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On Squares of Eigenfunctions for the Hyperbolic Plane and a New Bound on Certain L-Series

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1 Introduction

Various applications of L-series require knowledge of their behavior on their critical line. One usually needs to know the location of the poles and the various gamma factors that appear in the functional equation of the L-series. In [9] Sarnak treats the special case of the Rankin-Selberg convolution $L(\phi \otimes \phi, s)$ without such knowledge, where ϕ is an L^2-eigenfunction of the Laplace operator on the surface $\Gamma \setminus \mathbb{H}$, Γ a cofinite subgroup of $SL(2, \mathbb{R})$. The following bound is proved in [9]

$$\int_T^{T+1} |(\phi^2, E(z, 1/2 + it))|^2 \, dt \ll (T \log T)^2 e^{-\pi T} \tag{1}$$

as $T \to \infty$. The notation $\ll$ means that the left-hand side is (for sufficiently large T) less than a constant multiple of the right-hand side. Here $E(z, s)$ is the Eisenstein series corresponding to a cusp of Γ. Equation (1) implies that the Fourier coefficients a_n of an arbitrary Maaß cusp form ϕ satisfy the bound

$$|a_n| \ll_{\epsilon,\phi} |n|^{5/12+\epsilon} \tag{2}$$

for all $\epsilon > 0$. Here $\Delta\phi + (1/4 + \lambda^2)\phi = 0$, $\lambda \in \mathbb{R}$ or $\lambda \in i[-1/2, 1/2]$, and ϕ has the following Fourier expansion at the cusp $S^1 \times [a, \infty)$ with coordinates x, y:

$$\phi(x + iy) = \sum_{n \neq 0} a_n y^{1/2} K_{i\lambda}(2\pi|n|y) e^{2\pi i n x}. \tag{3}$$

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We can naturally assume that ϕ is real-valued and $\lambda \neq \pm i/2$, since we are not interested in the constant eigenfunction.

In [9] the case of compact surfaces is discussed as well. If the ϕ_k form an orthonormal basis for $L^2(\Gamma \backslash \mathbb{H})$, Γ a cocompact subgroup of $SL(2,\mathbb{R})$, and $\Delta\phi_k + (1/4 + r_k^2)\phi_k = 0$, then

$$(\phi^2, \phi_k) \ll (r_k \log r_k) e^{-\pi r_k/2} \tag{4}$$

as $j \to \infty$.

In the case that ϕ is a holomorphic cusp form of even integral weight $k > 2$, Good [4] proved that $a_n \ll n^{k/2-1/6}$ for arbitrary cofinite subgroups of $SL(2,\mathbb{R})$. We see that (2) falls short of the corresponding bound for the holomorphic cusp forms. This raises the issue of improving (1), (2), and (4). In this work we prove the following.

Theorem 1. If Γ is a cocompact subgroup of $SL(2,\mathbb{R})$ and $\lambda \neq 0$, i.e., the eigenvalue corresponding to ϕ is not $1/4$, then

$$(\phi^2, \phi_k) \ll r_k^{1/2} e^{-\pi r_k/2} \tag{5}$$

as $k \to \infty$. □

Theorem 2. If Γ is a cofinite subgroup of $SL(2,\mathbb{R})$ and the eigenvalue corresponding to ϕ is not $1/4$, then there exists an $\epsilon > 0$ such that

$$\int_t^{t+\epsilon} |(\phi^2, E(z, 1/2 + is))|^2 \, ds \ll t e^{-\pi t} \tag{6}$$

as $t \to \infty$. □

Corollary 1. The Fourier coefficients a_n of a Maaß cusp form ϕ satisfy

$$|a_n| \ll_{\epsilon,\phi} |n|^{2/5+\epsilon} \tag{7}$$

for all $\epsilon > 0$. □

If Γ is an arithmetic group of a special kind, like $SL(2,\mathbb{Z})$, much better bounds than (7) are known; see Bump et al. [1]. Even in these cases the bounds do not prove the Ramanujan conjecture $|a_n| \ll |n|^{\epsilon}$. In [9] Sarnak suggests that the Ramanujan conjecture may hold for arbitrary cofinite subgroups and is not a special feature of arithmetic. This makes improvements on (2) and (7) interesting to pursue. The method used to prove the theorems follows the same steps as [9]. Section 3 provides an outline of the method.

The restriction $\lambda \neq 0$, which follows from $i\lambda \notin \mathbb{Z}/2$, is purely technical. Lemma 1 does not apply for the eigenvalue 1/4. Equation (2.17) in [9], which gives the analytic continuation of the hypergeometric function, fails for $i\lambda \in \mathbb{Z}/2$. This problem first showed up in Helgason [5] and Lewis [6]. However, even in the case $\lambda = 0$, the sequence b_n in (11) increases at most polynomially in n; see Mazzeo [7, Th. 7.3]. The author has not investigated the order of growth of the b_n that follows from [7, Th. 7.3] for $\lambda = 0$. In any case, a generic cofinite or cocompact subgroup of $SL(2,\mathbb{R})$ does not have 1/4 in its L^2 spectrum; see [8].

2 Some general remarks

A point-pair invariant is a $K = SO(2)$ bi-invariant function $k(r)$, where r is the hyperbolic distance, and its Selberg-Harish-Chandra transform is given by

$$h(s) = 2\pi \int_0^\infty P_{-1/2+is}(\cosh r) k(r) \sinh r \, dr; \tag{8}$$

see Terras [11, 3.27, p. 149]. Here $P_{-1/2+is}(z)$ is the Legendre function of the first kind.

