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THE REMAINDER IN WEYL'S LAW FOR n-DIMENSIONAL HEISENBERG MANIFOLDS

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Abstract. We prove that the error term in Weyl's law for 'rational' $(2n+1)$-dimensional Heisenberg manifolds is of order $O(t^{n-7/41})$. In the 'irrational' case, for generic $(2n+1)$-dimensional Heisenberg manifolds with $n > 1$, we prove that the error term is of the order $O(t^{n-1/4}\log t)$. The polynomial growth is optimal.

1. Introduction

Let (M, g) be a closed n-dimensional Riemannian manifold with metric g and Laplace-Beltrami operator Δ. Let its eigenvalues be $0 = \lambda_0 < \lambda_1 \leq \cdots$. For the spectral counting function $N(t) = \#\{j, \lambda_j \leq t\}$ we have Hörmander's theorem

$$N(t) = \frac{\mathrm{vol}(B_n)\mathrm{vol}(M)}{(2\pi)^n} t^{n/2} + O(t^{(n-1)/2})$$

where $\mathrm{vol}(B_n)$ is the volume of the n-dimensional unit disk and by $O(t^{(n-1)/2})$ we mean a term with the growth not faster than $t^{(n-1)/2}$ as t tends to infinity.

The estimate of the error term in Hörmander's theorem, defined by

$$R(t) = N(t) - \frac{\mathrm{vol}(B_n)\mathrm{vol}(M)}{(2\pi)^n} t^{n/2},$$

is in general sharp, as the well-known example of the sphere S^n with its canonical metric shows [HÖ1]. However, the question of determining the optimal bound for this error term in any given example is difficult. Nevertheless, for certain types of manifolds, i.e. manifolds with integrable geodesic flows, some improvements have been obtained. The simplest compact manifold with integrable geodesic flow is the 2-torus $\mathbb{T}^2$. Hardy's conjecture for $\mathbb{T}^2$ [HA], i.e. for the Gauss circle problem, asserts that

$$R(t) = O_\epsilon(t^{1/4+\epsilon}), \quad \epsilon > 0.$$

Hardy [HA] has also proved that for $\mathbb{T}^2$, $R(t) = \Omega_-(t^{1/4}(\log t)^{1/4})$. See [HF] for the best Ω_- result. The point is that $R(t) \neq O(t^{1/4})$. Cramér's formula [CR] states

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that for $\mathbb{T}^2$:

$$\lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T |R(t)|^2 dt = C > 0,$$

which is consistent with Hardy's conjecture.

As the first natural non-commutative generalization of $\mathbb{T}^2$ consider H_1/r, the 3-dimensional Heisenberg manifold, which has completely integrable geodesic flow [BU]. Petridis and Toth [PT] have proved that, for certain 'arithmetic' Heisenberg metrics on H_1, $R(t) = O(t^{5/6+\epsilon})$. Later in [CPT] the exponent was improved and the result extended to all left-invariant Heisenberg metrics. It was conjectured in [PT] that for H_1/r,

$$R(t) = O(t^{3/4+\epsilon}). \tag{1}$$

Moreover, Petridis and Toth [PT] have proved the following L^2 result for H_1, by averaging over perturbations of the metric g and defining $M(u) = (H_1/\Gamma, g(u))$,

$$\int_{I^3} |R(t,u)|^2 du \le Ct^{3/2+\delta},$$

where $I = [1-\epsilon, 1+\epsilon]$.

The conjecture (1) follows from the standard conjectures on the growth of exponential sums; see [CPT]. The exponential sums that show up have convex phase and, consequently, van der Corput's method and the method of exponent pairs can be applied. In the case of $(2n+1)$-dimensional Heisenberg manifolds with $n > 1$, we face multiple sums with linear dependence on $n-1$ variables. Our main purpose in this paper is to prove the following pointwise estimates:

