THE REMAINDER IN WEYL'S LAW FOR HEISENBERG MANIFOLDS II

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Abstract. We prove that the error term $R(\lambda)$ in Weyl's law is $O(\lambda^{34/41})$ for all three-dimensional Heisenberg manifolds. This result uses the method of exponent pairs. The improved bound $R(\lambda) = O(\lambda^{119/146+\epsilon})$ follows from Huxley's work. We conjecture that $R(\lambda) = O_\epsilon(\lambda^{3/4+\epsilon})$ is a sharp upper bound for Heisenberg three-manifolds based on estimates on exponential sums. We investigate the error term numerically.

1. Introduction

Exponential sums have various applications in analytic number theory. Among them we mention the order of growth of the Riemann zeta function on its critical line, the Dirichlet divisor problem and the Gauss circle problem.

In this article we present an application on the remainder term in Weyl's law for 3-dimensional Heisenberg manifolds. This work is continuation of the paper [PT]. We briefly remind the reader that the Gauss circle problem can be interpreted in this spirit as follows: We consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ acting on functions on T^2, i.e., doubly periodic functions on $\mathbb{R}^2$ with period 1. The standard exponential functions $e^{2\pi i(mx+ny)}$, $m, n \in \mathbb{Z}$ are a basis of eigenfunctions and the eigenvalues are $4\pi^2(m^2 + n^2)$. The spectral counting function

$$N(\lambda) = \#\{\lambda_j \in \mathrm{Spec}(\Delta);\ \lambda_j \leq \lambda\}$$

is the function that counts the number of lattice points of $\mathbb{Z}^2$ inside a circle of radius $\sqrt{\lambda}/(2\pi)$. Consequently the error term of the spectral function is the error term in the Gauss circle problem.

More generally, we would like to study the error term in the asymptotic behavior of $N(\lambda)$ for other manifolds. Let (M^n, g) be a compact Riemannian manifold of dimension n with Laplace-Beltrami operator Δ and spectral counting function

$$N(\lambda)$$

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. Then, a celebrated theorem of Hörmander [HO] asserts that

$$N(\lambda) = c_n \mathrm{vol}(M)\, \lambda^{n/2} + O(\lambda^{(n-1)/2}), \tag{1.1}$$

for some constant c_n depending only on the dimension. Moreover, the estimate in (1.1) is sharp as can be seen by considering the round sphere, S^n. The question of determining the optimal bound for the error term

$$R(\lambda) = N(\lambda) - c_n \mathrm{vol}(M) \lambda^{n/2}$$

in any given example is a difficult problem which depends on the properties of the associated geodesic flow, and is far from being understood in detail.

The purpose of this article is to study Weyl's law for 3-dimensional Heisenberg manifolds. In [PT] we studied the error term for a specific metric on a Heisenberg manifold and proved the estimate $R(\lambda) = O(\lambda^{5/6} \log \lambda)$. In this work we prove the estimate $R(\lambda) = O(\lambda^{34/41})$ for *all* metrics. The important point is the reduction of the problem to a lattice-point counting problem with weights. Then the techniques on exponential sums come to play a role.

Heisenberg manifolds have interesting features: on the dynamical side they have integrable geodesic flows [Bu], on the analysis side they are together with other nilmanifolds among the few examples where the eigenvalues of the Laplace operator can be explicitly computed, see [GW], [Pe], [DS]. They have played an important role in the isospectral problem, see the work of Gordon, Wilson, DeTurck, [Go], [GW], [DTG], [DGGW].

We introduce notation and state our results. The 3-dimensional Heisenberg group H_1 consists of all matrices of the form

$$\gamma(x, y, t) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, t \in \mathbb{R}.$$

We are interested in the spectrum of Heisenberg manifolds. These are defined as $(\Gamma \setminus H_1, g)$, where Γ is a discrete subgroup of H_1 with compact quotient and where g is a left H_1-invariant metric. The classification theorem in [GW, 2.4] allows us to restrict our attention to subgroups Γ_r of the following type

$$\Gamma_r = \{\gamma(x, y, t) : x \in r\mathbb{Z}, y \in \mathbb{Z}, t \in \mathbb{Z}\}.$$

The left invariant metrics on H_1 are determined by the induced inner product on the Lie algebra $\mathcal{H}_1$. We can replace the metric g with $\phi^* g$, where ϕ is an inner automorphism, in such a way that the direct sum split of the Lie algebra $\mathcal{H}_1 = \mathbb{R}^2 + \mathfrak{z}$ is orthogonal, see [GW, 2.6(b)]. Here $\mathfrak{z}$ is the center of the Lie algebra and

$$\mathbb{R}^2 \equiv \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y \in \mathbb{R} \right\}.$$

With respect to this orthogonal split of $\mathcal{H}_1$ the metric g has the form

$$g = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{12} & h_{22} & 0 \\ 0 & 0 & g_3 \end{pmatrix}$$

where $g_3 > 0$ and $h_{11}h_{22} - h_{12}^2 > 0$. We set

$$\mathcal{L}_r = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x \in r\mathbb{Z}, y \in \mathbb{Z} \right\}.$$

