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Quantum unique ergodicity for $\mathbf{SL}_2(\mathcal{O})\backslash\mathbf{H}^3$ and estimates for L-functions

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To the memory of Ralph Phillips, teacher, collaborator and friend

1. Introduction

A basic problem in the theory of quantization of chaotic Hamiltonians is that of the behavior of the mass of eigenstates in the semiclassical limit. For the Hamiltonian which is the geodesic motion on the tangent space of an arithmetic hyperbolic surface the equidistribution of these masses has been established for some of the eigenstates [20], [25]. In this paper we show that the mass of the continuous spectrum of the Laplacian on arithmetic hyperbolic three manifolds becomes equidistributed in the large energy limit, that is, we establish the 'quantum unique ergodicity conjecture' [22] for these states. As in [20] this issue of equidistribution can be reduced to establishing subconvex estimates for GL_2 automorphic L-functions associated with corresponding imaginary quadratic fields, see [17] or [18] for this reduction. The main result of this paper establishes these estimates.

We turn to a more detailed description of our results. We stick to a specific hyperbolic three manifold, the results may be extended to any congruence subgroup of the Bianchi groups [24].

Let $K = \mathbf{Q}(\sqrt{-1})$ and $\mathcal{O} = \mathbf{Z}[\sqrt{-1}]$ be its ring of integers. The group $\Gamma = \mathrm{SL}_2(\mathcal{O})$ is a lattice in $\mathrm{SL}_2(\mathbf{C})$ and acts discontinuously on the hyperbolic 3-space $\mathbf{H}^3 \cong \mathrm{SL}_2(\mathbf{C})/\mathrm{SU}(2)$. The quotient $X_\Gamma = \Gamma\backslash\mathbf{H}^3$ is a non-compact hyperbolic 3-manifold of finite volume (the Picard manifold). The L^2-spectrum of the Laplacian Δ on functions on X_Γ consists of the continuous spectrum $[1,\infty)$ provided by the unitary Eisenstein series $E(w, 1/2 + it)$, $t \geq 0$, see (1.1) below, and a discrete spectrum corresponding to an orthonormal basis of eigenfunctions $\phi_0 = 1/\sqrt{\mathrm{vol}\,(X_\Gamma)}$ and cusp forms $\phi_1, \phi_2, \ldots$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. We parametrize $\mathbf{H}^3$ as $w = (y, z) \in \mathbf{H}^3$,

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$y > 0$, $z = x_1 + ix_2 \in \mathbf{C}$ and $\mathrm{SL}_2(\mathbf{C})$ acts on $\mathbf{H}^3$ by

$$g \cdot w = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} w = (y(g \cdot w), z(g \cdot w))$$

with

$$y(g \cdot w) = \frac{y}{|\gamma z + \delta|^2 + |\gamma y|^2}, \quad z(g \cdot w) = \frac{(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha \overline{\gamma} y^2}{|\gamma z + \delta|^2 + |\gamma|^2 y^2}.$$

Let

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, m \in \mathcal{O} \right\} \subset \Gamma$$

be the standard parabolic subgroup fixing the cusp at ∞. The Eisenstein series $E(w, s)$ is defined for $\Re s > 1$ by

$$E(w, s) = \sum_{g \in \Gamma_\infty \backslash \Gamma} (y(g \cdot w))^{2s}. \tag{1.1}$$

A Maaß cusp form $\phi(w)$ (such as one of the ϕ_j's above) has a Fourier expansion

$$\phi(w) = \sum_{0 \neq \nu \in \mathcal{O}} c(\nu) y K_{ir}(2\pi |\nu| y) e(\langle \nu, z \rangle) \tag{1.2}$$

where $\Delta\phi + (1 + r^2)\phi = 0$, $\Delta = y^2(4\partial_z \partial_{\overline{z}} + \partial_y^2) - y\partial_y$. To each cusp form ϕ we associate its standard L-function, which takes the form

$$L(s, \phi) = \sum_{(\nu) \neq 0} c(\nu) N(\nu)^{-s}. \tag{1.3}$$

Here $N(\nu) = \nu\bar{\nu}$ and (ν) is the principal ideal generated by ν. This L-function and its twists by Grossencharacters satisfy functional equations as follows: Let $\lambda((\alpha)) = (\alpha/|\alpha|)^4$ be the basic Grossencharacter on ideals (α) of $\mathcal{O}$. Then the twisted L-function is defined as

$$L(s, \phi \otimes \lambda^n) := \sum_{(\nu) \neq 0} \frac{c(\nu)\lambda^n(\nu)}{N(\nu)^s} \tag{1.4}$$

and is entire. The completed L-function

$$\Lambda(s, \phi \otimes \lambda^n) := \pi^{2s} \Gamma\left(s + \frac{|4n| + ir}{2}\right) \Gamma\left(s + \frac{|4n| - ir}{2}\right) L(s, \phi \otimes \lambda^n) \tag{1.5}$$

satisfies

$$\Lambda(1 - s, \phi \otimes \lambda^{-n}) = \Lambda(s, \phi \otimes \lambda^n).$$

The Phragmen-Lindelöf principle [3] or the approximate functional equation, see [14], implies for a fixed cusp form ϕ the bound

$$L(1/2+it,\phi\otimes\lambda^n) \ll_{\epsilon,\phi} (1+|n+it|)^{1+\epsilon}, \quad \epsilon>0 \tag{1.6}$$

for $L(s,\phi\otimes\lambda^n)$ on its critical line. This bound will be referred to as the convexity bound.