Let Γ be a cocompact or cofinite subgroup of $PSL(2,\mathbb{R})$. We set

$$K(w,w') = \sum_{\gamma\in\Gamma} k(r(\gamma w, w')).$$

If k satisfies the conditions explained in Selberg [10, p. 60], then the series above defines an operator K with integral kernel $K(w,w')$. The Fourier expansion of $K(w,w')$ is

$$K(w,w') = \sum_{k=0}^{\infty} h(r_k)\phi_k(w)\overline{\phi_k(w')} + \int_0^\infty h(s)E(w,1/2+is)E(w',1/2-is)\,ds, \tag{9}$$

where the ϕ_k's are an orthonormal basis of eigenfunctions for the discrete spectrum with corresponding eigenvalues $1/4 + r_k^2$. If Γ is cocompact the integral term is absent.

The metric on the disc model B^2 of hyperbolic space is $4|dz|^2/(1-|z|^2)^2$, $z = x+iy$. Denote an arbitrary eigenfunction on B^2 by ϕ. Assume its eigenvalue is $1/4+\lambda^2$. We write $\phi(r,\theta) = \sum_{-\infty}^{\infty} f_j(r)e^{ij\theta}$. Then

$$f_j(r) = b'_j \tanh^{|j|}(r/2)\left(\cosh(r/2)\right)^{-1-2i\lambda} F(1/2+i\lambda+|j|, 1/2+i\lambda, 1+|j|, \tanh^2(r/2))$$

for some constants b'_j.

3 Outline of the proof

Assume Γ is cocompact. If K is an integral operator as in Section 2, then

$$||K\phi^2||_2^2 = \sum_{j=0}^{\infty} |h(r_j)(\phi^2, \phi_j)|^2.$$

We will choose a family of operators K_t given by point-pair invariants $k_t(r)$ such that the corresponding transforms localize at r_j, i.e., $|h_{r_j}(r_j)| \geq c$ for all sufficiently large r_j, where c is a constant independent of j. If we prove that $||K_t(\phi^2)||_\infty \ll t^{1/2}\exp(-\pi t/2)$, then

$$r_k \exp(-\pi r_k) \gg \sum_{j=0}^{\infty} \left|h_{r_k}(r_j)(\phi^2, \phi_j)\right|^2 \geq \left|h_{r_k}(r_k)\right|^2 \left|(\phi^2, \phi_k)\right|^2 \geq c^2 \left|(\phi^2, \phi_k)\right|^2,$$

which implies that $(\phi^2, \phi_k) \ll r_k^{1/2}\exp(-\pi r_k/2)$ and proves Theorem 1. The choice of k_t will be explained later. The issue is to estimate the L^∞ norm of $K_t(\phi^2)$. Fix $w \in \mathbb{H}$. We switch to the disc model of hyperbolic space by a transformation that maps w to zero. All bounds will be uniform in w.

For $0 \leq r < \infty$ we define $B(r) = \int_{S^1} |\phi(r,\theta)|^2\, d\theta$. We set

$$C_j(r) = (\cosh(r/2))^{-1-2i\lambda} F(1/2 + i\lambda + |j|, 1/2 + i\lambda, 1 + |j|, \tanh^2(r/2)). \tag{10}$$

The functions $\tanh^{|j|}(r/2)C_j(r)$ are the associated spherical functions. Parseval's equality then gives

$$B(r) = \sum_{j=0}^{\infty} |b_j|^2 \tanh^{2j}(r/2)|C_j(r)|^2, \tag{11}$$

where $|b_j|^2 = |b'_j|^2 + |b'_{-j}|^2$, $j \geq 0$. The function $B(r)$ extends on the real line as an even function and, if $|\phi(r,\theta)| \leq M$, then $B(r) \leq 2\pi M^2$. It extends to an analytic function for $|\Im r| < \pi/2$; see Lemma 2. The crucial point is a lower bound of $C_j(q_j)$ as $j \to \infty$ for a certain sequence q_j that gives a rather sharp bound on b_j (see Lemma 1) and an upper bound on $C_j(r)$ as $j \to \infty$ for all r with $|\Im r| < \pi/2$ (see Lemma 3). This lemma is a sharper version of [9, Equation 2.21] and is the crucial new ingredient. Its proof is included in Appendix A.

The main point in Lemma 1 is that if an eigenfunction ϕ does not increase more than exponentially in the distance from the origin of the hyperbolic disc, then it corresponds to a distribution on the boundary of the disc and Lemma 1 gives a bound on the order, if ϕ is bounded, as long as $i\lambda \notin \mathbb{Z}/2$; see [5] and [6].

4 Proof of Theorem 1

4.1

We state the lemmas mentioned in Section 3.

Lemma 1. Assume that $i\lambda \notin \mathbb{Z}/2$. Then the sequence b_j in (11) is square integrable. □

Proof. See Helgason [5, Th. 4.24, p. 66] and Lewis [6]. ■

Lemma 2. The function $B(r)$ extends in the strip $|\Im r| < \pi/2$ as an even analytic function of r and satisfies the bound

$$B(r) \ll |\cosh(r/2)|^4.$$ □

Proof. We have

$$B(r) = \sum_{j=0}^{\infty} |b_j|^2 \tanh^j(r/2)\overline{\tanh^j(\overline{r}/2)}C_j(r)\overline{C_j(\overline{r})}.$$

We note that the hypergeometric function $F(a, b, c, z)$ is holomorphic in the region $|z| < 1$ and that the map $z = \tanh\left(r/2\right)$ is a conformal map from $|\Im r| < \pi/2$ to $|z| < 1$. By Lemma 1 it is enough to show that $|C(r)| \ll |\cosh(r/2)|^2$. This is equivalent to (12) in the following lemma, which captures the behavior of the hypergeometric function in (10) for large j. ■

Lemma 3. The following bound holds for $|z| < 1$:

$$\left|(1-z)^{3/2+i\lambda}F\left(1/2+i\lambda+j, 1/2+i\lambda, 1+j, z\right)\right| \ll 1. \tag{12}$$

We also have for $t > 0$

$$\left|(1-z)^{3/2}F\left(1/2, 1/2, 1-it, z\right)\right| \ll 1. \tag{13}$$

□

The proof of Lemma 3 is included in Appendix A.