The spectrum of the Laplace operator associated with this metric consists of two parts, see [GW, p. 258]:

(1) Type I eigenvalues: these are eigenvalues of the torus $\mathcal{L}_r \setminus \mathbb{R}^2$ with metric given by the matrix $h = (h_{ij})$, $i, j = 1, 2$, see [GW, Lemma 3.4].
(2) Type II eigenvalues: $\Sigma_2 = \{\mu(c,k) = 4\pi^2 g_3^{-1} c^2 + 2\pi d^2 c(2k+1);\ c, k \in \mathbb{Z}, c > 0, k \geq 0\}$. Here d is determined through the property that $\pm i d^2$ are the eigenvalues of $h^{-1}J$, where J is the standard symplectic 2×2 matrix. These eigenvalues have to be counted with multiplicity as follows: for every $c > 0$, the $\mu(c,k)$ is counted with multiplicity $2cr$ and, if it happens that we get the same eigenvalue from different pairs (c,k), the multiplicities are added.

It is the Type II eigenvalues that contribute the main term in Weyl's law. In the result below both Type I and Type II eigenvalues contribute to the improvement of Weyl's law.

Theorem 1.1. *The Heisenberg manifold $M = (\Gamma_r \setminus H_1, g)$ has Weyl Law*

$$N(\lambda) = c_3 vol(M) \lambda^{3/2} + O(\lambda^{34/41}).$$

Here $c_3 = (6\pi^2)^{-1}$ and $\mathrm{vol}(M) = r\sqrt{\det g}$, see [GW, Prop. 2.9]. The result in Theorem 1.1 is not optimal. In fact our analysis shows the stronger theorem:

Theorem 1.2. *If (k,l) is an exponent pair different from $(1/2, 1/2)$, then*

$$R(\lambda) = O(\lambda^{\frac{l+2k+1}{2k+2}}).$$

Theorem 1.1 follows using the exponent pair $(k, l) = (11/30, 16/30)$. For the definition of exponent pairs and their use is estimating exponential sums we include a discussion and refer the reader to [MO, pp. 56–60] and [GK]. Using the latest progress on estimating exponential sums, due to Huxley [HU] we can improve the result in theorem 1.1 to

$$R(\lambda) \ll \lambda^{119/146+\epsilon}$$

. Using the conjectural bounds for exponential sums, i.e., the conjectural exponent pair $(k, l) = (\epsilon, 1/2+\epsilon)$ for every $\epsilon > 0$ we are lead to the following conjecture:

Conjecture 1.3. The pointwise estimate $R(\lambda) = O_\delta(\lambda^{3/4+\delta})$ is sharp for 3-dimensional Heisenberg manifolds.

We provide numerical evidence for this conjecture.

2. Proof of the Theorems

Set l any positive number. Set $\psi(u) = u - [u] - 1/2$. We set

$$N(t) = \sum_{(x,y)\in\mathbb{Z}_+^2,\, y(y+l^{-1}x)\le t} 2y, \quad N_L(t) = \sum_{(x,y)\in\mathbb{Z}_+^2,\, y(y+2l^{-1}x)\le t} 2y.$$

It is important that we compute two-term asymptotic expansions, since we must see the cancellation of the λ^1 term that comes out of the counting function for Type I eigenvalues.

$$N(t) = \sum_{y\le\sqrt{t}} \sum_{x\le(ty^{-1}-y)l} 2y = \sum_{y\le\sqrt{t}} 2y[(ty^{-1}-y)l] \tag{2.1}$$

It follows from the Euler summation formula [FR, Satz 3, p. 187] that

$$\sum_{n\le u} n^a = \frac{u^{a+1}}{a+1} - \psi(u)u^a + O(u^{a-1}). \tag{2.2}$$

Using (2.2), we easily get that

$$\begin{aligned} N(t) &= \sum_{y\le\sqrt{t}} 2y(ty^{-1}-y)l - \sum_{y\le\sqrt{t}} y - \sum_{y\le\sqrt{t}} 2y\psi((ty^{-1}-y)l) \\ &= \sum_{y\le\sqrt{t}} (2tl - 2ly^2) - \left(\frac{t}{2} - \psi(\sqrt{t})\sqrt{t} + O(1)\right) - E(t) \end{aligned} \tag{2.3}$$

where we call the sum on right hand of (2.3) by $E(t)$. We get

$$\begin{aligned} N(t) &= 2tl(\sqrt{t}-\psi(\sqrt{t})-1/2)-2l\left(\frac{t^{3/2}}{3}-\psi(\sqrt{t})t+O(\sqrt{t})\right) \\ &\quad -E(t)-\frac{t}{2}+O(t^{1/2}) \qquad (2.4) \\ &= \frac{4lt^{3/2}}{3}-(l+1/2)t-E(t)+O(\sqrt{t}). \end{aligned}$$

