

# Hyperbolic lattice-point counting and modular symbols

par YIANNIS N. PETRIDIS et MORTEN S. RISAGER

RÉSUMÉ. Soit un sous-groupe  $\Gamma$  de  $\mathrm{SL}_2(\mathbb{R})$  cocompact et soit  $\alpha$  une forme harmonique réelle (non nulle). Nous étudions le comportement asymptotique de la fonction comptant des points du réseau hyperbolique  $\Gamma$  sous hypothèses imposées par des symboles modulaires  $\langle \gamma, \alpha \rangle$ . Nous montrons que les valeurs normalisées des symboles modulaires, ordonnées selon ce comptage possèdent une répartition gaussienne.

ABSTRACT. For a cocompact group  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{R})$  we fix a real non-zero harmonic 1-form  $\alpha$ . We study the asymptotics of the hyperbolic lattice-counting problem for  $\Gamma$  under restrictions imposed by the modular symbols  $\langle \gamma, \alpha \rangle$ . We prove that the normalized values of the modular symbols, when ordered according to this counting, have a Gaussian distribution.

## 1. Introduction

For a compact hyperbolic surface Huber [8] found the asymptotics in the hyperbolic lattice counting problem (see also Delsarte [3]). He also studied the distribution of the lengths of the closed geodesics. More precisely, let  $X$  be a hyperbolic surface. Let  $\Gamma = \pi_1(X)$  be its fundamental group. Let  $z$  and  $w$  be two points in the hyperbolic upper half-plane  $\mathbb{H}$ , the universal cover of  $X$ . Define the hyperbolic lattice counting function

$$N(z, w, x) = \{\gamma \in \Gamma, r(\gamma z, w) \leq x\},$$

where  $r(z_1, z_2)$  denotes the hyperbolic distance between  $z_1$  and  $z_2$  in  $\mathbb{H}$ . Huber [8] proved that

$$(1.1) \quad N(z, w, x) \sim \frac{\pi}{\mathrm{vol}(X)} e^x.$$

One may also obtain error terms. We do not investigate these but refer to [9, Theorem 12.1], [13, 6, 16, 2]. To every conjugacy class  $\{\gamma\}$  of  $\Gamma$

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corresponds a unique closed oriented geodesic of length  $l(\gamma)$ . Let  $\pi(x) = \#\{\{\gamma\} | \gamma \text{ prime}, l(\gamma) \leq x\}$ . The prime number theorem for closed geodesics states that

$$(1.2) \quad \pi(x) \sim e^x/x,$$

as  $x \rightarrow \infty$  and can be proved using the Selberg trace formula [9]. Generalizations of the hyperbolic lattice counting problem to infinite volume groups have been obtained by Lax and Phillips [11]. Recent applications of the hyperbolic lattice counting problem in this setting include sieving for almost primes among e.g. sums of squares, see [1] and [10].

On the other hand we have the Poincaré pairing between homology and cohomology:

$$H_1(X, \mathbb{R}) \times H_{\text{dR}}^1(X, \mathbb{R}) \rightarrow \mathbb{R}$$

and a projection  $\phi : \Gamma \rightarrow H_1(X, \mathbb{Z})$ . Let  $\langle \cdot, \cdot \rangle$  be the composition of the two maps:

$$\langle \gamma, \alpha \rangle = \int_{\phi(\gamma)} \alpha.$$

We fix a real 1-form  $\alpha$ . The numbers  $\langle \gamma, \alpha \rangle$  will be called modular symbols, even in the compact setting. See [4, 5] for their arithmetic significance in the cofinite case of congruence subgroups. In previous work the authors [14, 15, 18] have studied the distribution of the normalized values of the Poincaré pairing for compact and finite volume hyperbolic surfaces. In all articles we found as limiting distribution the normal Gaussian distribution. In all cases considered we were forced to group together elements in the same coset for appropriate coset decomposition of  $\Gamma$ . In [14] we ordered the group elements of a cofinite subgroup  $\Gamma$  by realizing  $\Gamma = \pi_1(X)$  as a discrete subgroup of  $\text{SL}_2(\mathbb{R})$ , setting  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and ordering the cosets  $\gamma \in \Gamma_\infty \backslash \Gamma$  according to  $c^2 + d^2$ . To find the asymptotics of the powers of the modular symbols in [14] we used Eisenstein series twisted by modular symbols [4, 5, 7]. These series are sums over the cosets of  $\Gamma_\infty \backslash \Gamma$ . In [18] the matrix elements are ordered according to  $(a^2 + b^2)(c^2 + d^2)$ , and grouped together in the same coset of a hyperbolic subgroup  $\Gamma_{\gamma_1}$ . To study the appropriate moments Risager defined in [18] and used hyperbolic Eisenstein series twisted by modular symbols.