Theorem 1.1. *Let $(H_n/\Gamma, g)$ be the $(2n+1)$-dimensional Heisenberg manifold where $n > 1$ and the metric g is in the orthogonal form*

$$g = \begin{pmatrix} h & 0 \\ 0 & g_{2n+1} \end{pmatrix}.$$

Let J_n be the standard symplectic matrix

$$J_n = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}.$$

Denote the eigenvalues of $h^{-1}J_n$ by $\pm\sqrt{-1}d_j^2, 1 \le j \le n$. If the ratios ${d_j}^2/{d_i}^2$ are rational, then

$$R(t) = O(t^{n-7/41}).$$

Remark 1. Conjecturally in the 'rational' case the best estimate, following from (19), is

$$R(t) = O_\epsilon(t^{n-1/4+\epsilon}). \tag{2}$$

Theorem 1.2. *Let $(H_n/\Gamma, g)$ and $\{\pm\sqrt{-1}d_j^2; 1 \le j \le n\}$ be as defined in Theorem 1.1. If there exists at least one irrational coefficient ${d_j}^2/{d_n}^2$, then for almost all metrics g, which are the ones where this irrational coefficient θ satisfies the Diophantine condition $\|j\theta\| \gg 1/(j\log^2 j)$, we have*

$$R(t) = O(t^{n-\frac{1}{4}}\log t).$$

Remark 2. There is nothing special in choosing the diophantine condition $\|j\theta\| \gg 1/(j\log^2 j)$. According to Khintchine's theorem for $f(x)$ increasing, positive with $\sum_n f(n)^{-1} < \infty$, the condition $\|j\theta\| \gg 1/f(j)$, for all j is satisfied by a generic θ in the sense of measure.

The analogue of Cramér's formula for Heisenberg manifolds is part of the first author's Ph.D. thesis at McGill University, [Kh].

2. Background on Heisenberg manifolds

2.1. **Definitions and notation.** For a row vector x and a column vector y in $\mathbb{R}^n$ let $\gamma(x,y,t)$ and $X(x,y,t)$ be the $(n+2)\times(n+2)$ matrices

$$\gamma(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix}, \ X(x,y,t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The real $(2n+1)$-dimensional Heisenberg group H_n is the Lie subgroup of $\mathrm{Gl}_{n+2}(\mathbb{R})$ consisting of all matrices in the form $\gamma(x,y,t)$, i.e.

$$H_n = \{\gamma(x,y,t) : x,y \in \mathbb{R}^n, t \in \mathbb{R}\}.$$

Its Lie algebra $\mathfrak{h}_n$ is the Lie subalgebra of $\mathfrak{gl}_{n+2}(\mathbb{R})$ consisting of all matrices $X(x,y,t)$, i.e.

$$\mathfrak{h}_n = \{X(x,y,t) : x,y \in \mathbb{R}^n, t \in \mathbb{R}\}.$$

Let $\mathfrak{z}_n = \{X(0,0,t), t \in \mathbb{R}\}$ be the center and the derived subalgebra of $\mathfrak{h}_n$. It is also convenient to identify the subspace $\{X(x,y,0), x,y \in \mathbb{R}^n\}$ of $\mathfrak{h}_n$ with $\mathbb{R}^{2n}$. Thus $\mathfrak{h}_n$ is the direct sum of these subspaces: $\mathfrak{h}_n = \mathbb{R}^{2n} \oplus \mathfrak{z}_n$.

Define $Z = X(0,0,1)$. Then the standard basis of $\mathfrak{h}_n$ is $\delta = \{X_1, X_2, ..., Y_1, ..., Y_n, Z\}$, where the first $2n$ elements are the standard basis of $\mathbb{R}^{2n}$. The nonzero brackets among the elements of δ are thus given by $[X_i, Y_i] = Z$ for $1 \le i \le n$. These are the standard commutation relations for positions and momenta in n-space in quantum mechanics. This justifies the use of the word Heisenberg to describe the group manifolds at hand.

Definition 2.1. A Riemannian Heisenberg manifold is a pair $(H_n/\Gamma, g)$ where Γ is a uniform discrete subgroup of H_n, i.e. the quotient H_n/Γ is compact, and g is a Riemannian metric on H_n/Γ whose lift to H_n is left H_n-invariant.

Heisenberg manifolds are circle bundles over tori.

2.2. **Classification of the uniform discrete subgroups of H_n.** For every n-tuple $r = (r_1, r_2, ..., r_n) \in \mathbb{Z}_+^n$ such that $r_i | r_{i+1}$ for every i, let $r\mathbb{Z}^n$ denote the n-tuples $x = (x_1, x_2, ..., x_n)$ where $x_i \in r_i\mathbb{Z}$. Define

$$\Gamma_r = \{\gamma(x,y,t) : x \in r\mathbb{Z}^n, y \in \mathbb{Z}^n, t \in \mathbb{Z}\}.$$

It is clear that Γ_r is a uniform discrete subgroup of H_n.