Clearly the estimate (2.4) implies

$$N_L(t)=\frac{2lt^{3/2}}{3}-(l/2+1/2)t-E_L(t)+O(\sqrt{t}), \tag{2.5}$$

where

$$E_L(t)=\sum_{y\le\sqrt{t}} 2y\psi((ty^{-1}-y)l/2).$$

We take $l=2\pi/(g_3d^2)$, $t=g_3\lambda/(4\pi^2)$. Equations (2.4) and (2.5) imply for the spectral counting function $N_{II}(\lambda)$ for Type II eigenvalues:

$$\begin{aligned} N_{II}(\lambda) &= rN(g_3\lambda/(4\pi^2))-rN_L(g_3\lambda/(4\pi^2)) \\ &= \frac{r\sqrt{g_3}}{6\pi^2d^2}\lambda^{3/2}-\frac{r}{4\pi d^2}\lambda-E(t)+E_L(t)+O(\sqrt{\lambda}) \qquad (2.6) \end{aligned}$$

We notice that $E(t)$ and $E_L(t)$ are weighted sums of the ψ function with monotone weight. Present methods do not give any saving when we average over the length of the sum. A summation by parts, see the next subsection, shows that the way to estimate the weighted sum is to take the maximum of the weight times the maximum of the sums of *psi* over the subinterval. In particular, an application of van der Corput's method shows that

$$\max(|E(t)|,|E_L(t)|)\ll t^{5/6}\log t. \tag{2.7}$$

On the other hand the eigenvalues of type I give

$$N_I(\lambda)=\mathrm{vol}(\mathcal{L}_r\setminus\mathbb{R}^2)\frac{\lambda}{4\pi}+O(\lambda^{1/2}).$$

The volume of the torus is $r\sqrt{\det(h)}$. Because $\pm id^2$ are the eigenvalues of $h^{-1}J$, we have $d^4=\det\,(h)^{-1}$, and the volume of the torus is rd^{-2}. We see that the λ terms cancel and we get as improvement in the Weyl Law $R(\lambda)\ll\lambda^{5/6}\log\lambda$. Although this is a weaker result than Theorem 1.1, we include it, as van der Corput's method is more easily accessible.

Remark 2.1. We also see that $r\sqrt{g_3}d^{-2}$ is the volume of the Heisenberg manifold.

2.1. **Application of Van der Corput's method.** We show only the bound $E(t) \ll t^{5/6}\log t$. The bound for $E_L(t)$ follows immediately from this.

A summation by parts gives

$$E(t) = 2\sqrt{t}\sum_{n=1}^{\sqrt{t}} \psi\left((tn^{-1}-n)l\right) - \int_1^{\sqrt{t}} \sum_{n\le x} \psi\left((tn^{-1}-n)l\right)\, dx. \tag{2.8}$$

We need to show that the sums in (2.8) are $\ll t^{1/3}\log t$ for $x \le \sqrt{t}$. The main point in Van der Corput's method can be summarized in the following proposition, see [FR, Satz 1, p. 41]

Proposition 2.2. Let $f(u)$ be a twice-differentiable function on the interval $[a,b]$ and satisfies either $f''(u) \ge \lambda$ for all $u \in [a,b]$, or, $f''(u) \le -\lambda$ for all $u \in [a,b]$, where $0 < \lambda \le 1$. Then

$$\sum_{a\le l\le b} \psi(f(l)) \ll |f'(b) - f'(a)|\lambda^{-2/3} + \lambda^{-1/2}, \tag{2.9}$$

with the implied constants being absolute.

Remark 2.3. The relation of $\psi(u)$ with exponential sums comes through the Fourier series

$$\psi(u) = -\sum_{k\ne 0} \exp(2\pi iku)/(2\pi ik).$$

This transforms the sum in (2.9) into a sum

$$\sum_{a\le l\le b} e(f(l)).$$

With the help of the Euler-Maclaurin formula this is transformed to an integral

$$\int_a^b e((f(x))\, dx \ll \lambda^{-1/2},$$

if $|f''| \ge \lambda > 0$ on $[a,b]$, by van der Corput's Lemma.

We set $f(u) = (tu^{-1} - u)l$, so that $f'(u) = -tu^{-2} - l$ and $f''(u) = 2t/u^3 \ge 2t/b^3$ for $a \le u \le b$. Theorem 2.2 gives for $2t/b^3 \le 1$

$$\begin{aligned} \sum_{a\le m\le b} \psi\left((tm^{-1}-m)l\right) &\ll \left(\frac{t}{4a^2} - \frac{t}{4b^2}\right)\left(\frac{2t}{b^3}\right)^{-2/3} + \left(\frac{2t}{b^3}\right)^{-1/2} \\ &\ll t^{1/3}(a^{-2} - b^{-2})b^2 + t^{-1/2}b^{3/2}. \end{aligned} \tag{2.10}$$