THEOREM 1.1. *Fix ϕ as above. Then*

(i) *for m fixed we have*

$$L(1/2+it,\phi\otimes\lambda^m) \ll_{m,\phi} (1+|t|)^{159/166},$$

(ii) *for t fixed*

$$L(1/2+it,\phi\otimes\lambda^m) \ll_{t,\phi} (1+|m|)^{159/166}.$$

REMARK 1.2. The dependence of the implied constants on the eigenvalue parameter r and m in (i) and r and t in (ii) are polynomial in m and r (respectively t and r). Also there is nothing special about the exponents, which can somewhat be improved by the methods below.

Our application to quantum unique ergodicity mentioned earlier uses part (i) only. As is shown in [17], [18] (i) implies the following equidistribution result.

Let μ_t be the measures on X_Γ (quantum mechanical densities) that correspond to the Eisenstein series, i.e., the continuous states masses, defined as follows

$$\mu_t = |E(w,1/2+it)|^2\, d\mathrm{vol}(w). \tag{1.7}$$

Note that the energy (eigenvalue) corresponding to $E(w,1/2+it)$ is $1+t^2$.

THEOREM 1.3. *For K_1 and K_2 compact Jordan measurable subsets of X_Γ we have*

$$\lim_{t\to\infty}\frac{\mu_t(K_1)}{\mu_t(K_2)} = \frac{vol(K_1)}{vol(K_2)}.$$

The sets K_1 and K_2 can be taken to be geodesic balls for instance. Theorem 1.3 asserts that the continuous spectrum of X_Γ is 'quantum ergodic' and confirms the general conjecture in [22] in this case. It is the main result of the paper.

As with all the recent developments concerning subconvexity estimates for L-functions [12], [5], [25], [15], we use families. We establish an averaged version of the expected sharp bound, that is the Lindelöf Hypothesis:

$$L(1/2+it,\phi\otimes\lambda^m) \ll_\epsilon (|t+im|+1)^\epsilon$$

over a sufficiently small family. With this and positivity, the subconvexity estimates follow. For the case at hand it was shown in [23] that

$$\sum_{|m|\le T}\int_{-T}^{T}|L(1/2+it,\phi\otimes\lambda^m)|^2\,dt \ll T^2\log T. \tag{1.8}$$

Note that the inequality (1.8) recovers (1.6) when we drop all but one term and use the fact that the other terms are positive. We proceed by reducing the size of this averaging. We show that for $R^{76/83}\le H\le R$ and $\epsilon>0$ we have

$$\sum_{R-H\le|m+2it|\le R+H}\int|L(1/2+it,\phi\otimes\lambda^m)|^2\,dt \ll (RH)^{1+\epsilon}. \tag{1.9}$$

Thus (1.9) establishes the Lindelöf bound in the mean over this family. Note that for technical reasons we maintain radial symmetry in $|m+2it|$. Theorem 1.1 follows from (1.9) with $H=T^{76/83}$. The crucial point is the extension of (1.8) in the form (1.9) with H a power of T less than 1. This involves facing off-diagonal terms in the analysis, which is a familiar feature with such subconvexity bounds. The burden of this estimation is then transferred to cancellations in sums of products of the shifted coefficients $c(\alpha)$. Precisely we have the following theorem.

THEOREM 1.4. *Fix a cusp form* ϕ. *For* $h\in\mathcal{O}$, $h\ne 0$ *the Dirichlet series*

$$D_\phi(s,h)=\sum_{\alpha,\alpha+h\ne 0}\frac{c(\alpha)\overline{c(\alpha+h)}}{N(\alpha)^s} \tag{1.10}$$

extends to an analytic function in the region $\sigma=\Re s>11/18$ *and satisfies the estimate*

$$D_\phi(s,h)\ll_\epsilon |h|^{11/9+\epsilon}(|t|+1)+(|t|+1)^{11/2}|h|^{1-2\sigma+2/9+\epsilon} \tag{1.11}$$

in this region.