Theorem 2.2 ([G–W]). *The subgroups Γ_r classify the uniform discrete subgroups of H_n up to automorphism. In other words, for every uniform discrete subgroup of H_n there exists a unique n-tuple r and an automorphism of H_n which maps Γ to Γ_r. Also if two subgroups Γ_r and Γ_s are isomorphic, then r and s are equal.*

Corollary 2.3 ([G–W]). *Given any Riemannian Heisenberg manifold $M = (H_n/\Gamma, g)$ there exists a unique n-tuple r as before and a left-invariant metric $\tilde{g}$ on H_n such that M is isometric to $(H_n/\Gamma_r, \tilde{g})$.*

Since every left-invariant metric g on H_n is uniquely determined by an inner product on $\mathfrak{h}_n$, we can identify the left-invariant metrics with their matrices relative to the standard basis δ of $\mathfrak{h}_n$.

For any g we can choose an inner automorphism φ of H_n such that $\mathbb{R}^{2n}$ is orthogonal to $\mathfrak{z}_n$ with respect to $\varphi^* g$. Therefore $(H_n/\Gamma, g)$ will be isometric to $(H_n/\Gamma, \varphi^* g)$, and we can replace every left-invariant metric g by $\varphi^* g$ and always assume that the metric g is in the form

$$g = \begin{pmatrix} h & 0 \\ 0 & g_{2n+1} \end{pmatrix},$$

where h is a positive-definite $2n \times 2n$ matrix and g_{2n+1} is a positive real number.

The volume of the Heisenberg manifold is given by $\mathrm{vol}(H_n/\Gamma_r, g) = |\Gamma_r|\sqrt{\det(g)}$, where $|\Gamma_r| = r_1 \cdot r_2 \cdots r_n$ for $r = (r_1, r_2, ..., r_n)$.

Notation. The matrix $h^{-1}J_n$ is similar to the skew-symmetric matrix $h^{-1/2}J_n h^{-1/2}$, so it has pure imaginary eigenvalues which we denote by $\pm\sqrt{-1}d_j^2; 1 \leq j \leq n$.

2.3. **The spectrum of Heisenberg manifolds.** Denote the spectrum of $M = (H_n/\Gamma_r, g)$ by $\Sigma(r, g)$, that is, the collection of all eigenvalues of the Laplacian, counting the multiplicities. Then $\Sigma(r, g) = \Sigma_1 \cup \Sigma_2$, where Σ_1 contains the eigenvalues of the first type corresponding to the $2n$-dimensional tori as a submanifold of M, and Σ_2 is the second part resulting from the non-commutative structure of the Heisenberg manifold.

More precisely, let $\mathfrak{L}_r = \{X(x, y, z), x_i \in r_i\mathbb{Z}, y \in \mathbb{Z}^n, z \in \mathbb{Z}\}$ be a lattice in the Lie algebra $\mathfrak{h}$. Then $\Sigma_1(r, h)$ is the spectrum of the Laplace operator on the flat torus $(\mathbb{R}^{2n}/\mathfrak{L}_r, h)$, see [G–W, p. 259].

The second part of the spectrum, Σ_2 contains the eigenvalues of the form:

$$\mu(c, k) = 4\pi^2 c^2/g_{2n+1} + \sum_{i=1}^{n} 2\pi c d_i^2 (2k_i + 1)$$

and

$$\Sigma_2(r, g) = \{\mu(c, k) : c \in \mathbb{Z}_+, k \in (\mathbb{Z}_+ \cup \{0\})^n\},$$

where every $\mu(c, k)$ is counted with the multiplicity $2c^n|\Gamma_r|$.

3. Proof of Theorem 1.1

3.1. **Computation of the error term.** The spectral counting function corresponding to type II eigenvalues is defined by

$$N_{II}(t) = \#\{\mu(c, k);\ \mu(c, k) \leq t\}, \tag{3}$$

where every $\mu(c, k)$ on the right-hand side of (3) is counted $2c^n|\Gamma_r|$ times, for each pair (c, k) such that $\mu = \mu(c, k)$.