We choose L to be the largest integer with $2^{-L}x \geq t^{1/3} \geq x^{2/3}$, i.e., $L \leq [\log x/(3\log 2)]$. In particular $L \ll \log x \ll \log t$. We have, using a dyadic decomposition,

$$\begin{aligned}
\sum_{n\leq x} \psi\left((tn^{-1}-n)l\right) &= \sum_{n\leq 2^{-L}x} \psi\left(f(n)\right) + \sum_{l=1}^{L} \sum_{2^{-l}x\leq n\leq 2^{-l+1}x} \psi\left(f(n)\right) + O(L) \\
&= O(2^{-L}x) + \sum_{l=1}^{L} \sum_{2^{-l}x\leq n\leq 2^{-l+1}x} \psi\left((tn^{-1}-n)l\right) + O(L) \\
&\ll t^{1/3} + \sum_{l=1}^{L} t^{1/3}\left(x^{-2}-2^{-2}x^{-2}\right)2^2x^2 + t^{-1/2}(2^{-l+1}x)^{3/2} \\
&\ll t^{1/3}\log t.
\end{aligned}$$

using (2.10) to estimate the inner sum in the second line of (2.11). Also we used that

$$\lambda = \frac{2t}{b^3} = \frac{2t}{(2^{-l+1}x)^3} \leq \frac{2t}{8(t^{1/3})^3} \leq \frac{1}{4} \leq 1.$$

2.2. **The method of exponent pairs.** This method is due to Phillips' [PH], [GK, p. 30–31]. It treats exponential sums where $(a,b) \subset [N,2N]$, and where $f'(x)$ is approximately tx^{-s} for some $t>0$ and $s>0$. More precisely, we define $\mathbf{F}(N,P,s,t,\epsilon)$ to be the set of functions f such that f is defined and has P continuous derivatives on $[a,b]\subset[N,2N]$ and such that the following estimate holds for $0\leq p\leq P-1$, $a\leq x\leq b$:

(2.11)
$$|f^{(p+1)}(x)-(-1)^p s(s+1)\cdots(s+p-1)tx^{-s-p}| < \epsilon s(s+1)\cdots(s+p-1)tx^{-s-p}.$$

Definition 2.4. Let k and l be real numbers such that $0\leq k\leq 1/2\leq l\leq 1$. Suppose that for every $s>0$, there is some $P=P(k,l,s)$ and some $\epsilon=\epsilon(k,ls)<1/2$ such that for every $N>0$, every $t>0$ and every $f\in\mathbf{F}(N,P,s,t,\epsilon)$, the estimate

$$\sum_{n\in(a,b]} e(f(n)) \ll_{k,l,s} (tN^{-s})^k N^l + t^{-1}N^s \tag{2.12}$$

holds. Then we say that (k,l) is an exponent pair.

It is known [GK, p. 21], [MO, p. 46–59] that there exist two processes called A and B which produce from the exponent pair (k,l) two new exponent pairs

$$A(k,l) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right)$$

and

$$B(k,l) = (l-1/2, k+1/2).$$

A repeated application of the two processes produces the exponent pair $BA^3B(0,1) = (11/30, 16/30)$, from the trivial exponent pair $(0,1)$. Van der Corput's estimate (2.7) corresponds to the exponent pair $AB(0,1) = (1/6, 2/3)$. Process A is based on the Weyl-van der Corput's inequality, while process B is based on stationary phase and the Poisson summation formula. The main application of exponent pairs to sums of the $\psi(u)$ function can be summarized in the proposition, see [GK, Lemma 4.3, p. 39]:

Proposition 2.5. Suppose that (k,l) is an exponent pair. If $f \in \mathbf{F}(N, P, s, t, \epsilon)$ and f is defined on $[a,b] \subset [N, 2N]$, then

$$\sum_{n\in[a,b]} \psi(f(n)) \ll t^{k/(k+1)} N^{((1-s)k+l)/(k+1)} + t^{-1}N^s. \tag{2.13}$$

We apply this proposition to $f(x) = (tx^{-1} - x)l$. We easily check that $f \in \mathbf{F}(N, 2, 2, tl, \epsilon)$ for every ϵ and every t sufficiently large, since $f'(x) = (tx^{-2} - 1)l$, $f''(x) = -2tlx^{-3}$. Using a dyadic decomposition as in the application of van der Corput's method above we get

$$\sum_{n\in[2^{-j-1}u, 2^{-j}u]} \psi(f(n)) \ll t^{k/(k+1)} (2^{-j-1}u)^{(-k+l)/(k+1)} + t^{-1}(2^{-j-1}u)^2,$$

for $u \leq \sqrt{t}$. If $k < l$ the series $2^{-j(-k+l)/(k+1)}$ converges and we get the estimate

$$\sum_{n\leq u} \psi(f(n)) \ll t^{k/(k+1)} u^{(l-k)/(k+1)} + t^{-1}u^2 \ll t^{(k+l)/(2k+2)}.$$

This completes the proof of Theorem 1.2.