In [15] we considered the distribution of the values of the Poincaré pairing, when we order the conjugacy classes of  $\Gamma$  according to the lengths  $l(\gamma)$ . We proved that, if

$$[[\gamma, \alpha]] = \sqrt{\frac{\text{vol}(X)}{2 \|\alpha\|^2 l(\gamma)}} \int_{\phi(\gamma)} \alpha,$$

then

$$\frac{\#\{\gamma \in \pi_1(X) \mid [[\gamma, \alpha]] \in [a, b], l(\gamma) \leq x\}}{\#\{\gamma \in \pi_1(X) \mid l(\gamma) \leq x\}} \rightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt$$

as  $x \rightarrow \infty$ . To prove this theorem we used the Selberg trace formula, via the Selberg zeta function and its derivatives in character varieties.

In this paper we show how we can calculate the moments of the modular symbols in another ordering of the group elements  $\gamma \in \Gamma$  *but without using any coset decomposition*. This is the same ordering used by Huber in the hyperbolic lattice counting problem. Moreover, we have two parameters to vary: the points  $z$  and  $w$  in the upper half-plane  $\mathbb{H}$ . The results are as follows.

**Theorem 1.1.** *Fix  $\alpha$  a nonzero real harmonic 1-form on  $X$  of norm  $\|\alpha\|$  and two points  $z, w \in \mathbb{H}$ . Let*

$$[\gamma, \alpha] = \sqrt{\frac{\text{vol}(X)}{2\|\alpha\|^2 r(\gamma z, w)}} \int_{\phi(\gamma)} \alpha.$$

Then

$$\frac{\#\{\gamma \in \Gamma \mid [\gamma, \alpha] \in [a, b], r(\gamma z, w) \leq x\}}{\#\{\gamma \in \Gamma \mid r(\gamma z, w) \leq x\}} \rightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt$$

as  $x \rightarrow \infty$ .

The normalization depends on  $z, w$  (but this is suppressed in the notation). However, the limiting distribution is universal. Along the way of proving Theorem 1.1 we introduce a new type of Dirichlet series twisted by modular symbols. Their study and analytic properties are of independent interest. Here are the results:

**Theorem 1.2.** *Let*

$$G^{(n)}(z, w, s, 0) = \sum_{\gamma \in \Gamma} \frac{(-i)^n \langle \gamma, \alpha \rangle^n}{(\cosh(r(\gamma z, w)))^s}.$$

*For each  $n \in \mathbf{N} \cup \{0\}$  this series converges absolutely for  $\Re(s) > 1$ . It admits analytic continuation in the whole complex plane with poles included in the set*

$$P = \{-2m + s_j \mid m \in \mathbf{N} \cup \{0\}, s_j(1 - s_j) \in \text{Spec}(\Delta)\}.$$

*The order of the pole at  $s = 1$  is less than or equal to  $[n/2] + 1$ , with order exactly  $[n/2] + 1$ , for  $n$  even. The poles at the other points of  $P$  are of order less than equal to  $n + 1$ . The functions  $G^{(n)}(z, w, s, 0)$  are also bounded polynomially on vertical lines when  $\Re(s) > 1/2$ .*

Using Theorem 1.2 and the fact that we can find explicit expressions for the leading term in the polar expansions when  $n$  is even we can calculate all the asymptotic moments of the (normalized) modular symbols. Theorem 1.1 then follows, see Section 5.

In principle we can find also the leading term in the polar expansion when  $n$  is odd. This gives asymptotic moments of the non-normalized modular symbols. As an example (see Remark 5) the first moment is given by the following:

$$(1.3) \quad \sum_{\substack{\gamma \in \Gamma \\ r(\gamma z, w) \leq x}} \langle \gamma, \alpha \rangle = - \int_w^z \alpha \frac{\pi}{\text{vol}(X)} e^x.$$

## 2. Dirichlet series

Let  $\Gamma$  be a discrete subgroup of  $\text{PSL}_2(\mathbb{R})$  with compact quotient  $X = \Gamma \backslash \mathbb{H}$ . Here  $\mathbb{H}$  is the upper half-plane. Let  $\chi$  be a unitary character of  $\Gamma$ . For our purposes we need in fact a family of characters

$$\chi(\cdot, \epsilon) : \Gamma \rightarrow S^1 \\ \gamma \mapsto \exp(-i\epsilon \langle \gamma, \alpha \rangle).$$

We have the following lemma:

**Lemma 2.1.**

$$\langle \gamma, \alpha \rangle = \int_{\phi(\gamma)} \alpha = O(\|\alpha\|_\infty r(\gamma z, w)),$$

where the implied constant depends on  $z, w$ .