In the calculations for $N_{II}(t)$, without loss of generality, we assume that $r = (1, 1, ..., 1)$. In the general case, the only change is a coefficient $|\Gamma_r|$ in $2c^n|\Gamma_r|$, for the multiplicity of each $\mu(c, k)$, which also appears in the coefficients of $\mathrm{vol}(M)$ and $\mathrm{vol}(\mathbb{R}^{2n}/\mathfrak{L}_r, h) = |\Gamma_r|\det(h)$. Therefore, we continue with the computation of $N_{II}(t)$ only for $r = (1, 1, ..., 1)$ and we count every $\mu(c, k)$ with multiplicity $2c^n$. We compute asymptotics with 2 terms in the expansions, since we need to see that the second term of order t^n cancels the contribution of the main term of type I (torus) eigenvalues. The calculations with 2 terms require the Euler summation formula [GK], which we quote in its only use in this paper:

$$\sum_{n\leq u} n^a = \frac{u^{a+1}}{a+1} - \psi(u)u^a + O(u^{a-1}). \tag{4}$$

Here $\psi(u)$ is the first Bernoulli function (row of teeth function) defined by $\psi(u) = u - [u] - 1/2$. Now $\mu(c,k) \leq t$ if and only if

$$c\left(c + \sum d_i^2 g_{2n+1} k_i/\pi + \sum d_i^2 g_{2n+1}/(2\pi)\right) \leq t g_{2n+1}/(4\pi^2).$$

Let $b_i = d_i^2 g_{2n+1}/(2\pi)$. Then

$$\mu(c,k) \leq 4\pi^2 t/g_{2n+1}$$

if and only if $c(c + \sum 2b_i k_i + \sum b_i) \leq t$. So

$$N_{II}(4\pi^2 t/g_{2n+1}) = \sum_{c(c+2\sum b_i k_i + \sum b_i)\leq t} 2c^n = 2\sum_{c\leq\sqrt{t}} c^n \sum_{\sum b_i k_i \leq \frac{t}{2c} - \frac{c}{2} - \frac{\sum b_i}{2}} 1.$$

Define

$$\alpha = \frac{t}{2c} - \frac{c}{2} - \frac{1}{2}\sum_{i=1}^{n} b_i \quad \text{and} \quad s_i = \sum_{j=1}^{i} b_j k_j. \tag{5}$$

We adopt the following notation. When a sum is indexed by the variable k_i, this means that the range of k_i is $0 \leq k_i \leq (\alpha - s_{i-1})/b_i$. We have

$$N_2(t) = \frac{1}{2} N_{II}(4\pi^2 t/g_{2n+1}) = \sum_{0<c\leq\sqrt{t}} c^n \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} 1. \tag{6}$$

Evaluating the last sum on the right-hand side of (6), we get

$$\sum_{k_n} 1 = \frac{(\alpha - s_{n-2})}{b_n} - \frac{b_{n-1}k_{n-1}}{b_n} - \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + \frac{1}{2}. \tag{7}$$

Continuing with the next summation in (6), we get

$$\sum_{k_{n-1}}\sum_{k_n} 1 = \left(\frac{\alpha - s_{n-2}}{b_n} + \frac{1}{2}\right)\sum_{k_{n-1}} 1 - \sum_{k_{n-1}} \frac{b_{n-1}k_{n-1}}{b_n} - \sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right).$$

Evaluating $\sum_{k_{n-1}} 1$ and $\sum_{k_{n-1}} k_{n-1}$, using the Euler summation (4), we obtain

$$\sum_{k_{n-1}}\sum_{k_n} 1 = \frac{(\alpha - s_{n-2})^2}{2b_n b_{n-1}} + \frac{1}{2}(\alpha - s_{n-2})\frac{(b_n + b_{n-1})}{b_n b_{n-1}} - \sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + O(1).$$

By induction we get

$$\begin{aligned}\sum_{k_1,\ldots,k_n} 1 &= \frac{\alpha^n}{n! b_1 b_2 \ldots b_n} + \frac{(b_1 + \cdots + b_n)\alpha^{n-1}}{2(n-1)! b_1 b_2 \ldots b_n} \\ &\quad - \sum_{k_1,\ldots,k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + O(\alpha^{n-2}).\end{aligned}$$

We set $\beta = \sum_i^n b_i$. Hence,

$$\begin{aligned}\sum_{\substack{0<c\le\sqrt{t}\\ k_1,\dots,k_n}} c^n &= \sum_{0<c\le\sqrt{t}} \frac{c^n\alpha^n}{n!b_1b_2...b_n} + \sum_{0<c\le\sqrt{t}} \frac{\beta c^n \alpha^{n-1}}{2(n-1)!b_1b_2...b_n}\\ &- \sum_{\substack{0<c\le\sqrt{t}\\ k_1,\dots,k_n}} c^n\psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + \sum_{0<c\le\sqrt{t}} c^n\cdot O(\alpha^{n-2}).\end{aligned} \tag{8}$$