Remark 2.6. The exponent pair we used $(k,l) = (11/30, 16/30)$ is not the best known. Based on the work of Huxley [HU], which provides the best estimates known for the lattice counting problem, we can estimate

$$\sum_{n\leq x} \psi(f(n)) = O(\lambda^{23/73} \log^A \lambda)$$

for $x \leq \lambda^{1/2}$. This gives $R(\lambda) \ll \lambda^{119/146} \log^B \lambda$.

Remark 2.7. Conjecture 2 in [MO, p. 59] states that $(\epsilon, 1/2+\epsilon)$ is an exponent pair for every $\epsilon > 0$. This gives the conjecture

$$R(\lambda) \ll \lambda^{3/4+\epsilon}$$

for every left-invariant metric on a 3-dimensional Heisenberg manifold. We remark that $5/6 = 0.833333$, $34/41 = 0.8292682$, $119/146 = 0.8150684932$ and $3/4 = 0.75$.

3. Numerical investigation

In this section we include numerical investigations for the error term in Weyl's law for the Heisenberg manifold $(\Gamma_1 \setminus H_1, g_1)$, where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\pi \end{pmatrix}.$$

In [PT] we explained how the spectral function for this metric is related to counting the number of lattice points under the hyperbola $xy = c$ and below the line $y = x$ with weight the y coordinate. One considers the standard lattice $\mathbb{Z}^2$ and the lattice $L = \{(x, y) \in \mathbb{Z}^2, x \equiv y \bmod 2\}$. As we explained above we expect the error to be of order $t^{3/4+\epsilon}$ in both cases. We show both the absolute error $E(t)$ and $E_L(t)$ and the relative error $E(t)/t^{3/4}$ and $E_L(t)/t^{3/4}$. For comparison we include the numerics for the error term $\Delta(x)$ in the Dirichlet divisor problem:

$$\sum_{n \leq x} \tau(x) = x \log x + (2\gamma - 1)x + \Delta(x).$$

Here $\tau(n)$ is the number of divisors of n and the sum counts the number of lattice points in the first quadrant below the hyperbola $x_1 x_2 = x$. Here the conjectural bound is $\Delta(x) = O_\epsilon(x^{1/4+\epsilon})$. We also include the numerics for Gauss' circle problem

$$\left| \{(x, y) \in \mathbb{Z}^2, x^2 + y^2 \leq t\} \right| = \pi t + R(t)$$

where $R(t)$ is also expected to be of order $t^{1/4+\epsilon}$.

We also show the histograms showing the distribution of the normalized errors for the standard lattice, the lattice L, the Dirichlet divisor problem and the Gauss circle problem. Heath-Brown [HB] proved that in the last two problems the normalized error has a distribution function. We hope to address this issue for Heisenberg manifolds in future work.

All the numerical investigations were performed on Matlab.

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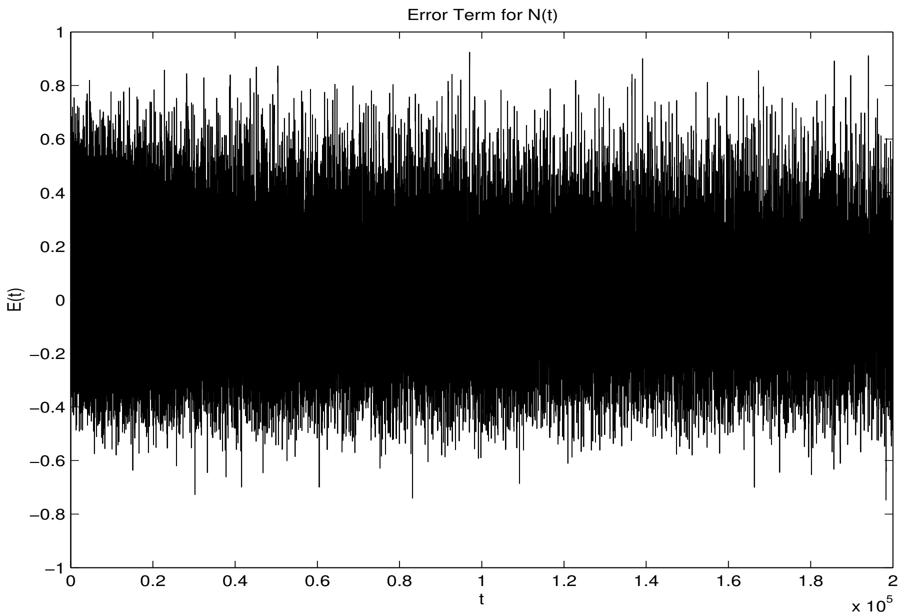