REMARK 1.5. Note that the Rankin-Selberg method implies the bound

$$\sum_{N(\alpha)\le X}|c(\alpha)|^2\ll_\phi X.$$

Thus the series $D_\phi(s,h)$ converges absolutely for $\Re s>1$. The key is the analytic continuation and the polynomial bounds in t and h, as these give the desired cancellation in smooth sums approximating $\sum c(\alpha)\overline{c(\alpha+h)}$. In the analogous setting in the hyperbolic plane $\mathbf{H}^2$, Good [11] was the first to establish such results for special forms ϕ and improve them in [10]. His method involves bounds on $\langle y^k|\phi|^2,\phi_j\rangle$, where ϕ is a holomorphic cusp form. Later it was shown in [5] that one could establish such cancellation more generally in $\mathbf{H}^2$

using a variant of Kloosterman's method together with the Voronoi summation and Weil's bound on Kloosterman sums. The recent general triple products bounds for eigenfunctions, see [26], [21] and [1], allow for a simple treatment of these sums. This is carried out in Section 2. This method has the advantage (at least at present) of being general. In particular, it can be applied to the case at hand as well as to general number fields, see [2]. It also relates $D_\phi(s,h)$ directly to the spectrum of $L^2(\Gamma\backslash\mathbf{H})$, see [27], thus allowing us to use recent bounds towards the Ramanujan conjectures, see [19], [16].

REMARK 1.6. One should be able to deal with the more general series

$$D_\phi(s,h,m) = \sum_{\alpha\neq 0} \frac{c(\alpha)\overline{c(\alpha+h)}}{N(\alpha)^s}\lambda^m(\alpha),$$

though we have not done so and, in fact, we have worked hard to avoid them. Indeed in order to deal with the most general L-function $L(s,\phi)$, for ϕ a cusp form on $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbf{A})$, where K is an imaginary quadratic field, one needs to deal with $m\neq 0$ as well.

2. Poincaré series

This section is devoted to proving Theorem 1.4. For $h\neq 0$, $h\in\mathcal{O}$ we define the Poincaré series

$$P_h(w,s) = \sum_{g\in\Gamma_\infty\backslash\Gamma} y(g\cdot w)^{2s} e(\langle h, z(g\cdot w)\rangle), \tag{2.1}$$

where $w=(y,z)$. These functions are slight modifications of those introduced in [24] and their analogs for the hyperbolic upper half plane were used for a similar purpose in [11]. They converge absolutely for $\Re s>1$, as they are majorized by $E(w,\sigma)$, and, moreover, define analytic (in s) automorphic functions of moderate growth. For the fixed cusp form $\phi(w)$ in Theorem 1.4 we consider the integral

$$I(s) = \int_{\Gamma\backslash\mathbf{H}^3} \phi(w)^2 P_h(w,s)\, d\mathrm{vol}(w). \tag{2.2}$$

This integral converges absolutely and defines an analytic function of s for $\Re s>1$. We unfold as in the Rankin-Selberg method to get

$$\begin{aligned} I(s) &= \int_0^\infty\int_0^1\int_0^1 \phi(w)^2 y^{2s} e(\langle h,z\rangle)\,\frac{dx_1dx_2dy}{y^3} \\ &= \sum_\nu c(\nu)c(\nu+h)\int_0^\infty K_{ir}(2\pi|\nu|y)K_{ir}(2\pi|\nu+h|y)y^{2s}\frac{dy}{y}. \end{aligned} \tag{2.3}$$

The integral may be evaluated, see [13, 6.576, 4] to give

$$I(s) \;=\; \frac{\Gamma(s+ir)\Gamma(s-ir)\Gamma(s)^2}{8\pi^{2s}\Gamma(2s)} \times \sum_{\nu\neq 0} \frac{c(\nu)c(\nu+h)}{|\nu|^{2s}} \left|\frac{\nu+h}{\nu}\right|^{ir} F(s+ir, s, 2s, 1-|1+h/\nu|^2). \tag{2.4}$$

LEMMA 2.1. *The function*

$$J(s) := \sum_{\nu\neq 0} \frac{c(\nu)c(\nu+h)}{|\nu|^{2s}} \left|\frac{\nu+h}{\nu}\right|^{ir} F(s+ir, s, 2s, 1-|1+h/\nu|^2)$$

is analytic for $\Re s > 11/8$ *and satisfies the bound*

$$J(s) \ll_\epsilon (1+|t|)^{11/2}|h|^{1-2\sigma+2/9+\epsilon}$$

in this region.

Proof. According to (2.4) we have

$$J(s) = \frac{8\pi^{2s}\Gamma(2s)}{\Gamma(s+ir)\Gamma(s-ir)\Gamma(s)^2} I(s). \tag{2.5}$$

Now, while $P_h(\cdot, s)$ is not in $L^2(\Gamma\backslash\mathbf{H}^3)$, it is of moderate growth and the expansion of $I(s)$ via the Parceval formula is easily justified. We remark that $\langle P_h, \phi_0\rangle = 0$. We have

$$I(s) = \sum_{j=1}^{\infty}\langle\phi^2, \phi_j\rangle\langle P_h(\cdot, s), \phi_j\rangle + \frac{1}{4\pi}\int_{-\infty}^{\infty}\langle\phi^2, E(\cdot, 1/2+it)\rangle \times\langle P_h(\cdot, s)E(\cdot, 1/2+it)\rangle dt. \tag{2.6}$$