For the first sum on the right-hand side of (8) we substitute $\alpha = t/(2c) - c/2 - \beta/2$, use the binomial theorem and (4), and obtain

$$\begin{aligned}\sum_{0<c\le\sqrt{t}} c^n\alpha^n &= \frac{t^{n+1/2}}{2^n}\sum_{j=0}^n \frac{(-1)^j}{2j+1}\binom{n}{j} - \frac{t^n\psi(\sqrt{t})}{2^n}\sum_{j=0}^n (-1)^j\binom{n}{j}\\ &+\frac{t^n}{2^{n+1}}\{-1-\beta+\beta\sum_{j=0}^n(-1)^j\binom{n}{j}\} + O(t^{n-1/2}).\end{aligned}$$

Here we notice that the sums involving the binomial coefficients can actually be calculated explicitly. By plugging $x = 1$ into the expansion of $(1-x)^n$ we get that $1-\binom{n}{1}+\binom{n}{2}-\cdots = 0$, which shows that the term with the row-of-teeth function disappears. By integrating over $[0,1]$ the expansion of $(1-x^2)^n$ we get

$$1-\binom{n}{1}/3+\binom{n}{2}/5-\cdots = \int_0^1 (1-x^2)^n\,dx = \int_0^{\pi/2}\sin^{2n+1}u\,du = \frac{(2n)!!}{(2n+1)!!};$$

see [GR, 3.621.4, p. 412]. This gives

$$\sum_{0<c\le\sqrt{t}} c^n\alpha^n = t^{n+1/2}\frac{2^n n!n!}{(2n+1)!} + \frac{t^n}{2^{n+1}}(-1-\beta) + O(t^{n-1/2}). \tag{9}$$

In the second summation in (8) we use $\frac{1}{n} = \int_0^1 (1-x)^{n-1}\,dx = \sum_{i=0}^{n-1}\frac{(-1)^i}{i+1}\binom{n-1}{i}$ to get

(10)

$$\begin{aligned}\sum_{0<c\le\sqrt{t}} c^n\alpha^{n-1} &= \sum_{0<c\le\sqrt{t}} c^n\Big\{\left(\frac{t}{2c}\right)^{n-1} - \binom{n-1}{1}\left(\frac{t}{2c}\right)^{n-2}\left(\frac{c}{2}+\frac{\beta}{2}\right)+\cdots\Big\}\\ &=\frac{t^n}{n2^n} + O(t^{n-1/2}).\end{aligned}$$

For the fourth summation in (8), using $\alpha \le t/(2c)$, we have

$$\sum_{0<c\le\sqrt{t}} c^n\alpha^{n-2} = O\left(\sum_{0<c\le\sqrt{t}} t^{n-2}c^2\right) = O(t^{n-1/2}). \tag{11}$$

Substituting the results from (9), (10) and (11) back into (8), we have

$$\begin{aligned}N_2(t) =& \frac{2^n n!}{(2n+1)!b_1b_2...b_n}t^{n+\frac{1}{2}} - \frac{t^n}{2^{n+1}n!b_1b_2...b_n} \\ &- \sum_{\substack{0<c\le\sqrt{t} \\ k_1,\dots,k_{n-1}}} c^n\psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) + O(t^{n-\frac{1}{2}}).\end{aligned}$$

Since $N_{II}(t) = 2N_2(g_{2n+1}t/(4\pi^2))$ and $b_j = d_j^2 g_{2n+1}/(2\pi)$, we have proved that

$$\begin{aligned}N_{II}(t) =& t^{n+1/2}\frac{\sqrt{g_{2n+1}}2^{n+1}n!}{(2\pi)^{n+1}(2n+1)!d_1^2d_2^2...d_n^2} \\ &- t^n\frac{1}{(2\pi)^n 2^n n! d_1^2d_2^2...d_n^2} - R(t) + O(t^{n-1/2}),\end{aligned}$$

where

$$R(t) = \sum_{0<c\le\sqrt{t}} c^n \sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-1}}\psi\left(\frac{\alpha - s_{n-1}}{b_n}\right). \tag{12}$$

On the other hand, we denote the spectral counting function, corresponding to type I eigenvalues, by $N_I(t)$. Since $N_I(t)$ represents the spectral counting function of the $2n$-dimensional torus T equipped with the metric h, we have

$$N_I(t) = \frac{\pi^n}{n!}\sqrt{\det(h)}\frac{t^n}{(2\pi)^{2n}} + O(t^{n-1/2}) = \frac{1}{2^{2n}\pi^n n! d_1^2d_2^2...d_n^2}t^n + O(t^{n-1/2}). \tag{13}$$

