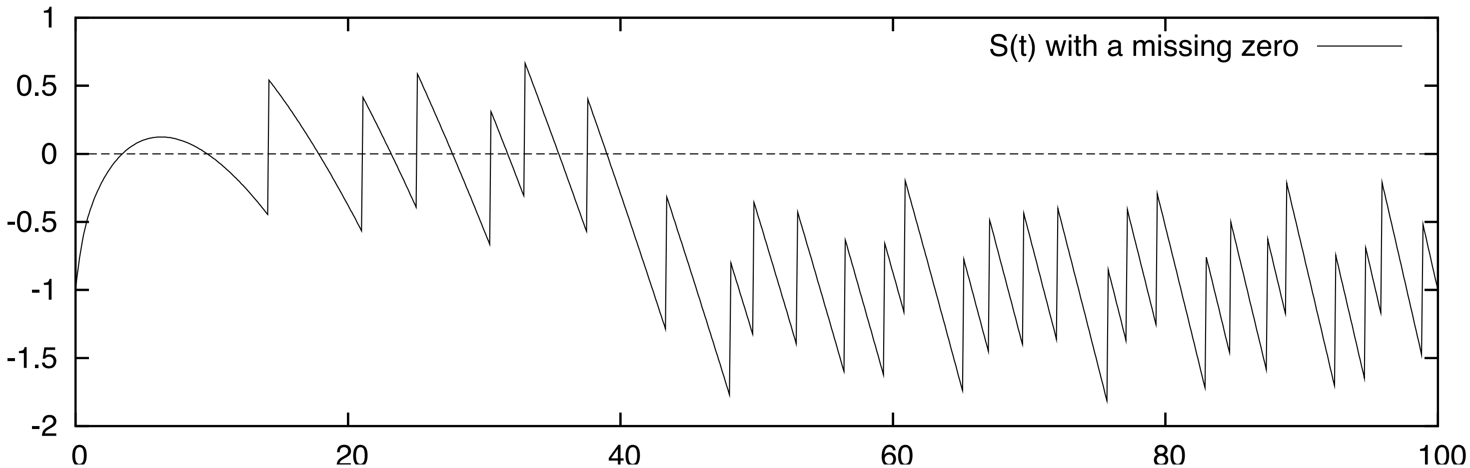
2. Show that all zeros up to height $T$ are accounted for by computing $N(T)$.

## Turing's idea

**Theorem** (Littlewood)**.** $S(t)$ *has mean value 0, i.e.*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T S(t)\,dt = 0.$$

Thus, the graph of $S(t)$ tends to oscillate around 0. Therefore, if we plot the graph using measured data, any *missing zeros* would show up as jumps.




To make this precise, we need an explicit version of Littlewood's theorem:

**Theorem** (Turing). *For any* $h > 0$ *and* $T > 168\pi$,

$$\left| \int_T^{T+h} S(t)\, dt \right| \le 2.3 + 0.128 \log \frac{T+h}{2\pi}.$$

Roughly speaking, this means that in order to verify the Riemann hypothesis up to height $T$ we need to compute values of $Z(t)$ for $t$ up to $T + c \log T$ for a modest constant $c$.

## A few remarks

- Turing's bound is not sharp; the coefficient of $\log(T+h)$ is limited by our knowledge about the growth rate of the $\zeta$-function along the line $\Re(s) = \frac{1}{2}$.

- The Lindelöf hypothesis, which is the conjecture that $Z(t) = O(t^{\varepsilon})$, is equivalent to the integral being $o(\log(T+h))$ as $T + h \to \infty$.

- RH implies the bound $O\left(\frac{\log(T+h)}{(\log\log(T+h))^2}\right)$.

- Heuristics based on random matrix theory suggest that the true maximum size of the integral is closer to $\sqrt{\log(T+h)}$.

- However, Turing's bound is already more than enough for numerics.

## Generalisations

Let $K$ be a number field (= finite extension of $\mathbb{Q}$) and $\mathfrak{o}_K$ its ring of integers. The *Dedekind $\zeta$-function* of $K$ is

$$\zeta_K(s) := \sum_{\substack{\text{ideals} \\ \mathfrak{a}\subset\mathfrak{o}_K}} N(\mathfrak{a})^{-s} = \prod_{\substack{\text{prime ideals} \\ \mathfrak{p}\subset\mathfrak{o}_K}} \frac{1}{1-N(\mathfrak{p})^{-s}}.$$

Analytic theory of $\zeta$ extends verbatim to $\zeta_K$:

- analytic continuation and functional equation
- prime ideal theorem: asymptotic for the number of prime ideals of norm $\leq x$
- Riemann hypothesis: All zeros of a "completed" form of $\zeta_K$ should have real part $\frac{1}{2}$

Natural question:
Can one check the Riemann hypothesis for $\zeta_K$?