Remark 1. On the horizontal lines $r = x + i\left(\pi/2 - 1/t\right)$ and $r = x - i\left(\pi/2 - 1/t\right)$, $x \in \mathbb{R}$, we have the bound

$$B(r) \ll e^{2|x|}. \tag{14}$$

4.2

We now come to the choice of the point-pair invariants $k_t(r)$:

$$k_t(r) = \frac{tP_{-1/2+it}(\cosh r)\sinh^2 r}{\cosh^8(r)}. \tag{15}$$

Remark 2. The intuition behind the choice of the point-pair invariants is as follows: The inversion formula for the Harish-Chandra transform

$$k(r) = \frac{1}{2\pi^2}\int_0^\infty h(\lambda)P_{-1/2+i\lambda}(\cosh r)|c(\lambda)|^{-2}\,d\lambda,$$

where $c(\lambda)$ is the Harish-Chandra c function, suggests that in order to localize $h_t(\lambda)$ at t, say $h_t(\lambda) = \delta_t(\lambda)$, $k_t(r)$ has to be essentially $P_{-1/2+it}(\cosh r)|c(t)|^{-2}$. One sees that $|c(t)|$ is asymptotic to $\pi^{-1/2}t^{-1/2}$; see Appendix B. However, since we do not want to work with distributions and need to define integral operators as in Section 2, we use a factor to make $k_t(r)$ rapidly decreasing. We choose it to be $\sinh^m r/\cosh^n r$, for m and n natural numbers. These point-pairs are smooth at $r = 0$ and can be odd or even functions of r and have order of vanishing at zero as high as needed by adjusting m and n. These options are important in order to generalize to the other rank-one symmetric spaces. Moreover, the fact that we incorporate, in the point-pair invariants, the spherical function allows us to avoid the fractional integral in [10]. With the exception of the odd-dimensional real hyperbolic spaces, all other rank-one spaces have Harish-Chandra transform that involves fractional integration and multiple integrals; see Venkov [12, p. 31].

We need to know that the point-pair invariants (15) are of moderate growth. We show that k_t has majorant $k_1(x, y) = te^{-(13/2)r}$. It is enough to study the behavior of $P_{-1/2+it}(\cosh r)$ for $r \in \mathbb{R}$. The Legendre function of the first kind $P_{-1/2+it}(z)$ is real for z real. Formula (26) in [2, p. 128] gives

$$\begin{aligned} P_{-1/2+it}(\cosh r) = {} & \frac{1}{\sqrt{\pi}}\frac{\Gamma(-it)}{\Gamma(1/2-it)}\frac{e^{(1/2-it)r}}{(e^{2r}-1)^{1/2}}F\left(1/2, 1/2, 1+it, \frac{1}{1-e^{2r}}\right) \\ & + \frac{1}{\sqrt{\pi}}\frac{\Gamma(it)}{\Gamma(1/2+it)}\frac{e^{(1/2+it)r}}{(e^{2r}-1)^{1/2}}F\left(1/2, 1/2, 1-it, \frac{1}{1-e^{2r}}\right) \end{aligned} \tag{16}$$

for $r > (1/2)\ln 2$. We also have $F(1/2, 1/2, 1+it, 1/(1-e^{2r})) = 1 + \xi(t, r)$, for $r > \ln 2$ and $t > 1$, where

$$|\xi(t, r)| \le 2e^{-2r}. \tag{17}$$

This follows from the series expansion of the hypergeometric function. We note that the bound on $\xi(t, r)$ is independent of t. From (16) we deduce that $P_{-1/2+it}(\cosh r) \ll_t e^{-1/2r}$, as $r \to \infty$.

4.3

Now we prove that $h_t(t) \geq c_0$ for all t sufficiently large, i.e., the Selberg-Harish-Chandra transform of k_t localizes at t. Using (8) and the fact that the spherical function $P_{-1/2+it}(\cosh r)$ is real, we get for m sufficiently large

$$\frac{1}{\pi}h_t(t) \geq t\int_m^\infty \left[P_{-1/2+it}(\cosh r)\right]^2 (\cosh r)^{-5}\, dr,$$

where m is to be determined later, independently of t. The issue is to show that the integral giving $h_t(t)$, which is positive and decreases as $t \to \infty$, decreases at most like $1/t$ and not more quickly. Using (16) and (17) we get

$$\begin{aligned} h_t(t) \geq{}& t\int_m^\infty \frac{\Gamma^2(-it)}{\Gamma^2(1/2-it)}\frac{1}{\cosh^5 r}\frac{e^{(1-2it)r}}{e^{2r}-1}\left(1+\xi(t,r)\right)^2\, dr \\ &+ t\int_m^\infty \frac{\Gamma^2(it)}{\Gamma^2(1/2+it)}\frac{1}{\cosh^5 r}\frac{e^{(1+2it)r}}{e^{2r}-1}\left(1+\overline{\xi}(t,r)\right)^2\, dr \\ &+ 2t\int_m^\infty \frac{\Gamma(-it)\Gamma(it)}{\Gamma(1/2-it)\Gamma(1/2+it)}\frac{1}{\cosh^5 r}\frac{e^{r}}{e^{2r}-1}\left(1+\xi(t,r)\right)(1+\overline{\xi}(t,r))\, dr. \end{aligned} \tag{18}$$

The idea suggested by the asymptotics of $P_{-1/2+it}(\cosh r)$ is that the main contribution comes from the integral

$$\int_m^\infty \frac{1}{\cosh^5 r}\frac{e^r}{e^{2r}-1}\, dr \geq \frac{1}{6}e^{-6m}.$$

By expanding the products and the squares in equation (18), we get nine integrals, which we estimate using (17). Those containing $\xi(t,r)$ or its conjugate are $\ll e^{-8m}$. The integrals

$$\int_m^\infty \frac{e^{(1\pm 2it)r}}{(e^{2r}-1)\cosh^5 r}dr$$

are estimated by $t^{-1}e^{-6m}$ using an integration by parts. The asymptotic behavior of the gamma function $\Gamma(x+iy)$ for large $|y|$,

$$\lim_{|y|\to\infty} |\Gamma(x+iy)|e^{\pi|y|/2}|y|^{1/2-x} = (2\pi)^{1/2} \tag{19}$$

(see [2, (6), p. 47]), implies that the absolute value of the gamma factors in (18) is asymptotic to $1/t$. We now choose m sufficiently large and use the triangle inequality and (18) to deduce that

$$\liminf_{t\to\infty} h_t(t) > 0, \tag{20}$$

which concludes the claims about the choice of the point-pair invariants.