Therefore, if $N(t)$ denotes the spectral counting function of the Heisenberg manifold (M,g), then $N(t) = N_I(t) + N_{II}(t)$. From (12) and (13), we have

$$N(t) = t^{n+1/2}\frac{\sqrt{g_{2n+1}}2^{n+1}n!}{(2\pi)^{n+1}(2n+1)!d_1^2d_2^2...d_n^2} - R(t) + O(t^{n-1/2}),$$

where $R(t)$ is defined by (12). Since $\mathrm{vol}(H_n/\Gamma) = \sqrt{\det(h)\cdot g_{2n+1}}$, we get the correct constant in the main term in Weyl's law for a $(2n+1)$-dimensional manifold.

3.2. **Proof of Theorem 1.1.** Suppose that b_{n-1}/b_n is a rational number, i.e. $b_{n-1}/b_n = p_{n-1}/q_{n-1}$ where p_{n-1} and q_{n-1} are two positive integers such that $(p_{n-1}, q_{n-1}) = 1$. Then, using the fact that $\psi(u)$ has period 1, we get

$$\begin{aligned}\sum_{k_{n-1}}\psi((\alpha - s_{n-1})/b_n) &= \sum_{k_{n-1}}\psi\left(\frac{\alpha - s_{n-2} - b_{n-1}k_{n-1}}{b_n}\right) \\ &= \sum_{j=0}^{q_{n-1}-1}\psi\left(\frac{\alpha - s_{n-2} - jb_{n-1}}{b_n}\right)\times\left(\left[\frac{\alpha - s_{n-2}}{q_{n-1}b_{n-1}}\right] + O(1)\right).\end{aligned}$$

We substitute back into (12). The $O(1)$ term contributes $O(t^{n-3/2})$ as it gives the sum in (8) with 2 fewer variables. We get

$$R(t) = \sum_{\substack{0<c\le\sqrt{t} \\ k_1,\dots,k_{n-2}}}\sum_{j=0}^{q_{n-1}-1} c^n\left(\frac{\alpha - s_{n-2}}{q_{n-1}b_n}\right)\psi\left(\frac{\alpha - s_{n-2} - jb_{n-1}}{b_{n-1}}\right) + O(t^{n-\frac{3}{2}}). \tag{14}$$

Without loss of generality, we continue with estimating the first summation on the right-hand side of (14) with $j = 0$:

$$\sum_{k_{n-2}} c^n \,(\alpha - s_{n-2})\, \psi(\frac{\alpha - s_{n-2}}{b_n}) = \sum_{k_{n-2}} c^n(\alpha - s_{n-3})\psi(\frac{\alpha - s_{n-2}}{b_n}) - \sum_{k_{n-2}} c^n b_{n-2}k_{n-2}\psi(\frac{\alpha - s_{n-2}}{b_n}). \tag{15}$$

To evaluate the first term on the right-hand side of (15), we proceed as in (14). That is, since b_{n-2}/b_n is a rational number, we can write it as $b_{n-2}/b_n = p_{n-2}/q_{n-2}$, where p_{n-2} and q_{n-2} are two relatively prime, positive integers. So

$$\sum_{k_{n-2}} \psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) = \sum_{j=0}^{q_{n-2}-1} \psi\left(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n}\right) \times \left(\left[\frac{\alpha - s_{n-3}}{q_{n-2}b_{n-2}}\right] + O(1)\right). \tag{16}$$

For the second term in (15) summation by parts gives

$$\sum_{k_{n-2}} k_{n-2}\psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) = \frac{\alpha - s_{n-3}}{b_{n-2}} \sum_{k_{n-2}} \psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) - \int_1^{\frac{\alpha - s_{n-3}}{b_{n-2}}} \Big(\sum_{0\le k_{n-2}\le x} \psi(\frac{\alpha - s_{n-2}}{b_n})\Big)dx. \tag{17}$$

The first sum on the right-hand side of (17) has been evaluated in (16). The second term is equal to

$$\begin{aligned}
&\int_1^{\frac{\alpha - s_{n-3}}{b_{n-2}}} \Big(\sum_{0\le k_{n-2}\le x} \psi(\frac{\alpha - s_{n-3} - b_{n-2}k_{n-2}}{b_n})\Big)dx \\
&= \int_1^{\frac{\alpha - s_{n-3}}{b_{n-2}}} \Big(\sum_{j=0}^{q_{n-2}-1} \psi(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n})\Big) \times \Big([\frac{x}{q_{n-2}}] + O(1)\Big)dx \\
&= \Big(\sum_{j=1}^{q_{n-2}-1} \psi(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n})\Big) \times \Big(\frac{1}{2q_{n-2}}\left(\frac{\alpha - s_{n-3}}{b_{n-2}}\right)^2 + O(\alpha)\Big).
\end{aligned}$$