FIGURE 1. The relative error for the standard lattice $E(t)/t^{3/4}$

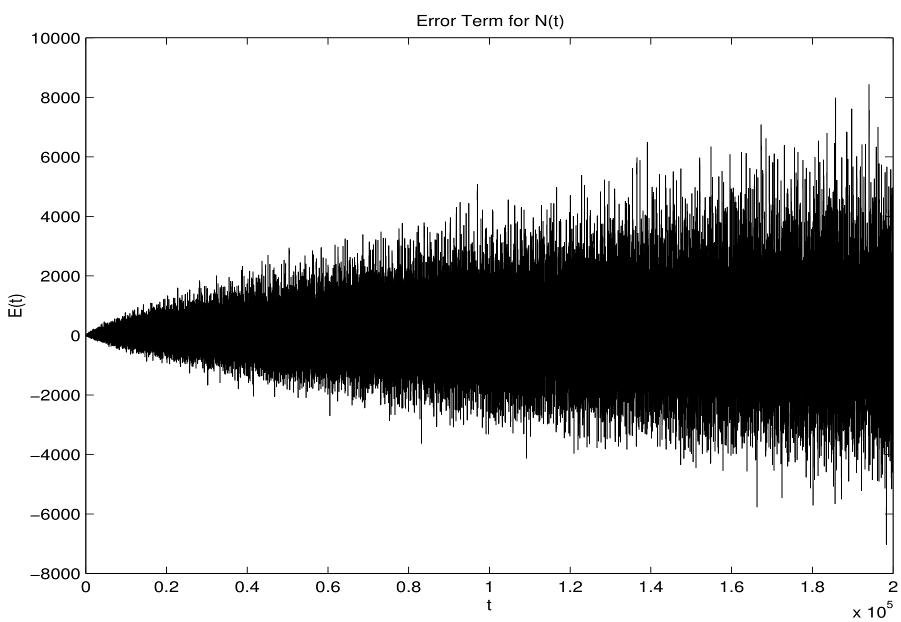

FIGURE 2. The absolute error for the standard lattice

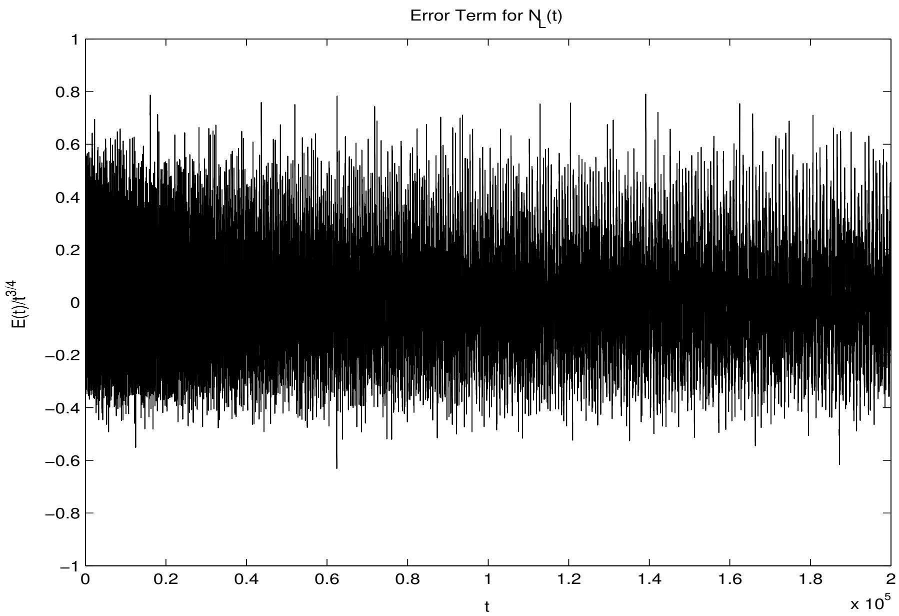

FIGURE 3. The relative error for the lattice L

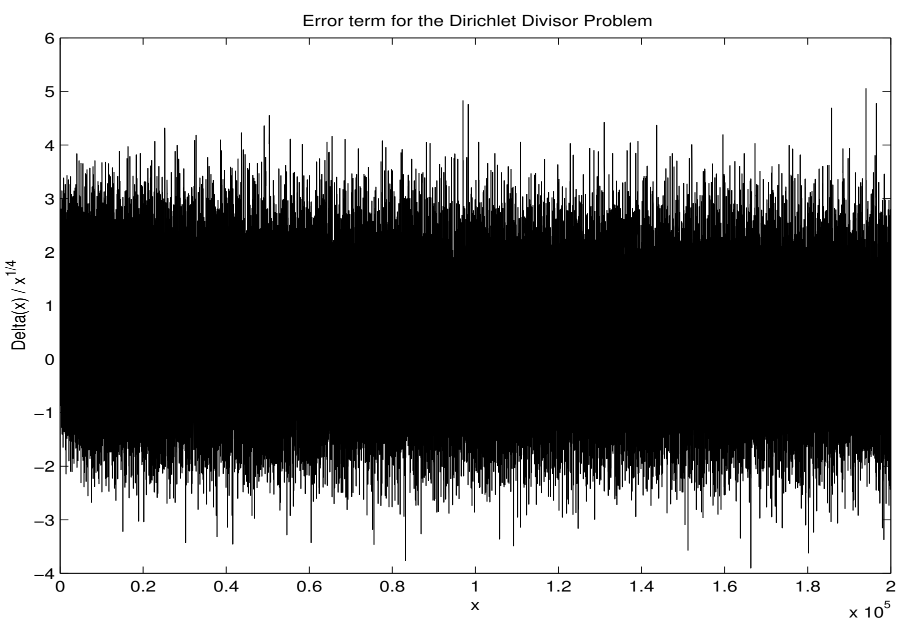