4.4

We come back to estimate the L^∞ norm of $K_t(\phi^2)$. We have, by using polar coordinates,

$$K(\phi^2)(w) = I = \int_0^\infty k_t(r)B(r)\sinh r\, dr = \int_0^\infty \frac{tP_{-1/2+it}(\cosh r)}{\cosh^8(r)} B(r)\sinh^3 r\, dr.$$

Formula 3.3.1 (3) in [2, p. 140] gives

$$\pi^{-1} i \coth(t\pi)\left(Q_{-1/2+it}(z) - Q_{-1/2-it}(z)\right) = P_{-1/2+it}(z), \tag{21}$$

where $Q_\nu(z)$ is the Legendre function of the second kind. We split the integral in equation (21) as I_1 and I_2 to get

$$\begin{aligned} I = I_1 - I_2 = &\int_0^\infty \frac{it}{\pi}\coth(t\pi)\frac{Q_{-1/2+it}(\cosh r)}{\cosh^8(r)} B(r)\sinh^3 r\, dr \\ &- \int_0^\infty \frac{it}{\pi}\coth(t\pi)\frac{Q_{-1/2-it}(\cosh r)}{\cosh^8(r)} B(r)\sinh^3 r\, dr. \end{aligned} \tag{22}$$

The Legendre functions of the second kind $Q_\nu^\mu(z)$ and the Legendre functions of the first kind $P_\nu^\mu(z)$ are not single-valued in the plane. One must introduce a cut from $-\infty$ to 1. However, when μ is an even integer, we can reduce the cut for $P_\nu^\mu(z)$ to $(-\infty, -1]$; see [2, p. 143]. We see that in the strip $|\Im r| < \pi/2$, the cut $[0, 1]$ corresponds to $i[-\pi/2, \pi/2]$ and that the conformal map $z = \cosh r$ opens the cut $[0, 1]$ so that approaching $[0, 1]$ from above (below) corresponds to approaching $i[0, \pi/2]$ ($i[-\pi/2, 0]$). We denote the new branches of $Q_\nu^\mu(z)$ when we go around the branch point one clockwise (counterclockwise) by $Q_\nu^\mu(z, 1-)$ ($Q_\nu^\mu(z, 1+)$). The relation between $Q_\nu^\mu(z)$, $Q_\nu^\mu(z, 1\pm)$, and $P_\nu^\mu(z)$ is described by the equations

$$\begin{aligned} &Q_\nu^\mu(z, 1-) - e^{-i\mu\pi}Q_\nu^\mu(z) = \pi i e^{i\mu\pi}P_\nu^\mu(z) \\ &Q_\nu^\mu(z, 1+) - e^{i\mu\pi}Q_\nu^\mu(z) = -\pi i e^{i\mu\pi}P_\nu^\mu(z); \end{aligned} \tag{23}$$

see [2, 3.3.2 (19), p. 142].

In (23) we subtract and pass to the limit $\mu \to 0$ to get

$$Q_\nu(z, 1+) - Q_\nu(z, 1-) = -2\pi i P_\nu(z). \tag{24}$$

Now we shift the contour of integration for I_1, I_2 as follows: For I_1 (I_2) we first go along the negative real axis from zero to $-\infty$ and on the lower (upper) cut of the plane, called the path γ_1 (γ_3), and then along the line γ_2 (γ_4) given by the equation $r = x - i\left(\pi/2 - 1/t\right)$ ($r = x + i(\pi/2 - 1/t)$). We set

$$\tilde{Q}_{-1/2\pm it}(\cosh r) = \begin{cases} Q_{-1/2\pm it}(\cosh r), & \Re r \geq 0, \\ Q_{-1/2\pm it}(\cosh r, 1\mp), & \Re r < 0. \end{cases}$$

Then

$$\begin{aligned} I = &- \int_{-\infty}^{0} \frac{it}{\pi} \coth(t\pi) Q_{-1/2+it}(\cosh r, 1-) \frac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ &+ \int_{-\infty}^{0} \frac{it}{\pi} \coth(t\pi) Q_{-1/2-it}(\cosh r, 1+) \frac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ &+ \int_{\gamma_2} - \int_{\gamma_4} . \end{aligned}$$

Moreover,

$$\begin{aligned} I = &\int_0^{\infty} \frac{it}{\pi} \coth(t\pi) \left[Q_{-1/2-it}(\cosh r, 1+) - Q_{-1/2+it}(\cosh r, 1-)\right] \frac{-B(r)\sinh^3 r}{\cosh^8 r} dr \\ &+ \int_{\gamma_2} - \int_{\gamma_4} , \end{aligned} \tag{25}$$

because $B(r)$ is even. Since

$$P_{-1/2-it}(\cosh r) = \frac{1}{\pi} \tan(-1/2 - it)\pi \left[Q_{-1/2-it}(\cosh r) - Q_{-1/2+it}(\cosh r)\right]$$

for $r > 0$, we get by analytic continuation when we cross the cut $i[0, \pi/2]$

$$P_{-1/2-it}(\cosh r) = \frac{1}{\pi} \tan(-1/2 - it)\pi \left[Q_{-1/2-it}(\cosh r, 1+) - Q_{-1/2+it}(\cosh r, 1+)\right],$$

which gives, together with (24), (25),

$$\begin{aligned} I = &\int_{\gamma_2} - \int_{\gamma_4} \\ &+ \int_0^{\infty} t P_{-1/2-it}(\cosh r) \frac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ &- \int_0^{\infty} \frac{it}{\pi} \coth(t\pi)(-2\pi i) P_{-1/2+it}(\cosh r) \frac{B(r)\sinh^3 r}{\cosh^8 r} dr. \end{aligned} \tag{26}$$