Taking the results from the last equation and (16), (17) back into (15), we have proved that

$$\begin{aligned}
&\sum_{\substack{0<c\le\sqrt{t}\\ k_1,\dots,k_{n-2}}} c^n \,(\alpha - s_{n-2})\, \psi\left(\frac{\alpha - s_{n-2}}{b_n}\right) \\
&= \sum_{\substack{0<c\le\sqrt{t}\\ k_1,\dots,k_{n-3}}} \sum_{j=0}^{q_{n-2}-1} c^n\psi\left(\frac{\alpha - s_{n-3} - jb_{n-2}}{b_n}\right) O((\alpha - s_{n-3})^2).
\end{aligned}$$

We use the last result in (14) to get

$$R(t) = O\left(\sum_{0<c\le\sqrt{t}} c^n \sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-3}} (\alpha - s_{n-3})^2 \left(\psi(\frac{\alpha - s_{n-3}}{b_n})\right)\right).$$

Finally, by induction, after $n-1$ steps, and given $\alpha = t/(2c) - c/2 - \beta/2$ we get

$$\begin{aligned} R(t) =& O\left(\sum_{0<c\leq\sqrt{t}} c^n \alpha^{n-1} \psi(\frac{\alpha}{b_n}) \right) \\ =& O\left(\sum_{0<c\leq\sqrt{t}} c^n (\frac{t}{c} - c - \beta)^{n-1} \psi\left(\frac{t}{2cb_n} - \frac{c}{2b_n} - \frac{\beta}{2b_n} \right) \right). \end{aligned}$$

If (k,l) is an exponent pair [GK], by [GK, Lemma 4.3, p. 39], if $f(x)$ satisfies the properties in the definition of exponent pairs, then

$$\sum_{m\in[a,b]} \psi(f(m)) \ll t^{k/(k+1)} N^{((1-s)k+l)/(k+1)} + t^{-1} N^s. \tag{18}$$

We apply (18) to $f(x) = (tx^{-1} - x - \beta)/(2b_n)$. Using a dyadic decomposition we get

$$\sum_{m\in[2^{-j-1}u, 2^{-j}u]} \psi(f(m)) \ll t^{k/(k+1)} (2^{-j-1}u)^{(-k+l)/(k+1)} + t^{-1}(2^{-j-1}u)^2,$$

for $u \leq \sqrt{t}$. If $k < l$ the series $2^{-j(-k+l)/(k+1)}$ converges and we get the estimate

$$\sum_{m\leq u} \psi(f(m)) \ll t^{k/(k+1)} u^{(l-k)/(k+1)} + t^{-1}u^2 \ll t^{(k+l)/(2k+2)}.$$

This implies, using summation by parts, that

$$R(t) = O(t^{n-1/2+(k+l)/(2k+2)}). \tag{19}$$

The exponent pair $(11/30, 16/30)$ (see [GK]) gives the statement of Theorem 1.1. The conjectural best exponent pairs $(\epsilon, 1/2+\epsilon)$ gives the conjecture 2.

4. Proof of Theorem 1.2

In Theorem 1.2 we assume that at least one of the coefficients ${d_j}^2/{d_n}^2, 1 \leq j \leq n-1$, is irrational. Without loss of generality, we can assume that this happens for $j = n-1$. In fact, obtaining the formula (12) was based on an optional ordering for the summations over $k_1, k_2, \ldots, k_{n-1}$ in (6).

According to Vaaler's theorem [Va] (see also [GK, p.116]), for every positive integer J, there exist constants $\{\gamma_j; 1 \leq |j| \leq J\}$, satisfying the property $|\gamma_j| \ll 1/|j|$, such that for every real number ω,

$$\psi(\omega) - \sum_{1\leq|j|\leq J} \gamma_j e^{2\pi i(j\omega)} \ll \frac{1}{J}. \tag{20}$$

Therefore, by fixing J and taking $\omega = (\alpha - s_{n-1})/b_n$ in Vaaler's theorem, we have

$$\sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n} \right) \ll \sum_{1\leq|j|\leq J} |\gamma_j| \Big| \sum_{k_{n-1}} \exp(2\pi i j(\alpha - s_{n-1})/b_n)) \Big| + \frac{\alpha - s_{n-2}}{J}. \tag{21}$$