*Proof.* By well-known properties of line integrals

$$\left| \int_{\phi(\gamma)} \alpha \right| \leq \|\alpha\|_\infty l(\gamma).$$

Since  $l(\gamma) = \inf_{x \in X} r(x, \gamma x)$ , we have

$$\left| \int_{\phi(\gamma)} \alpha \right| \leq \|\alpha\|_\infty r(z, \gamma z) \\ \leq \|\alpha\|_\infty (r(z, w) + r(w, \gamma z)) \leq \|\alpha\|_\infty (D + r(\gamma z, w)),$$

where  $D$  is the diameter of  $X$ . □

Huber [8] introduced the series

$$G(z, w, s) = \sum_{\gamma \in \Gamma} \frac{1}{(\cosh r(\gamma z, w))^s},$$

which converges absolutely for  $\Re(s) > 1$ . Moreover, see [8, Satz 1], he proved that for fixed  $w$  it is a continuous and automorphic function of  $z$ . It is twice differentiable in  $z$ . We introduce the series

$$(2.1) \quad G(z, w, s, \epsilon) = \sum_{\gamma \in \Gamma} \frac{\chi(\gamma, \epsilon)}{(\cosh r(\gamma z, w))^s}$$

and its  $n$ -th derivative with respect to  $\epsilon$ , evaluated at  $\epsilon = 0$ , namely

$$(2.2) \quad G^{(n)}(z, w, s, 0) = \sum_{\gamma \in \Gamma} \frac{(-i)^n \langle \gamma, \alpha \rangle^n}{(\cosh(r(\gamma z, w)))^s}.$$

The series  $G(z, w, s, \epsilon)$  converges absolutely for  $\sigma = \Re(s) > 1$  by comparison with  $G(z, w, \sigma)$ , as the character  $\chi(\cdot, \epsilon)$  is unitary. The series  $G^{(n)}(z, w, s, 0)$  converges absolutely in the same region, using Lemma 2.1. Since  $G(z, w, s, \epsilon)$  are automorphisms of  $(\cosh r)^{-s}$ , they satisfy the shifted eigenvalue equation

$$(2.3) \quad \Delta G(z, w, s, \epsilon) + s(1-s)G(z, w, s, \epsilon) = -s(s+1)G(z, w, s+2, \epsilon).$$

Here

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace operator for the upper half-plane, acting in the  $z$  variable. The equation (2.3) appeared already in [8]. The series (2.1), (2.2) are analogous to the series

$$E(s, \epsilon) = \sum_{\{\gamma_0\}} \frac{\chi(\gamma_0, \epsilon) \ln(N(\gamma_0))}{N(\gamma_0)^s},$$

and its derivatives at  $\epsilon = 0$

$$E^{(n)}(s, 0) = \sum_{\{\gamma_0\}} \frac{(-i)^n \langle \gamma_0, \alpha \rangle^n \ln(N(\gamma_0))}{N(\gamma_0)^s},$$

studied in our previous work [15]. Here the sums are over only primitive conjugacy classes  $\{\gamma_0\}$  of  $\Gamma$ .

We need to show that (2.2) admits meromorphic continuation to the whole complex plane, find the pole order at  $s = 1$  (Section 3) and find the leading term in the Laurent expansion of  $G^{(n)}(z, w, s, 0)$  (Theorem 3.1). We also need to control these derivatives as functions of  $s$  on vertical lines for  $\Re(s) > 1/2$  to be able to use contour integration techniques from analytic number theory (Section 4). Theorem 1.2 summarizes (less precisely) these results.

We consider the space

$$L^2(\Gamma \backslash \mathbb{H}, \bar{\chi}(\cdot, \epsilon))$$

of  $(\Gamma, \bar{\chi}(\cdot, \epsilon))$ -automorphic functions, i.e. functions  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfying

$$f(\gamma z) = \bar{\chi}(\gamma, \epsilon) f(z),$$

and

$$\int_{\Gamma \backslash \mathbb{H}} |f(z)|^2 d\mu(z) < \infty.$$

Here  $d\mu(z) = y^{-2} dx dy$  is the invariant Riemannian measure on  $\mathbb{H}$  derived from the Poincaré metric  $ds^2 = y^{-2}(dx^2 + dy^2)$ . We shall denote by  $\|\cdot\|$  the usual norm in the Hilbert space  $L^2(\Gamma \backslash \mathbb{H}, \bar{\chi}(\cdot, \epsilon))$ . It is easy to see that  $G(z, w, s, \epsilon) \in L^2(\Gamma \backslash \mathbb{H}, \bar{\chi}(\cdot, \epsilon))$ . The automorphic Laplacian  $\tilde{L}(\epsilon)$  is the closure of the operator  $\Delta$  acting on smooth functions in  $L^2(\Gamma \backslash \mathbb{H}, \bar{\chi}(\cdot, \epsilon))$ . The spectrum of  $\tilde{L}(\epsilon)$  is discrete and  $-\tilde{L}(\epsilon)$  is nonnegative with eigenvalues

$$0 \leq \lambda_0(\epsilon) \leq \lambda_1(\epsilon) \leq \dots$$

As usual we set  $\lambda_j(\epsilon) = s_j(\epsilon)(1 - s_j(\epsilon))$ . The first eigenvalue is zero if and only if  $\epsilon = 0$  and in this case it is a simple eigenvalue.