Since $P_{-1/2-it}(z) = P_{-1/2+it}(z)$ (see [2, 3.3.1 (1), p. 140]) we get from (26)

$$I = \frac{e^{t\pi} - e^{-t\pi}}{2(e^{t\pi} + e^{-t\pi})} \left(\int_{\gamma_2} - \int_{\gamma_4} \right).$$

Therefore, it is enough to prove that

$$\int_{\gamma} \frac{it}{\pi} \coth(t\pi) \tilde{Q}_{-1/2 \pm it}(\cosh r) \frac{B(r) \sinh^3 r}{\cosh^8 r} dr = O(t^{1/2} e^{-(\pi/2)t})$$

where γ is γ_2 (γ_4) respectively. Since $B(r)$ is real for real $r > 0$, we have $B(r) = \overline{B(\overline{r})}$ on the strip $|\Im r| < \pi/2$, and we see that the integrand $\mathcal{C}(r)$ in $\int_{\gamma_2}$ satisfies $\mathcal{C}(r) = \overline{\mathcal{D}(\overline{r})}$, where $\mathcal{D}(r)$ is the integrand for $\int_{\gamma_4}$. So it is enough to look at $\int_{\gamma_4} \mathcal{D}(r) dr$. We have

$$Q_{-1/2-it}(\cosh r) = \sqrt{\pi/2} \frac{\Gamma(1/2 - it)}{\Gamma(1 - it)} \frac{e^{itr}}{\sqrt{\sinh r}} F\left(1/2, 1/2, 1 - it, \frac{1}{1 - e^{2r}}\right)$$

for $r > (1/2) \ln 2$ [2, 3.2 (44), p. 136]. This formula holds by analytic continuation in the domain: $\{r \mid \pi/4 < \Im r < \pi/2, -\infty < \Re r < \ln 2\} \cup \{r \mid -\pi/4 < \Im r < \pi/2, \Re r \geq \ln 2\}$. On this domain we have $\left|1/(1 - e^{2r})\right| < 1$, so we can apply (13). On the line γ_4 we have $|\sinh r| \ll e^{|x|}$, $|F(1/2, 1/2, 1 - it, 1/(1 - e^{2r}))| \ll |1 - e^{-2r}|^{3/2} \ll e^{3|x|}$, $|\cosh r| \gg e^{|x|}$ for $t > 3/\pi$. Using (19), (14) we finally get

$$\int_{\gamma_4} \frac{it}{\pi} \coth(t\pi) \tilde{Q}_{-1/2-it}(\cosh r) \frac{B(r) \sinh^3 r}{\cosh^8 r} dr \ll t^{1/2} e^{-(\pi/2)t} \int_{-\infty}^{\infty} e^{-|x|/2} dx,$$

which gives the result. This completes the proof of Theorem 1.

5 Noncompact surfaces

We need the following property of the point-pair invariants $k_t(r)$ defined in (15).

Claim. There exist $\epsilon > 0$, $\epsilon_0 > 0$, and $t_0 > 0$ such that $|h_t(s)| \geq \epsilon_0$ for all $t \geq t_0$ and $|s - t| < \epsilon$. □

This property is proved in Appendix B.

The spectral decomposition of the integral kernel in this case is given by (9). Parseval's identity now gives

$$\||K\varphi^2||_2^2 = \sum_{j=0}^{\infty} |h(r_j)|^2 |(\varphi^2, \varphi_j)|^2 + \frac{1}{4\pi} \int_0^{\infty} |h(s)|^2 |(\varphi^2, E(z, 1/2 + is))|^2 \, ds. \tag{27}$$

The rest of the proof remains unchanged, and we look now at the integral on the right-hand side of (27) over the short interval $[t, t+\epsilon]$ to deduce (6) and complete the proof of theorem 2. In order to study the Fourier coefficients of Maaß cusp forms for Γ, we follow the method used in [4, p. 546] to study the Fourier coefficients of holomorphic cusp forms. We define ψ_U, U sufficiently large, to be a nonnegative C^∞-function on $\mathbb{R}$ with

$$\psi_U(\tau) = \begin{cases} 1, & \text{if } \tau \le 1 - 1/U, \\ 0, & \text{if } \tau \ge 1 + 1/U, \end{cases}$$

and $\psi_U^{(j)}(\tau) \ll U^j$ for $j = 0, 1, \ldots$. We will work with the Mellin transform of ψ_U given by $R_U(s) = \int_0^\infty \psi_U(\tau)\tau^{s-1}\, d\tau$ for $\sigma = \Re s > 0$. We have

$$R_U(s) = \frac{1}{s} + O\left(\frac{1}{U}\right). \tag{28}$$

Integration by parts gives

$$R_U(s) = \frac{(-1)^j}{s(s+1)\cdots(s+j-1)} \int_0^\infty \tau^{s+j-1}\psi_U^{(j)}(\tau) d\tau \ll \frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^{j-1} \tag{29}$$

for $j = 1, 2, \ldots$. The estimates (28) and (29) are uniform for σ bounded. Now by interpolation it is easy to see that for all $c \ge 0$ we have

$$R_U(s) \ll \frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^c \tag{30}$$

again uniformly for σ bounded. We assume the Maaß cusp form $\phi(z)$ has the Fourier expansion (3) at the cusp and its eigenvalue is $1/4 + \lambda^2$. The L-series $D(s) = \sum |a_n|^2 |n|^{-s}$ converges absolutely for $\Re s > 2$ by the Hecke bound $a_n = O(|n|^{1/2})$. The Rankin-Selberg method provides the analytic continuation of $D(s)$ to the whole plane. A standard argument gives

$$D(s) = \frac{2\pi^s \Gamma(s)}{\Gamma(s/2)^2 \Gamma(s/2 + i\lambda)\Gamma(s/2 - i\lambda)} \int_{\Gamma\backslash\mathbb{H}} \phi^2 E(z, s)\, dz. \tag{31}$$