We proceed to analyze the discrete spectrum sum, since the analysis of the continuous spectrum is similar and, in fact, the bounds towards the Ramanujan conjecture used below (cf (2.9)) are not needed, since the coefficients of the unitary Eisenstein series satisfy the optimal Ramanujan bounds. In fact, one can explicitly write the Fourier expansion of the Eisenstein series for the group $\Gamma = \mathrm{SL}_2(\mathbf{Z}[\sqrt{-1}])$, see [8, 2.17, 2.18]

$$E(w, s) \;=\; 2y^{2s} + \frac{2\pi}{2s-1}\frac{\zeta_K(2s-1)}{\zeta_K(2s)}y^{2-2s} + \frac{4\pi^{2s}}{\Gamma(2s)\zeta_K(2s)}\sum_{\omega\neq 0}|\omega|^{2s-1}\sigma_{1-2s}(\omega)yK_{2s-1}(2\pi|\omega|y)e^{2\pi i\langle\omega,z\rangle},$$

where $\zeta_K(s)$ is the Dedekind zeta function of the number field, ω runs over the Gauss integers $\mathbf{Z}[i]$ and

$$\sigma_s(\omega) = \sum_{(c)|(\omega)} (N(c))^s.$$

We have, see [13, 6.561, 16],

$$\begin{aligned}\langle P_h, \phi_j\rangle &= \int_0^\infty \int_0^1 \int_0^1 \overline{\phi_j(w)} e(\langle h, z\rangle) y^{2s} \frac{dx_1 dx_2 dy}{y^3} \\ &= c_j(h) \int_0^\infty y^{2s+1} K_{ir_j}(2\pi|h|y) \frac{dy}{y^3} \\ &= \frac{c_j(h)}{|h|^{2s-1}} 2^{2s-3}(2\pi)^{1-2s}\Gamma(s-1/2+ir_j/2)\Gamma(s-1/2-ir_j/2), \end{aligned} \tag{2.7}$$

where the Fourier expansion of the L^2-normalized ϕ_j is

$$\phi_j(w) = \sum_{\nu\neq 0} c_j(\nu) y K_{ir_j}(2\pi|\nu|y) e(\langle \nu, z\rangle).$$

We can assume that the orthonormal basis $\phi_j(w)$ consists of Hecke eigenforms. Denote by $\lambda_j(\nu)$ the eigenvalue of the Hecke operator T_ν, $\nu \neq 0$. Proceeding as in [9] one has for every $\epsilon > 0$

$$|c_j(\nu)| \ll_\epsilon \frac{|r_j|^\epsilon |\lambda_j(\nu)|}{|\Gamma(1+ir_j)|}. \tag{2.8}$$

We now invoke the strongest bounds towards the Ramanujan conjectures in this case [16]

$$|\Im r_j| \le 2/9, \quad |\lambda_j(\nu)| \ll_\epsilon |\nu|^{2/9+\epsilon}. \tag{2.9}$$

Actually for $\Gamma = \mathrm{SL}_2(\mathbf{Z}[i])$ it is known that $\Im r_j = 0$, see [7], however, for the more general Γ only (2.9) is known. Combining (2.7), (2.8), (2.9), (2.6) we deduce that $I(s)$ is analytic for $\sigma > 11/18$ and satisfies

$$I(s) \ll_\epsilon |h|^{1-2\sigma+2/9+\epsilon} \sum_{j\neq 0} \frac{|\Gamma(s-1/2+ir_j/2)\Gamma(s-1/2-ir_j/2)|}{|\Gamma(1+ir_j)|} |\langle \phi^2, \phi_j\rangle|. \tag{2.10}$$

Now the main result in [26] concerning the precise exponential decay in r_j of $\langle \phi^2, \phi_j\rangle$ asserts that

$$|\langle \phi^2, \phi_j\rangle| \ll (|r_j|+1)^3 e^{-\pi|r_j|/2}. \tag{2.11}$$

Using this in (2.10) together with Stirling's formula for $|\Gamma(1+ir_j)|$, we are lead to

$$I(s) \ll_\epsilon |h|^{1-2\sigma+2/9+\epsilon} \sum_j (1+|r_j|)^{5/2} \times |\Gamma(s-1/2+ir_j/2)\Gamma(s-1/2-ir_j/2)|. \tag{2.12}$$

Hence

$$J(s) \ll_\epsilon \left| \frac{\Gamma(2s) h^{1-2\sigma+2/9}}{\Gamma(s+ir)\Gamma(s-ir)\Gamma(s)^2} \right| \sum_j (1+|r_j|)^{5/2} |\Gamma(s-1/2+ir_j/2) \times \Gamma(s-1/2-ir_j/2)|. \tag{2.13}$$