FIGURE 4. The relative error in the Dirichlet divisor problem

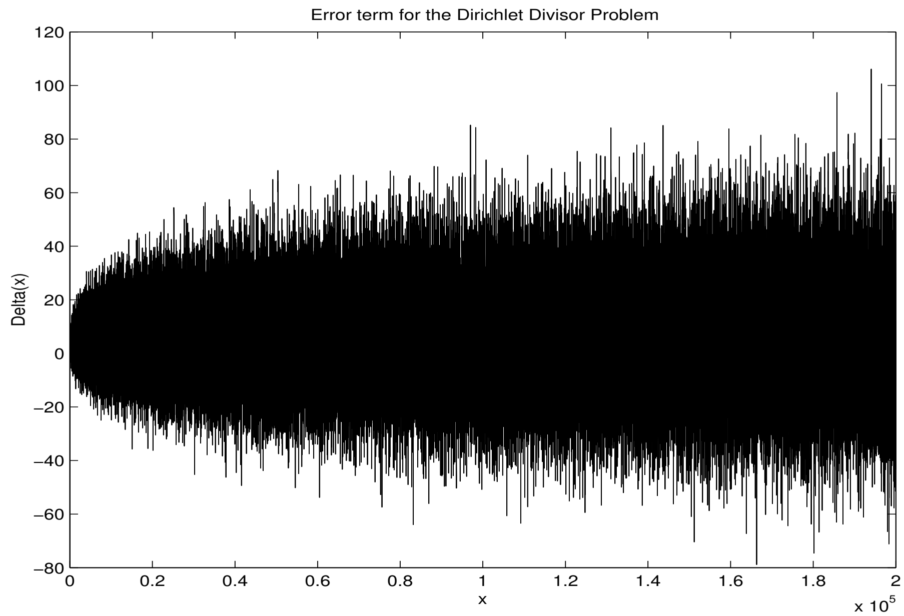

FIGURE 5. The absolute error in the Dirichlet divisor problem

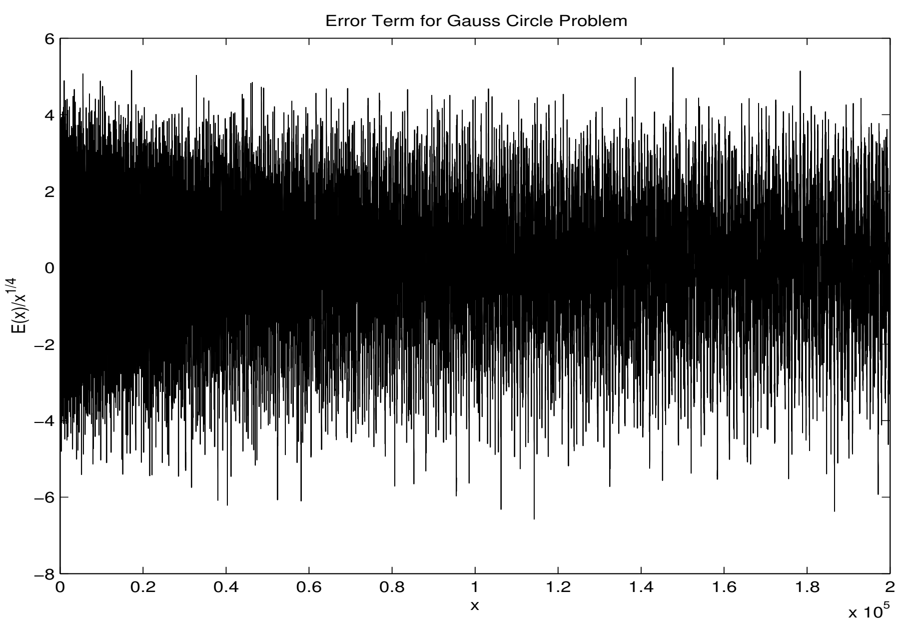

FIGURE 6. The relative error term in Gauss' circle problem