To estimate the right-hand side of (21), we have

$$\begin{aligned}\left|\sum_{k_{n-1}} e^{2\pi i(j\frac{\alpha-s_{n-1}}{b_n})}\right| &= \left|e^{2\pi i(j\frac{\alpha-s_{n-2}}{b_n})}\sum_{k_{n-1}} e^{-2\pi i(j\frac{b_{n-1}k_{n-1}}{b_n})}\right| \\ &= \left|\frac{1-e^{-2\pi i(j\frac{b_{n-1}}{b_n})((\frac{\alpha-s_{n-2}}{b_{n-1}})+1)}}{1-e^{-2\pi i(j\frac{b_{n-1}}{b_n})}}\right| \\ &\leq \frac{1}{|\sin(\frac{\pi j b_{n-1}}{b_n})|} \leq \frac{1}{2\|\frac{jb_{n-1}}{b_n}\|},\end{aligned} \tag{22}$$

where, for every real number θ, $\|\theta\|$ is the distance between θ and the nearest integer. For almost all irrational θ there exists a constant K_θ such that $\|j\theta\| \geq \frac{K_\theta}{j\log^2 j}$ for every positive integer j. Therefore, applying this approximation for the right-hand side of (22), we have

$$\Big|\sum_{k_{n-1}} \exp(2\pi ij(\alpha - s_{n-1})/b_n)\Big| \leq \frac{1}{2\|jb_{n-1}/b_n\|} \leq K|j|\log^2 j, \tag{23}$$

where K is a positive constant, dependent on b_{n-1}/b_n.

Substituting (23) in (21), and noticing that $|\gamma_j| \ll 1/|j|$, we obtain

$$\sum_{k_{n-1}} \psi\left(\frac{\alpha - s_{n-1}}{b_n}\right) \ll J\log^2 J + \frac{\alpha - s_{n-2}}{J}. \tag{24}$$

Substituting (24) in (12), we have

$$R(t) \ll \sum_{0<c\leq\sqrt{t}} c^n \sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-2}}\left(J\log^2 J + \frac{\alpha - s_{n-2}}{J}\right). \tag{25}$$

For the last summation on the right-hand side of (25), we have

$$\begin{aligned}\sum_{k_{n-2}}\left(J\log^2 J + \frac{\alpha - s_{n-2}}{J}\right) &= \sum_{k_{n-2}}\left(J\log^2 J + \frac{\alpha - s_{n-3} - b_{n-3}k_{n-3}}{J}\right) \\ &\ll (\alpha - s_{n-3})J\log^2 J + (\alpha - s_{n-3})^2 J^{-1}.\end{aligned} \tag{26}$$

Therefore, by induction, we have

$$\sum_{k_1}\sum_{k_2}\cdots\sum_{k_{n-2}}\left(J\log^2 J + \frac{\alpha - s_{n-2}}{J}\right) \ll \alpha^{n-2}J\log^2 J + \alpha^{n-1}J^{-1}. \tag{27}$$

Substituting (27) in (25) and using (5), we see that

$$\begin{aligned}R(t) &\ll \sum_{0<c\leq\sqrt{t}} c^n(\alpha^{n-2}J\log^2 J + \alpha^{n-1}J^{-1}) \\ &\ll \sum_{0<c\leq\sqrt{t}}\left\{c^n(\frac{t}{c})^{n-2}J\log^2 J + c^n(\frac{t}{c})^{n-1}\frac{1}{J}\right\} \\ &= t^{n-2}\sum_{0<c\leq\sqrt{t}}\big(c^2 J\log^2 J + tcJ^{-1}\big).\end{aligned} \tag{28}$$

We take $J = c^{\rho}$ on the right-hand side of (28) and get

$$\begin{aligned} R(t) \ll & t^{n-2} \sum_{0<c\leq\sqrt{t}} c^{2+\rho}\rho^2 \log^2 c + t^{n-1} \sum_{0<c\leq\sqrt{t}} c^{1-\rho} \\ \ll & \rho^2 t^{n+(-1+\rho)/2} \log^2 t + t^{n-\rho/2}. \end{aligned} \tag{29}$$

So, to optimize the estimate on the right-hand side of (29), we choose $\rho = 1/2 - 2\log\log t/\log t$ and we are done: $R(t) \ll t^{n-1/4}\log t$.

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