For $s = \sigma + it$ we use Stirling's formula and the Weyl law for the distribution of eigenvalues

$$\sum_{|r_j| \le R} 1 \sim c_\Gamma R^3$$

to get

$$J(s) \ll_\epsilon (1+|t|)^{11/2} |h|^{1-2\sigma+2/9+\epsilon}. \tag{2.14}$$

This completes the proof of Lemma 2.1. □

To make effective use of Lemma 2.1 we first transform the Gauss hypergeometric function using

$$F\left(a, b, 2b, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a} F(a, a-b+1/2, b+1/2, z^2),$$

see [13, 9.134, 3]. Hence

$$F(s+ir, s, 2s, 1-|1+h/\nu|^2) = \left(\frac{1+|1+h/\nu|}{2}\right)^{-2s-2ir} \times F\left(s+ir, ir+1/2, s+1/2, \frac{(1-|1+h/\nu|)^2}{(1+|1+h/\nu|)^2}\right).$$

In this form we can expand the hypergeometric functions in its Taylor series uniformly for $1/2 \le \Re s \le 2$ to get the bound

$$\left(\frac{1+|1+h/\nu|}{2}\right)^{-2s-2ir} (1+O(|h/\nu|)) = 1 + O(|h|(|s|+1)/|\nu|).$$

Also $|1+h/\nu| = 1 + O(|h/\nu|)$, hence

$$J(s) = \sum_{\nu \ne 0} \frac{c(\nu)c(\nu+h)}{|\nu|^{2s}} + O\left(\sum_\nu \frac{|c(\nu)c(\nu+h)|h|(|s|+1)}{|\nu|^{2\sigma+1}}\right).$$

We deduce that

$$J(s) = D_\phi(s,h) + G_\phi(s,h)$$

where $G_\phi(s)$ is analytic in $\Re s > 1/2$ and satisfies

$$G_\phi(s) \ll_\epsilon |h|^{11/9+\epsilon}(|t|+1) \tag{2.15}$$

for $\sigma \geq 11/18$. Combining Lemma 2.1 with (2.15) we conclude that $D_\phi(s,h)$ is analytic in $11/18 < \sigma \leq 2$ and satisfies the bound

$$D_\phi(s,h) \ll_\epsilon |h|^{11/9+\epsilon}(|t|+1) + (|t|+1)^{11/2}|h|^{1-2\sigma+2/9+\epsilon}.$$

This completes the proof of Theorem 1.4.

3. Subconvexity

In this section we prove (1.9) after which Theorem 1.1 follows easily. Using the approximate functional equation for $L(s,\phi\otimes\lambda^m)$, see [4], [14], and a dyadic smooth partition of the sums in it we can bound $L(1/2+it,\phi\otimes\lambda^m)$ by at most $O(\log|t+im|)$ sums of the form

$$S_X(t,m) = \sum_\alpha \frac{c(\alpha)}{N(\alpha)^{1/2+it}}\left(\frac{\alpha}{|\alpha|}\right)^m G\left(\frac{|\alpha|}{X}\right) \tag{3.1}$$

where G is a real-valued smooth function supported in, say, the interval $(1/2,2)$ and X is of size at most R, where $|m+it| \leq R$. In fact it is the case X is of size R that is the critical case. Thus (1.9) follows if we can establish that for a fixed smooth $\psi \geq 0$, supported in $(1/2,2)$ with $\psi(1)=1$, and $R^{76/83} \leq H \leq R$, the bound

$$A := \sum_m \int_{-\infty}^{\infty} \psi\left(\frac{|m+it|-R}{H}\right)|S_X(t,m)|^2\,dt \ll_\epsilon (RH)^{1+\epsilon}. \tag{3.2}$$

We begin with the restriction $H^{1/2} \leq R \leq H$ and write

$$\frac{\alpha}{|\alpha|} = e^{2\pi i\theta_\alpha}, \quad \theta_\alpha \in \mathbf{R}/\mathbf{Z},$$

that is $\arg(\alpha) = 2\pi\theta_\alpha$.We have

$$\begin{aligned} A = \sum_{\alpha,\beta} G(|\alpha|/X)G(|\beta|/X)\frac{c(\alpha)\overline{c(\beta)}}{|\alpha\beta|} \\ \times \sum_m \int_{-\infty}^{\infty}\psi\left(\frac{|m+2it|-R}{H}\right)e((\theta_\alpha-\theta_\beta)m + it\log(|\alpha|/|\beta|)/\pi)\,dt. \end{aligned} \tag{3.3}$$