On the critical line $\Re s = 1/2$ the factor $f(s) = 2\pi^s\Gamma(s)\Gamma(s/2)^{-2}\Gamma(s/2+i\lambda)^{-1}\Gamma(s/2-i\lambda)^{-1}$ is asymptotic to $e^{\pi t/2}t$, as $t \to \infty$, as follows from equation (19). The inversion formula for the Mellin transform gives

$$\sum_{|n| \le X(1-1/U)} |a_n|^2 \le \sum_{|n|} |a_n|^2 \psi_U(|n|/X) = \frac{1}{2\pi i} \int_{\Re s = 2+\epsilon} D(s) X^s R_U(s)\, ds. \tag{32}$$

We shift the countour of integration in the integral in (32) to the line $\Re s = 1/2$. The function $D(s)$ has poles coming from the residues of the Eisenstein series on the interval

$(1/2, 1]$. Let us assume these are at the points s_l with residues the noncuspidal eigenfunctions $u_l(z)$. We estimate the integral along the line $\Re s = 1/2$ as follows: We choose m an integer with $1/m < \epsilon$. Then, using (6) and (19),

$$\int_{-\infty}^{\infty} D(1/2+it)X^{1/2+it}R_U(1/2+it)dt$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=1}^{m}\int_{n+(k-1)/m}^{n+k/m} f(1/2+it)(\varphi^2, E(z,1/2+it))X^{1/2+it}R_U(1/2+it)dt$$
$$\ll \sum_{n,k}\left(\int_{n+(k-1)/m}^{n+k/m}|(\varphi^2, E(z,1/2+it))|^2dt\int_{n+(k-1)/m}^{n+k/m}|f(1/2+it)|^2X|R_U(1/2+it)|^2dt\right)^{1/2}$$
$$\ll \sum_{n=-\infty}^{\infty}\sum_{k=1}^{m} e^{-\pi|n|/2}|n|^{1/2}\left(e^{\pi|n|}X\left(\frac{U}{1+|n|}\right)^{2c}\right)^{1/2} = X^{1/2}\sum_{n=-\infty}^{\infty}|n|^{1/2}U^c(1+|n|)^{-c}.$$

To make the last series converge, we choose $c > 3/2$, say, $c = 3/2+\epsilon'$ with $\epsilon' > 0$. Then the integral is estimated by $X^{1/2}U^{3/2+\epsilon'}$. Therefore,

$$\sum_{|n|\le X(1-1/U)}|a_n|^2 = \sum_{1/2\le s_l\le 1}(u_l, \phi^2)f(s_l)X^{s_l}\frac{1}{s_l} + O(X/U) + O(X^{3/2}U^{1/2+\epsilon'}),$$

since $X^{s_l} \ll X$ and (28) holds. The pole of Eisenstein series at $s = 1$ gives the constant eigenfunction and we conclude

$$\sum_{|n|\le X(1-1/U)}|a_n|^2 = cX + O(X/U + X^{1/2}U^{3/2+\epsilon'}). \tag{33}$$

We choose U so that the two error terms are equal, i.e., $U = X^{1/(5+2\epsilon')} = X^{1/5-\epsilon}$, and then the error term becomes $O(X^{4/5+\epsilon})$. Then $a_m = O(|m|^{2/5+\epsilon})$. This proves Corollary 1.

Appendix A

Proof of Lemma 3. Using the fundamental integral representation for the hypergeometric function [2, 2.1.3(10), p. 59], we get

$$F(1/2+i\lambda+j, 1/2+i\lambda, 1+j, z) = \Gamma(1+j)\Gamma(1/2+i\lambda+j)^{-1}\Gamma(1/2-i\lambda)^{-1}$$
$$\times\int_0^1 s^{-1/2+i\lambda+j}[(1-s)(1-zs)]^{-1/2-i\lambda}ds. \tag{34}$$

We can assume that $0 \ge \Im\lambda > -1/2$, which is necessary for the integral representation to be valid. We can also assume that $|\arg(1-z)| < \pi$ and $|\arg(1-sz)| < \pi$. We study the hypergeometric integral in equation (34) using Laplace's method. For a similar approach to get uniform asymptotics of hypergeometric integrals, see [13]. We fix $\delta > 0$ small and set $u(s) = s^{-1/2+i\lambda+j}$, $\nu(s) = (1-s)^{-1/2-i\lambda}$, $P(s) = (1-zs)^{-1/2-i\lambda}$, $U(s) = s^{1/2+i\lambda+j}/(1/2+i\lambda+j)$. Then $U(0) = 0$, $\nu'(s) = (1/2+i\lambda)(1-s)^{-3/2-i\lambda}$, and $P'(s) = z(1/2+i\lambda)(1-zs)^{-3/2-i\lambda}$. We have

$$\begin{aligned}\int_0^1 u\nu P &= \int_0^{1-\delta} u\nu P + \int_{1-\delta}^1 u\nu P \\ &= U(1-\delta)\nu(1-\delta)P(1-\delta) - \int_0^{1-\delta} U(\nu' P + \nu P') + \int_{1-\delta}^1 u\nu P.\end{aligned} \tag{35}$$