## Can one check RH for $\zeta_K$?

- Not known in general!
- Basic problem: $\zeta_K$ can have multiple zeros
- Workaround: Use finite group representation theory, but that leads to other unsolved problems (Artin's conjecture)

**Theorem** (B, 2005). *Up to a certain group-theoretic hypothesis on* Gal($K/\mathbb{Q}$)*, there is an algorithm for checking the Riemann hypothesis for* $\zeta_K$*.*

One key ingredient:
Generalisation of Turing's method to arbitrary $L$-*functions*

## $L$-functions

- $L$-functions are generating functions for arithmetic data, e.g. $\zeta_K$ encodes information about the prime ideals in $K$.
- Example from arithmetic geometry: let $E : y^2 = x^3 + Ax + B$ be an elliptic curve defined over $\mathbb{Q}$. Given a prime $p$ (with finitely many exceptions), one can reduce the equation mod $p$ to get an elliptic curve over $\mathbb{F}_p$. Define

  $$\#E(\mathbb{F}_p) := \#\{(x, y) \in \mathbb{F}_p^2 : y^2 = x^3 + Ax + B\}.$$

  By a theorem of Hasse, $|p - \#E(\mathbb{F}_p)| < 2\sqrt{p}$.

  The $L$-function of $E$ combines the local data for each $p$:

  $$L(s, E) := \prod_{p \text{ prime}} \frac{1}{1 - \left(\frac{p - \#E(\mathbb{F}_p)}{\sqrt{p}}\right) p^{-s} + p^{-2s}}.$$

- Big theorem (Wiles, Taylor, et al.): $L(s, E)$ continues to an entire function and satisfies a functional equation relating $s$ to $1 - s$.
- Corollary: Fermat's last theorem

## Automorphic forms

Langlands' philosophy: $L$-functions with nice analytic properties should come from *automorphic forms*

Example: *Maass forms*
Let $\mathbb{H} = \{z = x + iy : y > 0\}$ be the hyperbolic upper half plane, with Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and Laplace operator
$\Delta = -\text{div} \circ \text{grad} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$

A Maass form $f$ is a function on $\mathbb{H}$ satisfying:

- $\Delta f = \left( \frac{1}{4} + r^2 \right) f$ for some $r \in \mathbb{R}$
- $f\left( \frac{az+b}{cz+d} \right) = f(z)$ for all $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z})$
- $\int_{\text{SL}(2,\mathbb{Z}) \backslash \mathbb{H}} |f(z)|^2 \frac{dx\,dy}{y^2} < \infty$

Fourier expansion: $f(z) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_{ir}(2\pi ny) \cos(2\pi nx)$