Applying the Poisson summation formula in m we get

$$\sum_m \psi\left(\frac{|m+2it|-R}{H}\right) e(m(\theta_\alpha-\theta_\beta))$$
$$= \sum_\nu \int_{-\infty}^{\infty} \psi\left(\frac{|x+2it|-R}{H}\right) e(x(\theta_\alpha-\theta_\beta)-\nu x)\,dx$$
$$= H\sum_\nu \int_{-\infty}^{\infty} \psi\left(|y+2it/H|-\frac{R}{H}\right) e(H(\theta_\alpha-\theta_\beta-\nu)y)\,dy$$
$$= H\sum_\nu \hat{\psi}_{R,H,t}(H(\theta_\alpha-\theta_\beta-\nu)), \tag{3.4}$$

where

$$\psi_{R,H,t}(y)=\psi(|y+2it/H|-R/H).$$

Repeated integration by parts shows that

$$\hat{\psi}_{R,H,t}(\xi)=\int_{-\infty}^{\infty}\psi\left(\sqrt{y^2+4t^2/H^2}-R/H\right)e(-y\xi)\,dy \ll_N (|\xi|+1)^{-N}$$

for any $N\geq 1$. Hence, if we choose $-1/2\leq\theta_\alpha-\theta_\beta\leq 1/2$, which we can assume, then only the term $\nu=0$ is significant in (3.4). That is

$$\sum_m \psi\left(\frac{|m+it|-R}{H}\right) e(m(\theta_\alpha-\theta_\beta))$$
$$=\int_{-\infty}^{\infty}\psi\left(\frac{|y+2it|-R}{H}\right)e(y(\theta_\alpha-\theta_\beta))\,dy+O_N(H^{-N}).$$

Returning to (3.3) we have

$$A = \sum_{\alpha,\beta} G(|\alpha|/X)G(|\beta|/X)\frac{c(\alpha)\overline{c(\beta)}}{|\alpha\beta|}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi\left(\frac{|y+2it|-R}{H}\right)$$
$$\times\, e\left((\theta_\alpha-\theta_\beta)y+\frac{it}{\pi}\log\frac{|\alpha|}{|\beta|}\right)dy\,dt+\text{small error},$$

which gives

$$A = \frac{H^2}{2}\sum_{\alpha,\beta}G(|\alpha|/X)G(|\beta|/X)\frac{c(\alpha)\overline{c(\beta)}}{|\alpha\beta|}$$
$$\times\hat{\psi}_{R,H}(2\pi(\theta_\alpha-\theta_\beta)H,\, H\log(|\alpha|/|\beta|))+\text{small}, \tag{3.5}$$

where $y/H = y_1$, $2t/H = x_1$ and

$$\psi_{R,H}(x_1, y_1) = \psi(|x_1 + iy_1| - R/H). \tag{3.6}$$

In particular $\psi_{R,H}$ is radial in (x_1, y_1) and hence so is its Fourier transform $\hat{\psi}$:

$$\hat{\psi}_{R,H}(\xi_1, \xi_2) = \hat{\psi}_{R,H}(|\xi|)$$

and so

$$\hat{\psi}_{R,H}(2\pi(\theta_\alpha - \theta_\beta), \log(|\alpha|/|\beta|)) = \hat{\psi}_{R,H}(|\log(\alpha/\beta)|). \tag{3.7}$$

Integration by parts in the definition of the Fourier transform of $\psi_{R,H}$, see (3.6), gives

$$|\hat{\psi}_{R,H}(|\xi|)| \ll_N \frac{R}{H}(1 + |\xi|)^{-N}$$

for any $N \geq 1$ and

$$|\hat{\psi}^{(\nu)}_{R,H}(|\xi|)| \ll \left(\frac{R}{H}\right)^{\nu+1} \tag{3.8}$$

for its ν-th derivative. From (3.7) we can write (3.5) as

$$A = \frac{H^2}{2} \sum_{\alpha,\beta} \frac{G(|\alpha|/X)G(|\beta|/X)c(\alpha)\overline{c(\beta)}}{|\alpha\beta|} \hat{\psi}_{R,H}(H|\log(\alpha/\beta)|) \tag{3.9}$$

with a small error. Hence, if $\delta > 0$ is arbitrarily small, the contribution to (3.9) of the terms with $|\log(\alpha/\beta)| \geq H^{\delta-1}$ is negligible. Also $|\alpha|$ and $|\beta|$ are of size X, so we have

$$A = \frac{H^2}{2} \sum_{|\alpha-\beta| \ll XH^{\delta-1}} G(|\alpha|/X)G(|\beta|/X)\frac{c(\alpha)\overline{c(\beta)}}{|\alpha\beta|} \hat{\psi}_{R,H}(H|\log(\alpha/\beta)|) \tag{3.10}$$

with small error. The contribution to (3.10) of the diagonal $\alpha = \beta$ is

$$\frac{H^2}{2} \sum_{\alpha} G(|\alpha|/X)^2 \frac{|c(\alpha)|^2}{|\alpha|^2} \hat{\psi}_{R,H}(0) \ll_\epsilon H^2 X^\epsilon R/H \ll (RH)^{1+\epsilon}. \tag{3.11}$$