The resolvent  $\tilde{R}(s, \epsilon) = (\tilde{L}(\epsilon) + s(1 - s))^{-1}$ , defined off the spectrum of  $\tilde{L}(\epsilon)$ , is a Hilbert-Schmidt operator on  $L^2(\Gamma \backslash \mathbb{H}, \bar{\chi}(\cdot, \epsilon))$ . It is a holomorphic (in  $s$ ) family of operators, and the operator norm of the resolvent is bounded as follows:

$$(2.4) \quad \left\| \tilde{R}(s, \epsilon) \right\|_{\infty} \leq \frac{1}{\text{dist}(s(s-1), \text{spec}(\tilde{L}(\epsilon)))} \leq \frac{1}{|t|(2\sigma-1)},$$

where  $s = \sigma + it$ ,  $\sigma > 1/2$ .

We fix  $z_0 \in \mathbb{H}$  and introduce unitary operators (see ([17])

$$(2.5) \quad \begin{aligned} U(\epsilon) : L^2(\Gamma \backslash \mathbb{H}) &\rightarrow L^2(\Gamma \backslash \mathbb{H}, \bar{\chi}(\cdot, \epsilon)) \\ f &\mapsto \exp\left(i\epsilon \int_{z_0}^z \alpha\right) f(z). \end{aligned}$$

We then define

$$(2.6) \quad L(\epsilon) = U^{-1}(\epsilon) \tilde{L}(\epsilon) U(\epsilon),$$

$$(2.7) \quad R(s, \epsilon) = U^{-1}(\epsilon) \tilde{R}(s, \epsilon) U(\epsilon).$$

This ensures that  $L(\epsilon)$  and  $R(s, \epsilon)$  act on the fixed space  $L^2(\Gamma \backslash \mathbb{H})$ . It is then easy to verify that

$$(2.8) \quad \begin{aligned} L(\epsilon)h &= \Delta h + 2i\epsilon \langle dh, \alpha \rangle - i\epsilon \delta(\alpha)h - \epsilon^2 \langle \alpha, \alpha \rangle h, \\ (L(\epsilon) + s(1-s))R(s, \epsilon) &= R(s, \epsilon)(L(\epsilon) + s(1-s)) = I. \end{aligned}$$

Here

$$\begin{aligned} \langle f_1 dz + f_2 d\bar{z}, g_1 dz + g_2 d\bar{z} \rangle &= 2y^2(f_1 \bar{g}_1 + f_2 g_2) \\ \delta(pdx + qdy) &= -y^2(p_x + q_y). \end{aligned}$$

We notice that  $\delta(\alpha) = 0$ , since  $\alpha$  is harmonic. We notice also that

$$(2.9) \quad L^{(1)}(\epsilon)h = 2i \langle dh, \alpha \rangle - 2\epsilon \langle \alpha, \alpha \rangle h,$$

$$(2.10) \quad L^{(2)}(\epsilon)h = -2 \langle \alpha, \alpha \rangle h,$$

$$(2.11) \quad L^{(i)}(\epsilon)h = 0, \quad \text{when } i \geq 3.$$

(We use superscript  $(n)$  to denote the  $n$ -th derivative in  $\epsilon$ .) Using Lemma 2.1 we see that

$$\int_{z_0}^{\gamma z} \alpha = O(r(\gamma z, w)).$$

In order to investigate the behavior of  $G^{(n)}(z, w, 0)$  we define the auxiliary series

$$D(z, w, s, \epsilon) = U(-\epsilon)G(z, w, s, \epsilon) = \sum_{\gamma \in \Gamma} \frac{\exp(-i\epsilon \int_{z_0}^{\gamma z} \alpha)}{(\cosh r(\gamma z, w))^s} \in L^2(\Gamma \backslash \mathbb{H}).$$

This series converges absolutely for  $\Re(s) > 1$ . Using (2.3) we see that

$$(2.12) \quad (L(\epsilon) + s(1-s))D(z, w, s, \epsilon) = -s(s+1)D(z, w, s+2, \epsilon).$$