L-function: $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$

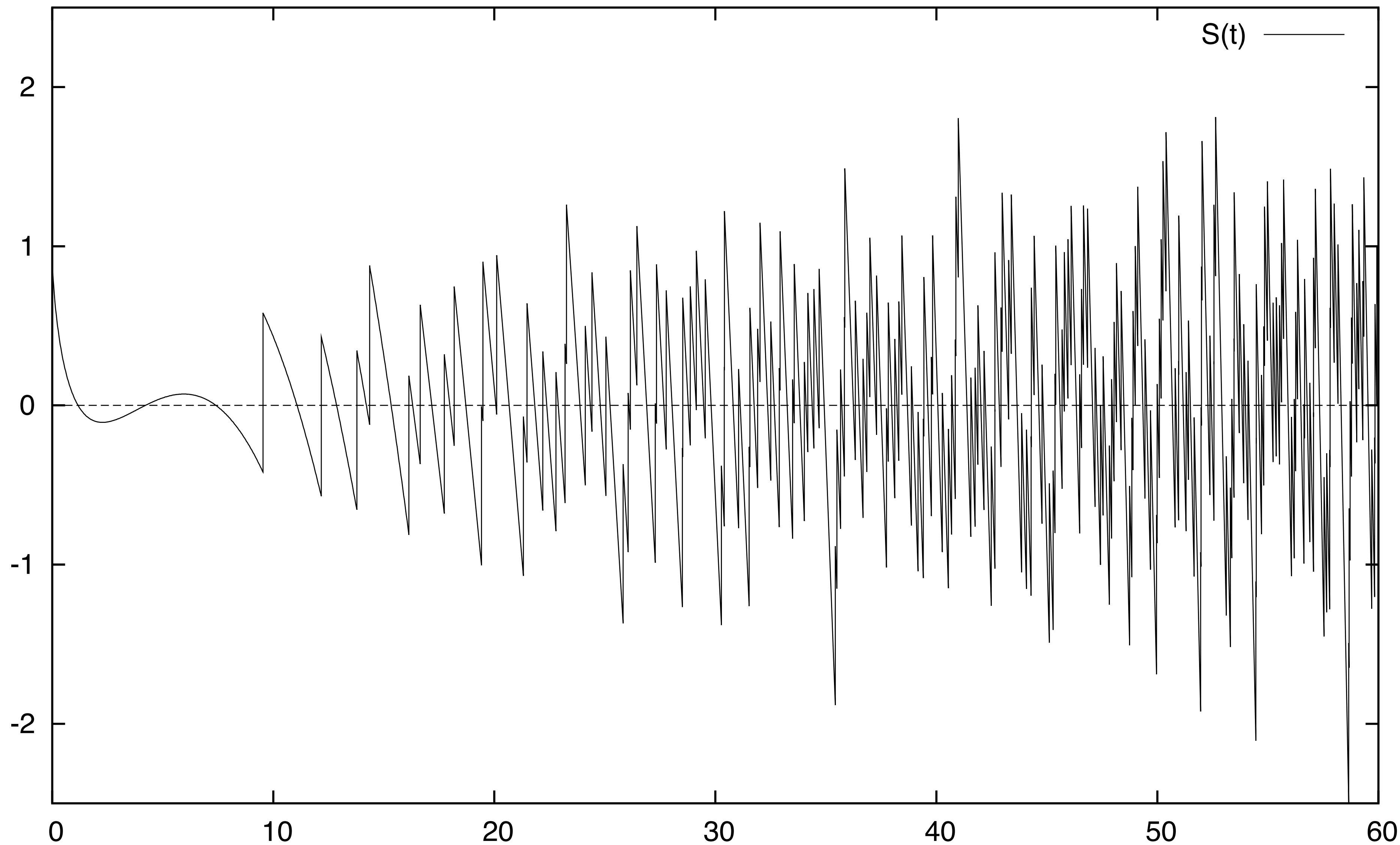
## The Selberg $\zeta$-function

$$Z(s) = \prod_{\substack{\text{primitive closed} \\ \text{geodesics } \mathcal{P} \\ \text{in SL}(2,\mathbb{Z})\backslash\mathbb{H}}} \prod_{k=0}^{\infty} \frac{1}{1 - N(\mathcal{P})^{-s-k}}$$

$Z$ encodes information about the geometry of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$, but is also intimately connected with its spectrum:
$Z(s) = 0 \Leftrightarrow s = \frac{1}{2} + ir$, where $\frac{1}{4} + r^2$ is an eigenvalue of $\Delta$.

$Z$ has many properties in common with $\zeta$:

- analytic continuation and functional equation
- prime geodesic theorem: asymptotic of number of primitive (prime) geodesics of length $\leq x$
- analogue of the Riemann hypothesis: known in this case!


S(t)
2
1
0
-1
-2
0
10
20
30
40
50
60

## Turing's method for the Selberg $\zeta$-function

**Theorem** (B-Strömbergsson, 2008). *Let* $N(t)$ *be the number of zeros of the Selberg* $\zeta$*-function with imaginary part* $\Im(s) \in [0, t]$, *and set*

$$S(t) := N(t) - \left( \frac{t^2}{12} - \frac{2t}{\pi} \log \frac{t}{e\sqrt{\frac{\pi}{2}}} - \frac{131}{144} \right)$$

*and*

$$E(t) := \left(1 + \frac{6.59125}{\log t}\right) \left(\frac{\pi}{12 \log t}\right)^2 .$$

*Then for* $T > 1$,

$$-2E(T) \leq \frac{1}{T} \int_0^T S(t)\, dt \leq E(T).$$

## References

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