This is in agreement with the required bound (3.2). For the off-diagonal terms we have $X \geq H^{1-\delta}$ (otherwise there are essentially no such terms). We write $\alpha = \beta + h$ with $h \neq 0$. According to (3.10) we have

$$|h| \ll XH^{\delta-1}. \tag{3.12}$$

We estimate the sum for each such h. Let

$$S(h) = \sum_{\beta} \frac{G(|\beta + h|/X)G(|\beta|/X)c(\beta)\overline{c(\beta + h)}}{|\beta(\beta + h)|} \hat{\psi}_{R,H}(H|\log(1 + h/\beta)|).$$

From (3.12) we have

$$\log(1+h/\beta) = h/\beta + O(H^{2\delta-2}).$$

Hence $|\log(1+h/\beta)| = |h|/|\beta| + O(H^{2\delta-2})$ and $|1+h/\beta| = 1 + O(H^{\delta-1})$. Hence

$$\begin{aligned} S(h) \ &= \ \sum_{\beta} G(|\beta|/X + O(H^{\delta-1}))G(|\beta|/X)\frac{c(\beta)\overline{c(\beta+h)}}{|\beta|^2} \\ &\quad \times (1+O(H^{\delta-1}))\hat{\psi}_{R,H}(H|h|/|\beta| + O(H^{2\delta-1})). \end{aligned}$$

Using (3.8) with $\nu = 1$ we get

$$S(h) = \sum_{\beta} G(|\beta|/X)^2 \frac{c(\beta)\overline{c(\beta+h)}}{N(\beta)}\hat{\psi}_{R,H}(H|h|/|\beta|) + O(H^{2\delta-1}R^2/H^2). \tag{3.13}$$

Finally we can write the first sum as follows

$$\begin{aligned} &\sum_{\beta} G(|\beta|/X)^2 \frac{c(\beta)\overline{c(\beta+h)}}{N(\beta)}\hat{\psi}_{R,H}(H|h|/|\beta|) \\ &\quad = \frac{1}{2\pi i}\int_{\Re s=2} X^{2s} D_{\phi}(h, s+1) B_{h,H,X}(s)\, ds, \end{aligned} \tag{3.14}$$

where

$$B_{h,H,X}(s) = \int_0^{\infty} \hat{\psi}_{R,H}(H|h|/(Xy))G(y)^2 y^{2s}\frac{dy}{y}. \tag{3.15}$$

For $-1 \le \sigma \le 2$ we integrate by parts N times in (3.15) and use (3.8) to get

$$B_{h,H,X}(\sigma+it) \ll_{N,\epsilon} (|t|+1)^{-N}(R/H)^{N+1+\epsilon}.$$

Now we shift the contour in (3.14) to $\Re s = -7/18 + \epsilon_1$, where ϵ_1 is arbitrarily small. According to Theorem 1.4 we pick up no poles. Moreover, if we apply the bounds from Theoren 1.4 we obtain

$$\begin{aligned} S(h) &\ll H^{2\delta-3}R^2 + \int_{-\infty}^{\infty} X^{-7/9}(|t|+1)^{-N}(R/H)^{N+1+\epsilon} \\ &\qquad \times (|h|^{11/9+\epsilon}(|t|+1) + (|t|+1)^{11/2})\, dt. \end{aligned} \tag{3.16}$$

Having gained the key cancellation from Theorem 1.4 we now proceed with somewhat crude estimations. We take $N = 7$ in (3.16) and get

$$S(h) \ll H^{2\delta-3}R^2 + X^{-7/9}(R/H)^{8+\epsilon}|h|^{11/9+\epsilon}.$$

We sum on h satisfying (3.12) to see that the off-diagonal contribution to A is

$$\ll H^{2\delta-1}R^2XH^{\delta-1} + H^2(XH^{\delta-1})^{20/9+\epsilon}X^{-7/9}(R/H)^{8+\epsilon}$$
$$\ll R^{3+\epsilon}/H^2 + R^{85/9+\epsilon}/H^{74/9}.$$

This satisfies the desired bound $O((RH)^{1+\epsilon})$ as long as

$$H \geq R^{76/83}. \tag{3.17}$$

This bound for the off-diagonal contributions together with the bound for the diagonal contribution (3.11) proves (3.2). As pointed out at the beginning of this section this implies (1.9). Applying (1.9) with $H = T^{76/83}$ shows that for m fixed

$$\int_{T-1}^{T+1} |L(1/2+it, \phi \otimes \lambda^m)|^2\, dt \ll_\epsilon T^{159/83+\epsilon}. \tag{3.18}$$

A standard argument, see [12], allows us to go from such mean-value estimates to the pointwise estimate

$$|L(1/2+it, \phi \otimes \lambda^m)| \ll_\epsilon T^{159/166+\epsilon}. \tag{3.19}$$

Actually we were very generous (or crude) in the estimations (3.13) and (3.17). One can easily improve the exponents, but, rather than doing so, we remark that this would lead to the removal of the ϵ in (3.19). This then proves Theorem 1.1, part (i). Part (ii) is deduced in the same way.