The first term in (35) is $O(1/j)$, since $|P(1-\delta)|$ is bounded, as $|z| < 1$. Since for $|z| < 1$ and $0 \le s \le 1$, $|1-z| \le 2|1-zs|$, we also have $|(1-z)^{3/2+i\lambda}P(s)| \le |1-z|c_1$ and $|(1-z)^{3/2+i\lambda}P'(s)| \le c_2$; therefore,

$$(1-z)^{3/2+i\lambda}\int_0^{1-\delta} U(\nu' P + P'\nu) \ll \int_0^{1-\delta} |U||\nu'| + |U||\nu| = O(1/j). \tag{36}$$

We now look at the third term in (35):

$$\begin{aligned}\int_{1-\delta}^1 u\nu P &= P(1)\int_0^1 u\nu - P(1)\int_0^{1-\delta} u\nu + \int_{1-\delta}^1 u\nu[P - P(1)] \\ &= (1-z)^{-1/2-i\lambda}\frac{\Gamma(1/2+i\lambda+j)\Gamma(1/2-i\lambda)}{\Gamma(1+j)} - P(1)U(1-\delta)\nu(1-\delta) \\ &\quad + P(1)\int_0^{1-\delta} U\nu' + \int_{1-\delta}^1 u\nu[P - P(1)],\end{aligned} \tag{37}$$

where we used the beta integral to evaluate $\int_0^1 u\nu$. The second and third terms in (37) multiplied by $(1-z)^{3/2+i\lambda}$ are clearly $O(1/j)$. Since

$$\lim_{s\to 1}(P(s) - P(1))/(s-1) = z(1/2+i\lambda)(1-z)^{-3/2-i\lambda},$$

we have

$$\int_{1-\delta}^1 u\nu[P - P(1)] = -(U\nu[P - P(1)])(1-\delta) - \int_{1-\delta}^1 U\{\nu'[P - P(1)] + \nu P'\}. \tag{38}$$

The first term in (38) is $O(1/j)$, when multiplied by $(1-z)^{3/2+i\lambda}$. Moreover,

$$\left|(1-z)^{3/2+i\lambda}\int_{1-\delta}^1 U\nu P'\right| \le \int_{1-\delta}^1 |U||\nu|, \tag{39}$$

which is $O(1/j)$, since ν is integrable on $[1-\delta, 1]$, as $\Re(-1/2-i\lambda) > -1$. The last term to consider in (38) is

$$\int_{1-\delta}^1 U\nu'[P - P(1)] = -(1/2+i\lambda)\int_{1-\delta}^1 U\nu[P - P(1)]/(s-1), \tag{40}$$

and the function $(1-z)^{3/2+i\lambda}[P-P(1)]/(s-1)$ is bounded for s close to one. This completes the study of the various terms. We now take into account the asymptotics of the gamma function [2, p. 47] to see that $\Gamma(1+j)/\Gamma(1/2+i\lambda+j)^{-1} \sim j^{1/2-i\lambda}$ as $j \to \infty$. Since $\Re(1/2-i\lambda) \leq 1/2$, all the terms in the expansion of the integral representation of the hypergeometric function tend to zero as $j \to \infty$, when we multiply by $(1-z)^{3/2+i\lambda}$, except

$$(1-z)^{-1/2-i\lambda}\Gamma(1/2+i\lambda+j)\Gamma(1/2-i\lambda)\Gamma(1+j)^{-1},$$

which, when multiplied by

$$(1-z)^{3/2+i\lambda}\Gamma(1+j)\Gamma(1/2+i\lambda+j)^{-1}\Gamma(1/2-i\lambda)^{-1},$$

remains bounded. This proves the estimate in (12). The second estimate in Lemma 3 is proved similarly. ■

Appendix B

In this appendix we prove the claim made at the beginning of Section 5. By equation (20) there exist $\epsilon_0 > 0$ and $t_0 > 0$ such that $|h_t(t)| > 2\epsilon_0$ for all $t > t_0$. If we prove that $|dh_t(s)/ds| \leq K$ for $|s-t| < \epsilon_1$, t sufficiently large and K independent of t, then the mean value theorem allows us to deduce $|h_t(s)| > \epsilon_0$ for $|s-t| < \epsilon = \min(\epsilon_1, \epsilon_0/K)$, t sufficiently large. Using (8) we get

$$\frac{dh_t(s)}{ds} = 2\pi \int_0^\infty \frac{d}{ds} P_{-1/2+is}(\cosh r) \frac{tP_{-1/2+it}(\cosh r)\sinh^3 r}{\cosh^8 r} dr.$$

We will prove that, for $|s-t| < \epsilon_1$, the integrand is bounded by a function of r which is integrable on $[0,\infty)$ (independent of s and t). We review some facts about the spherical function on the symmetric space $\mathbb{H}$ (see [3, pp. 144, 150–152]). The spherical function $P_{-1/2+i\lambda}(\cosh r)$ can be split as

$$P_{-1/2+i\lambda}(\cosh r) = \varphi_\lambda(r) = c(\lambda)\Phi_\lambda(r) + c(-\lambda)\Phi_{-\lambda}(r), \tag{41}$$

where $c(\lambda)$ is the Harish-Chandra c function and $\Phi_\lambda(r)$ is the unique solution of the equation $\partial^2\varphi/\partial r^2 + \coth r \partial\varphi/\partial r + (1/4+\lambda^2)\varphi = 0$ satisfying $\Phi_\lambda(r) = e^{(i\lambda-1/2)r}(1+o(1))$ as $r \to \infty$. We have $c(\lambda) = \Gamma(i\lambda)\Gamma(1/2+i\lambda)^{-1}\pi^{-1/2}$, whose absolute value is asymptotic to $\pi^{-1/2}\lambda^{-1/2}$ as $\lambda \to \infty$, by (19). Moreover,

$$\Phi_\lambda(r) = e^{(i\lambda-1/2)r} \sum_{m=0}^{\infty} \Gamma_m(\lambda) e^{-mr}, \tag{42}$$

where the $\Gamma_m(\lambda)$ satisfy the following recursion formula

$$4n(n-i\lambda)\Gamma_{2n} = \sum_{k=0}^{n-1}(2k-i\lambda+1/2)2\Gamma_{2k} \tag{43}$$

with $\Gamma_0 = 1$ and $\Gamma_{2n-1} = 0$. The convergence of (42) is uniform on $[c,\infty)$ for any $c > 0$ by the estimate $|\Gamma_m(\lambda)| \le K(1+m)^d$, for some K, $d > 0$. This is explained in Flensted-Jensen [3, Lemma 7] or Helgason [5, p. 57]. We set $a_n(\lambda) = \Gamma_{2n}(\lambda)$. We need more precise information. We have the following two lemmas.