Since  $D(z, w, s, \epsilon)$  is square integrable on  $\Gamma \backslash \mathbb{H}$ , it follows that

$$(2.13) \quad D(z, w, s, \epsilon) = -s(s+1)R(s, \epsilon)D(z, w, s+2, \epsilon),$$

and this implies the meromorphic continuation of  $D(z, w, s, \epsilon)$  in the whole complex plane with (at most) poles of first order at the spectral points  $s_j$  and  $-2n + s_j$ ,  $n = 0, 1, \dots$ . This is done in the standard way of extending the domain of analyticity on subsequent vertical strips of width 2 to the left, using the well-known meromorphic continuation of the resolvent (with poles of order 1 at the points  $s_j$ ). The series

$$D^{(n)}(z, w, s, 0) = \sum_{\gamma \in \Gamma} \frac{(-i)^n (\int_{z_0}^{\gamma z} \alpha)^n}{(\cosh r(\gamma z, w))^s}$$

converges absolutely for  $\Re(s) > 1$ . By differentiating (2.12)  $n$ -times and plugging  $\epsilon = 0$  we get

$$(2.14) \quad \begin{aligned} & (\Delta + s(1-s))D^{(n)}(z, w, s, 0) \\ & + \binom{n}{1} L^{(1)} D^{(n-1)}(z, w, s, 0) + \binom{n}{2} L^{(2)} D^{(n-2)}(z, w, s, 0) \\ & = -s(s+1)D^{(n)}(z, w, s+2, 0). \end{aligned}$$

This implies for  $n \geq 2$

$$(2.15) \quad \begin{aligned} D^{(n)}(z, w, s, 0) &= -s(s+1)R(s, 0)D^{(n)}(z, w, s+2, 0) \\ &\quad - \binom{n}{1} R(s, 0)L^{(1)}D^{(n-1)}(z, w, s, 0) \\ &\quad - \binom{n}{2} R(s, 0)L^{(2)}D^{(n-2)}(z, w, s, 0), \end{aligned}$$

while

$$(2.16) \quad \begin{aligned} D^{(1)}(z, w, s, 0) &= -s(s+1)R(s, 0)D^{(1)}(z, w, s+2, 0) \\ &\quad - R(s, 0)L^{(1)}D(z, w, s, 0). \end{aligned}$$

These equations prove the meromorphic continuation of  $D^{(n)}(z, w, s, 0)$  in the whole complex plane with poles at  $-2n + s_j$ ,  $n = 0, 1, \dots$  using induction on  $n$  and continuation in vertical strips from  $\Re(s) > 1$  to  $-1 < \Re(s) \leq 1$ , and subsequently to  $-3 < \Re(s) \leq -1$ , etc. It also follows that  $D^{(n)}(z, w, s, 0)$  is analytic for  $\Re(s) > 1$ , and that at points, where it is analytic, it is square integrable. We also see that

$$(2.17) \quad G^{(n)}(z, w, s, 0) = \sum_{j=0}^n \binom{n}{j} \left( i \int_{z_0}^z \alpha \right)^j D^{(n-j)}(z, w, s, 0).$$

### 3. The pole at $s = 1$ .

In this section we identify the pole order and the leading term of the functions  $G^{(n)}(z, w, s, 0)$  at  $s = 1$ . We keep  $w \in \mathbb{H}$  fixed. In  $G^{(n)}(z, w, s, \epsilon)$ ,  $R(s, \epsilon)$  and  $L^{(n)}(\epsilon)$  we shall often omit 0 from the notation when we set  $\epsilon = 0$ . We note that  $G(z, w, s)$  has a first order pole with residue  $2\pi / \text{vol}(\Gamma \backslash \mathbb{H})$ , as shown by Huber [8, Satz 2, p. 5]. We derive it below only for completeness.

We recall that close to  $s = 1$  the resolvent has a Laurent expansion

$$(3.1) \quad R(s) = \sum_{i=-1}^{\infty} R_i(s-1)^i, \quad R_{-1} = -P_0$$

and that  $R(s) - R_{-1}(s-1)^{-1}$  is holomorphic in  $\Re(s) > h$ . Here  $h = \Re(s_1)$ ,  $s_1(1-s_1) = \lambda_1$  is the first non-zero eigenvalue, and

$$P_0 f = \langle f, \text{vol}(\Gamma \backslash \mathbb{H})^{-1/2} \rangle \text{vol}(\Gamma \backslash \mathbb{H})^{-1/2}$$

is the projection of  $f$  to the zero eigenspace.



For  $n = 0$  it follows from (2.13) that the residue of  $D(z, w, s)$  at  $s = 1$  equals

$$\begin{aligned} \frac{-1(2)(-1)}{\text{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} D(z, w, 3) d\mu(z) &= \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} \int_0^\pi \int_0^\infty \frac{2 \sinh r}{(\cosh r)^3} dr d\phi \\ &= \frac{2\pi}{\text{vol}(\Gamma \backslash \mathbb{H})}. \end{aligned}$$

Here we have unfolded the integral  $\int_{\Gamma \backslash \mathbb{H}}$  to the whole upper half-plane and integrated in polar coordinates centered at  $w$ , where the invariant hyperbolic measure is  $2 \sinh r dr d\phi$ ,  $r \geq 0$ ,  $\phi \in [0, \pi]$ , see [9, p. 16].