REFERENCES

[1] BERNSTEIN, J. and REZNIKOV, A., *Analytic continuation of representations and estimates of automorphic forms*. Ann. of Math. (2) *150* (1999) no. 1, 329–352.

[2] COGDELL, J. W., PIATETSKI-SHAPIRO, I. I. and SARNAK, P., *Estimates for Hilbert Modular L-functions and Applications.* In preparation.

[3] CHANDRASEKHARAN, K. and NARASIMHAN, R., *The approximate functional equation for a class of zeta-functions.* Math. Ann. *152* (1963), 30–64.

[4] DUKE, W., *Some problems in multidimensional analytic number theory*. Acta Arith. *52* (1989) no. 3, 203–228.

[5] DUKE, W. FRIEDLANDER, J. and IWANIEC, H., *Bounds for Automorphic L-functions*. Invent. Math. *112* (1993) no. 1, 1–8.

[6] DUKE, W., FRIEDLANDER, J. and IWANIEC, H., *A quadratic divisor problem*. Invent. Math. *115* (1994) no. 2, 209–217.

[7] ELSTRODT, J., GRUNEWALD, F. and MENNICKE, J., *Some remarks on discrete subgroups of $SL_2(C)$*. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) *162* (1987), Avtomorfn. Funkts. i Teor. Chisel. III, 77–106, 189 translation in J. Soviet Math. *46* (1989) no. 2, 1760–1788.

[8] ELSTRODT, J., GRUNEWALD, F., MENNICKE, J., *Eisenstein series on three-dimensional hyperbolic space and imaginary quadratic number fields.* J. Reine Angew. Math. *360* (1985), 160–213.

[9] HOFFSTEIN, J. and LOCKHART, P., *Coefficients of Maass forms and the Siegel zero.* Ann. of Math. (2) *140* (1994) no. 1, 161–181.

[10] GOOD, A., *On various means involving the Fourier coefficients of cusp forms.* Math. Z. *183* (1983) no. 1, 95–129.

[11] GOOD, A., *Beiträge zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind.* J. Number Theory *13* (1981) no. 1, 18–65.

[12] GOOD, A., *The square mean of Dirichlet series associated with cusp forms.* Mathematika *29* (1982) no. 2, 278–295 (1983).

[13] GRADSHTEYN, I. S. and RYZHIK, I. M., *Table of Integrals, Series and Products*, Fifth edition, Alan Jeffrey, ed., Academic Press, San Diego, 1994.

[14] HUXLEY, M. N., *The large sieve inequality for algebraic number fields. II. Means of moments of Hecke zeta-functions.* Proc. London Math. Soc. (3) *21* (1970), 108–128.

[15] IWANIEC, H. and SARNAK, P., *Perspectives on the Analytic Theory of L-functions.* To appear in Geom. Funct. Anal.

[16] KIM, H. H. and SHAHIDI, F., *Cuspidality of Symmetric Powers with Applications.* Preprint, 2000.

[17] KOYAMA, S., *Arithmetic quantum chaos and quantum ergodicity.* Analytic number theory (Kyoto, 1995). Sūrikaisekikenkyūsho Kōkyūroku No. 961, 1996, 53–72.

[18] KOYAMA, S., *Arithmetic in quantum chaos and quantum ergodicity*, Comm. Math. Phys. *215* (2000) no. 2, 477–486.

[19] Luo, W., Rudnick, Z. and SARNAK, P., *On the generalized Ramanujan conjecture for* $\mathrm{GL}(n)$. Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 301–310, Proc. Sympos. Pure Math. *66*, Part 2, Amer. Math. Soc., Providence, RI, 1999.

[20] LUO, W. and SARNAK, P., *Quantum ergodicity of eigenfunctions on* $PSL_2(\boldsymbol{Z})\backslash\boldsymbol{H}^2$. Inst. Hautes Études Sci. Publ. Math. 1995, no. 81, 207–237.

[21] PETRIDIS, Y., *On squares of eigenfunctions for the hyperbolic plane and a new bound on certain L-series*, Internat. Math. Res. Notices, 1995, no. 3, 111–127.

[22] RUDNICK, Z. and SARNAK P., *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. *161* (1994) no. 1, 195–213.

[23] SARNAK, P., *Fourth moments of Grössencharakteren zeta functions*, Comm. Pure Appl. Math. *38* (1985) no. 2, 167–178.

[24] SARNAK, P., *The arithmetic and geometry of some hyperbolic three-manifolds*. Acta Math. *151* (1983) no. 3–4, 253–295.

[25] SARNAK, P., *Estimates for Rankin-Selberg L-functions and quantum unique ergodicity*. Preprint, 2000.

[26] SARNAK, P., *Integrals of products of eigenfunctions*. Internat. Math. Res. Notices, 1994, no. 6, 251–260.

[27] SELBERG, A., *On the estimation of Fourier Coefficients of Modular Forms*. 1965 Proc. Sympos. Pure Math., vol. VIII, pp. 1–15, Amer. Math. Soc., Providence, R. I.

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