Lemma 4. There is a constant $K > 0$ such that $|\Gamma_m(\lambda)| \le K$ for all $\lambda > 0$ and $m \in \mathbb{N}$. □

Proof. See [3]. Use induction and equation (43). ■

Lemma 5. For all $d > 0$, there exists a $K_1 > 0$ such that for all $m \in \mathbb{N}$, $\lambda > 0$

$$\left|\frac{d}{d\lambda}\Gamma_m(\lambda)\right| \le K_1 m^d.$$ □

Proof. We have $a_0'(\lambda) = 0$. We differentiate (43) and use the triangle inequality to get

$$\left|\frac{da_n(\lambda)}{d\lambda}\right| \le \sum_{k=0}^{n-1}\frac{|a_k(\lambda)|}{2n|n-i\lambda|} + \sum_{k=0}^{n-1}\frac{|4k-2i\lambda+1|}{4n|n-i\lambda|}\left|\frac{da_k(\lambda)}{d\lambda}\right| + \frac{|a_n(\lambda)|}{|n-i\lambda|}. \tag{44}$$

If we assume that $|da_k(\lambda)/d\lambda| \le K_1 k^d$ for $k < n$, we get using the previous lemma and (44)

$$\left|\frac{da_n(\lambda)}{d\lambda}\right| \le \frac{3K}{2n} + K_1\sum_{k=0}^{n-1}\frac{k^d}{n}.$$

For n sufficiently large, say $n > N_0$, $3K/(2n) + K_1\sum_{k=0}^{n-1}k^d/n \le K_1 n^d$, since $\sum_{k=0}^{n-1}(k/n)^d/n \to \int_0^1 x^d dx = 1/(d+1) < 1$. Equation (44) shows that we can bound $(da_n(\lambda)/d\lambda)$ for all $n \le N_0$ independently from λ, so we can start the induction and the inductive step is complete. ■

We are interested in the product

$$\frac{d\varphi_s(r)}{ds}\varphi_t(r) = \frac{d}{ds}\{c(s)\Phi_s(r) + c(-s)\Phi_{-s}(r)\}\cdot\left[c(t)\Phi_t(r) + c(-t)\Phi_{-t}(r)\right].$$

The products $|c(s)c(t)|$, $|c(-s)c(t)|$, $|c(s)c(-t)|$, and $|c(-s)c(-y)|$ are asymptotic to $\pi^{-1}s^{-1/2}t^{-1/2}$ as $t \to \infty$, $|s-t| < \epsilon_1$ $(s \to \infty)$. We study now

$$c'(s) = \frac{i\Gamma(is)}{\sqrt{\pi}\Gamma(1/2+is)}\left[\psi(is) - \psi(1/2+is)\right],$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. Using the asymptotics for $\psi(z)$ (see [2, 1.18 (7), p. 47]), we get $|c'(s)| = O(s^{-3/2})$. The products $|c'(s)c(t)|$, $|c'(-s)c(t)|$, $|c'(s)c(-t)|$, and $|c'(-s)c(-t)|$ are $O(s^{-3/2}t^{-1/2})$. Lemma 4 and equation (42) give

$$|\Phi_s(r)| \le e^{-r/2} \sum_{m=0}^{\infty} |\Gamma_m(s)| e^{-mr} \le K e^{r/2}/(e^r - 1),$$

which blows like K/r as $r \to 0$. We take $d = 1$ in Lemma 5. Then

$$\frac{d\Phi_s(r)}{ds} = e^{(is-1/2)r} \left(\sum_{m=0}^{\infty} \left[ir\Gamma_m(s) + \frac{d\Gamma_m(s)}{ds} \right] e^{-mr} \right)$$

and

$$\left| \frac{d\Phi_s(r)}{ds} \right| \le K r e^{-r/2}/(1 - e^{-r}) + K_1 e^{-r/2} \sum_{m=0}^{\infty} m e^{-mr},$$

which behaves like K_1/r^2 as $r \to 0$. For $r \ge c > 0$ we get for $\Phi_s(r)$ and $d\Phi_s(r)/ds$ bounds by exponentially decreasing functions of r with no dependence on s. The products $\Phi_s(r)\Phi_t(r)$, $\Phi_{-s}(r)\Phi_t(r)$, $\Phi_s(r)\Phi_{-t}(r)$, and $\Phi_{-s}(r)\Phi_{-t}(r)$ blow at most like c_1/r^2 as $r \to 0$ and the products $d\Phi_s/ds \cdot \Phi_t$, $d\Phi_{-s}/ds \cdot \Phi_t$, $d\Phi_s/ds \cdot \Phi_{-t}$ and $d\Phi_{-s}/ds \cdot \Phi_{-t}$ blow at most like c_2/r^3 ar $r \to 0$. Away from zero all these products can be bounded by a function of r that decreases exponentially as $r \to \infty$. Since $tO(s^{-1/2}t^{-1/2}) = O(1)$ and $tO(s^{-3/2}t^{-1/2}) = O(1/t)$ as $t \to \infty$, $|s - t| < \epsilon_1$, the function $d\varphi_s(r)/ds \cdot \varphi_t(r) t \sinh^3 r/\cosh^8 r$ can be bounded by an integrable function of r independently from s and t. This concludes the claim in Section 6.

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