Close to  $s = 1$  we also have the Laurent expansion of  $D(z, s, w)$

$$\frac{2\pi / \text{vol}(\Gamma \backslash \mathbb{H})}{s - 1} + D_0(z, w) + D_1(z, w)(s - 1) + \dots$$

The crucial observation is that

$$(3.2) \quad L^{(1)}P_0 = 0, \quad P_0L^{(1)} \subseteq 0.$$

The first equality follows from the fact that  $L^{(1)}$  is a differentiation operator while  $P_0$  projects to the constants. The second equality follows from the first by using the fact that both operators are self-adjoint.

For  $n = 1$  we look at (2.16). We notice that  $R(s)L^{(1)}D(z, w, s)$  is regular at  $s = 1$ : at  $s = 1$  its singular terms are

$$\frac{-P_0L^{(1)}(2\pi / \text{vol}(\Gamma \backslash \mathbb{H}))}{(s - 1)^2} + \frac{-P_0L^{(1)}D_0(z, w) + R_0L^{(1)}(2\pi / \text{vol}(\Gamma \backslash \mathbb{H}))}{s - 1} = 0.$$

We can now prove:

**Theorem 3.1.** *For  $n \geq 0$ ,  $D^{(n)}(z, w, s)$  and  $G^{(n)}(z, w, s)$  have a pole at  $s = 1$  of order at most  $[n/2] + 1$ . If  $n = 2m$  the pole is of order  $m + 1$  and the  $(m + 1)$ -term in the expansion around  $s = 1$  is*

$$(-1)^m \frac{(2\pi)(2m)! \|\alpha\|^{2m}}{\text{vol}(\Gamma \backslash \mathbb{H})^{m+1}} \frac{1}{(s - 1)^{m+1}}.$$

*Proof.* The claim has been shown above for  $n = 0$  and  $n = 1$ . We assume that the order of the pole of  $D^{(n-1)}(z, w, s)$  and  $D^{(n-2)}(z, w, s)$  at  $s = 1$  are less than or equal to  $[(n-1)/2] + 1$  and  $[(n-2)/2] + 1$  respectively. By (2.10), (2.15), and (3.2) the order of the pole of  $D^{(n)}(z, w, s)$  at  $s = 1$  is less than or equal to  $\max([(n-1)/2] + 1, [(n-2)/2] + 1 + 1) = [n/2] + 1$ .

Assume that for  $n = 2(m-1)$  we have proved the claim for the order of the pole and leading singularity. By (2.10), (2.15) and (3.2) the coefficient of  $(s-1)^{-m-1}$  in  $D^{(2m)}(z, w, s)$  is

$$-\binom{2m}{2} \int_{\Gamma \backslash \mathbb{H}} \frac{-1}{\text{vol}(\Gamma \backslash \mathbb{H})} (-2\langle \alpha, \alpha \rangle) \frac{(-1)^{m-1} (2\pi)(2m-2)! \|\alpha\|^{2m-2}}{\text{vol}(\Gamma \backslash \mathbb{H})^m} d\mu(z),$$

and this is

$$\frac{(2m)!}{2(2m-2)!} 2 \|\alpha\|^2 \frac{(-1)^m (2\pi)(2m-2)! \|\alpha\|^{2m-2}}{\text{vol}(\Gamma \backslash \mathbb{H})^{m+1}} = \frac{(-1)^m (2\pi)(2m)! \|\alpha\|^{2m}}{\text{vol}(\Gamma \backslash \mathbb{H})^{m+1}}.$$

The claim for  $G^{(n)}(z, w, s)$  follows from (2.17) and the result for  $D^{(n)}(z, w, s)$ .  $\square$

#### 4. Growth on vertical lines

**Lemma 4.1.** *For  $\sigma = \Re(s) \geq \sigma_0 > 1$ , we have  $G^{(n)}(z, w, \sigma + it) = O(1)$ .*

This follows from the absolute convergence of the series in this region.

**Lemma 4.2.** *For  $\sigma = \Re(s) > 1/2$  we have  $G(z, w, s) = O(|t|^{6(1-\sigma)+\epsilon})$ .*

*Proof.* By (2.3) and (2.4) we get

$$\|G(z, w, s)\|_2 \leq \frac{|s(s+1)|}{|t|(2\sigma-1)} \|G(z, w, s+2)\|_2 \ll |t|$$

and

$$\|\Delta G(z, w, s)\|_2 \ll |t|^3.$$

By the Sobolev embedding theorem, which provides the bound

$$(4.1) \quad \|f\|_\infty \ll \|f\|_2 + \|\Delta f\|_2,$$

we get

$$\|G(z, w, s)\|_\infty = O(|t|^3).$$

We apply the Phragmén-Lindelöf principle in the strip  $1/2 + \epsilon \leq \sigma \leq 1 + \epsilon$  to get the result.  $\square$

The right-hand side in (4.1) is the  $H^2$ -norm of  $f$ . Similarly one can define the  $H^1$ -norm using any first order differential operator and we have  $\|f\|_{H^1} \ll \|f\|_{H^2}$ .

**Lemma 4.3.** *For  $\Re(s) > 1/2$  we have*

$$D^{(n)}(z, w, s) = O(|t|^{6(n+1)(1-\sigma)+\epsilon}), \quad G^{(n)}(z, w, s) = O(|t|^{6(n+1)(1-\sigma)+\epsilon}).$$

*Proof.* Since the series of  $D^{(n)}(z, w, s)$  and  $G^{(n)}(z, w, s)$  converge absolutely for  $\Re(s) > 1$ , we get that they are  $O(1)$  in this region. To prove the lemma one works as in Lemma 4.2, by first proving

$$(4.2) \quad D^{(n)}(z, w, s) = O(|t|^{3(n+1)}), \quad G^{(n)}(z, w, s) = O(|t|^{3(n+1)})$$

for  $\Re(s) > 1/2$  and then using the Phragmén-Lindelöf principle. We prove inductively the estimates

$$(4.3) \quad \left\| D^{(n)}(z, w, s) \right\|_{\infty} = O(|t|^{3(n+1)}),$$

$$(4.4) \quad \left\| L^{(1)} D^{(n-1)}(z, w, \sigma + it) \right\|_2 = O(|t|^{3n+2}),$$

$$(4.5) \quad \left\| L^{(2)} D^{(n-2)}(z, w, s) \right\|_2 = O(|t|^{3(n-1)}).$$

Assume that the result is true for  $m \leq n-1$ . We prove it for  $n$ . Using (4.3) for  $n-2$  we have

$$\left\| D^{(n-2)}(z, w, s) \right\|_{\infty} = O(|t|^{3(n-1)}) \implies \left\| L^{(2)} D^{(n-2)}(z, w, s) \right\|_2 = O(|t|^{3(n-1)}),$$

as  $L^{(2)}$  is a multiplication operator on a compact set. This proves (4.5) for  $n$ . We have (for appropriate constants  $c_i$ )

$$\begin{aligned} \left\| L^{(1)} D^{(n-1)}(z, w, s) \right\|_2 &\leq c_1 \left\| D^{(n-1)}(z, w, s) \right\|_{H^1} \\ &\leq c_2 \left\| D^{(n-1)}(z, w, s) \right\|_{H^2} \\ &\leq c_3 \left( \left\| D^{(n-1)}(z, w, s) \right\|_2 + \left\| \Delta D^{(n-1)}(z, w, s) \right\|_2 \right) \\ &\ll |t|^{3n} + |t|^2 \left\| D^{(n-1)} \right\|_2 + \left\| L^{(1)} D^{(n-2)} \right\|_2 + \left\| L^{(2)} D^{(n-3)} \right\|_2 \\ &\ll |t|^{3n} + |t|^{3n+2} + |t|^{3n-1} + |t|^{3(n-2)} = O(|t|^{3n+2}). \end{aligned}$$

Here we have used the inductive hypothesis and (2.14). This proves (4.4) for  $n$ . We have by (2.15) and (2.4) and the inductive hypothesis

$$(4.6) \quad \left\| D^{(n)}(z, w, s) \right\|_2 \ll \frac{1}{|t|} (|t|^{3n+2} + |t|^{3(n-1)}) = O(|t|^{3n+1}).$$

Using (2.14) and (4.6) and the previous results we get

$$(4.7) \quad \begin{aligned} \left\| \Delta D^{(n)}(z, w, s) \right\|_2 &\ll |t|^2 \left\| D^{(n)}(z, w, s) \right\|_2 + \left\| L^{(1)} D^{(n-1)} \right\|_2 + \left\| L^{(2)} D^{(n-2)} \right\|_2 \\ &\ll |t|^{3n+3}. \end{aligned}$$

Equations (4.6) and (4.7) prove (4.3) for  $n$ .  $\square$

## 5. Calculating the moments

We are now ready to prove Theorem 1.1. The proof uses the method of asymptotic moments precisely as in [14, 18]. From Theorem 3.1, Lemma 4.3 and Lemma 2.1 we may conclude, using a more or less standard contour

integration argument (see [14, 18] for details), that, as  $T \rightarrow \infty$ ,

(5.1)

$$\begin{aligned} & \sum_{\substack{\gamma \in \Gamma \\ \cosh r(\gamma z, w) \leq T}} \langle \gamma, \alpha \rangle^n \\ &= \begin{cases} \frac{(2\pi)(2m)! \|\alpha\|^{2m}}{m! \operatorname{vol}(\Gamma \backslash \mathbb{H})^{m+1}} T(\log T)^m + O(T(\log T)^{m-1}), & n = 2m, \\ O(T(\log T)^m), & n = 2m + 1. \end{cases} \end{aligned}$$

Setting  $T = \cosh(x)$  and using  $\cosh(x) = e^x/2 + O(1)$  as  $x \rightarrow \infty$  we can formulate this as follows:

(5.2)

$$\sum_{\substack{\gamma \in \Gamma \\ r(\gamma z, w) \leq x}} \langle \gamma, \alpha \rangle^n = \begin{cases} \frac{\pi(2m)! \|\alpha\|^{2m}}{m! \operatorname{vol}(\Gamma \backslash \mathbb{H})^{m+1}} e^x x^m + O(e^x x^{m-1}), & n = 2m, \\ O(e^x x^m), & n = 2m + 1. \end{cases}$$

as  $x \rightarrow \infty$ . Let now

$$(5.3) \quad [\gamma, \alpha] = \sqrt{\frac{\operatorname{vol}(\Gamma \backslash \mathbb{H})}{2r(\gamma z, w) \|\alpha\|^2}} \langle \gamma, \alpha \rangle.$$

We then define the random variable  $Y_x$  with probability measure

$$(5.4) \quad P(Y_x \in [a, b]) = \frac{\#\{\gamma \in \Gamma | r(\gamma z, w) \leq x, [\gamma, \alpha] \in [a, b]\}}{\#\{\gamma \in \Gamma | r(\gamma z, w) \leq x\}}.$$

We want to calculate the asymptotic moments of these, i.e. find the limit of

$$(5.5) \quad M_n(Y_x) = \frac{1}{\#\{\gamma \in \Gamma | r(\gamma z, w) \leq x\}} \sum_{\substack{\gamma \in \Gamma \\ r(\gamma z, w) \leq x}} [\gamma, \alpha]^n$$

as  $x \rightarrow \infty$ . We note that by (1.1) or, alternatively, (5.1), the denominator is asymptotically  $\pi e^x / \operatorname{vol}(\Gamma \backslash \mathbb{H})$ . By partial summation we have

$$\begin{aligned} \sum_{r(\gamma z, w) \leq x} [\gamma, \alpha]^n &= \frac{\operatorname{vol}(\Gamma \backslash \mathbb{H})^{n/2}}{\|\alpha\|^n 2^{n/2}} \sum_{r(\gamma z, w) \leq x} \langle \gamma, \alpha \rangle^n \frac{1}{r(\gamma z, w)^{n/2}} \\ &= \frac{\operatorname{vol}(\Gamma \backslash \mathbb{H})^{n/2}}{\|\alpha\|^n 2^{n/2} x^{n/2}} \sum_{r(\gamma z, w) \leq x} \langle \gamma, \alpha \rangle^n + O(e^x x^{-1}), \end{aligned}$$

This may be evaluated by (5.2). We find

$$M_n(Y_x) \rightarrow \begin{cases} \frac{(2m)!}{m!2^m}, & \text{if } n = 2m, \\ 0, & \text{otherwise.} \end{cases}$$

We notice that the right-hand side coincides with the moments of the Gaussian distribution. Hence by a classical result due to Fréchet and Shohat, see [12, 11.4.C], we may conclude that

$$P(Y_x \in [a, b]) \rightarrow \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{t^2}{2}\right) dt \text{ as } x \rightarrow \infty.$$

This concludes the proof of Theorem 1.1.

**Remark.** Equation (1.3) follows from (2.16) and (2.17) as follows. We take  $z_0 = w$ . The pole of

$$\left(i \int_w^z \alpha\right) D^{(0)}(z, w, s, 0)$$

contributes a residue

$$i \int_w^z \alpha \frac{2\pi}{\text{vol}(\Gamma \backslash \mathbb{H})}.$$

On the other hand the term  $-s(s+1)R(s, 0)D^{(1)}(z, w, s, 0)$  contributes a residue 0, since

$$\int_{\Gamma \backslash \mathbb{H}} D^{(1)}(z, w, 3, 0) d\mu(z) = \int_{\mathbb{H}} \frac{-i \int_w^z \alpha}{\cosh r(z, w)^3} d\mu(z).$$

Now an integration in polar coordinates centered at  $w$  produces the average of  $F(z) = \int_w^z \alpha$  on a circle around  $w$  and this is the average of a harmonic function on a circle. The mean-value theorem gives that it is equal to  $2\pi F(w) = 0$ .

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Department of Mathematics, University College London, Gower Street, London WC1E 6BT

The Graduate Center, Mathematics Ph.D. Program, 365 Fifth Avenue, Room 4208, New York, NY 10016-4309

*E-mail* : [petridis@member.ams.org](mailto:petridis@member.ams.org)

Department of Mathematical Sciences, University of Aarhus, Ny Munkegade Building 530, 8000 Aarhus, Denmark

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark

*E-mail* : [risager@math.ku.dk](mailto:risager@math.ku.dk)