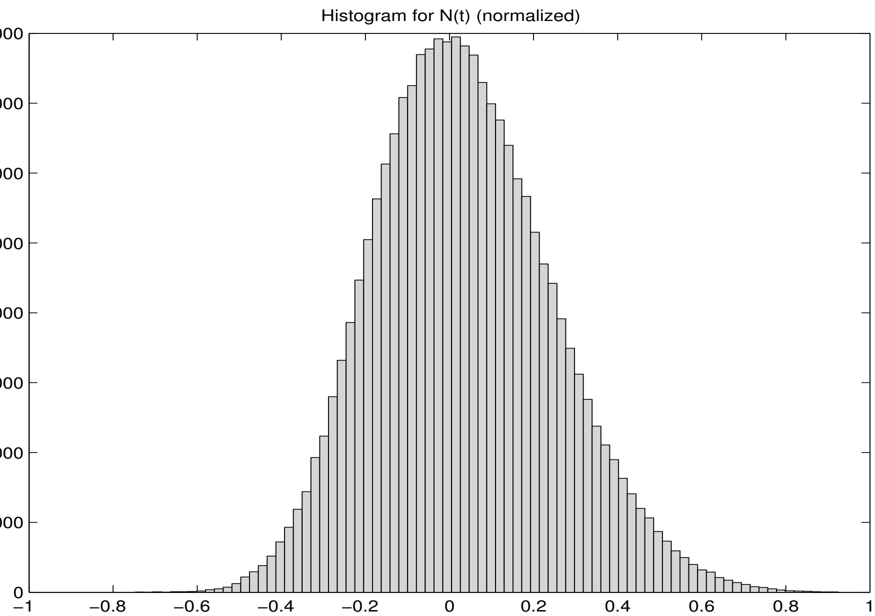

FIGURE 7. The Histogram for the standard lattice

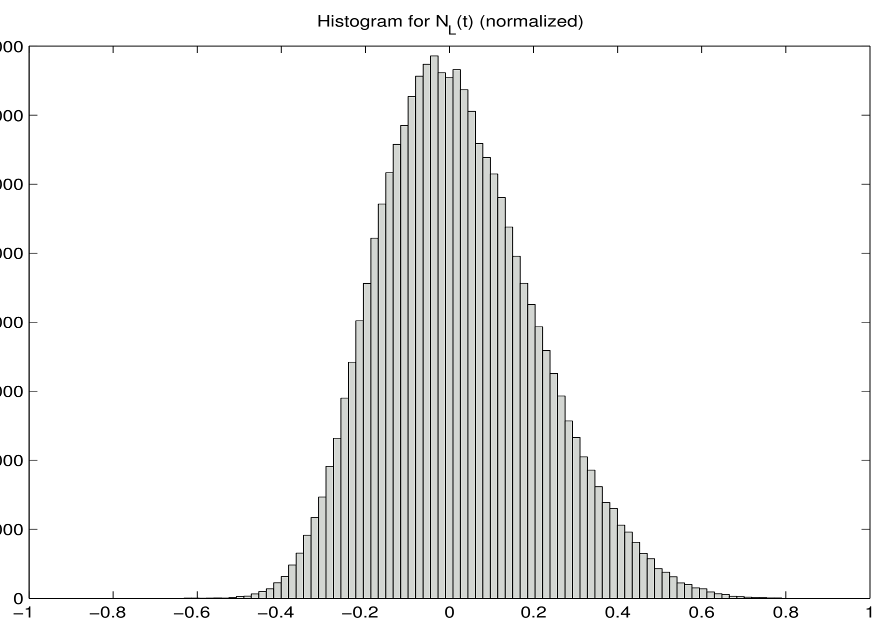

FIGURE 8. Histogram for the Lattice L

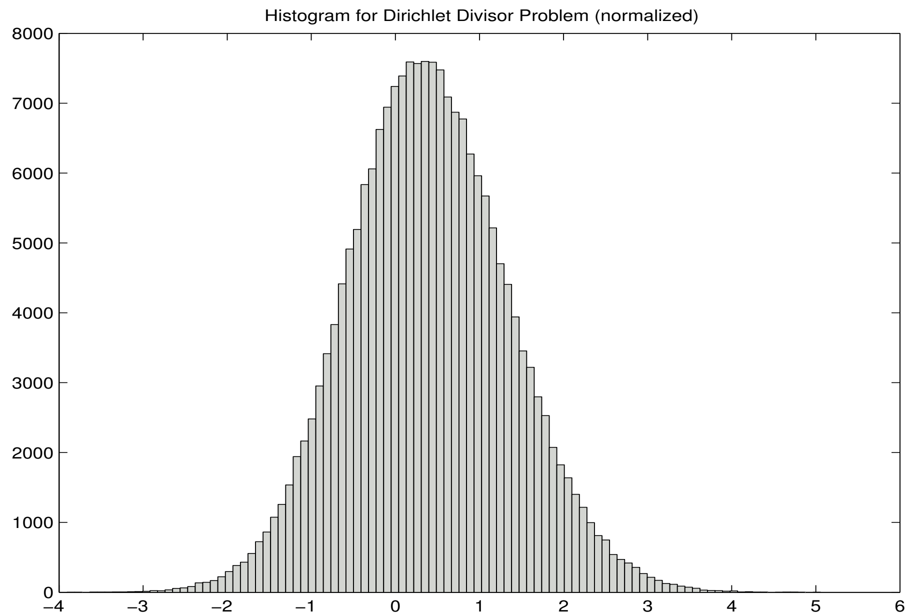

FIGURE 9. Histogram for Dirichlet's Divisor problem

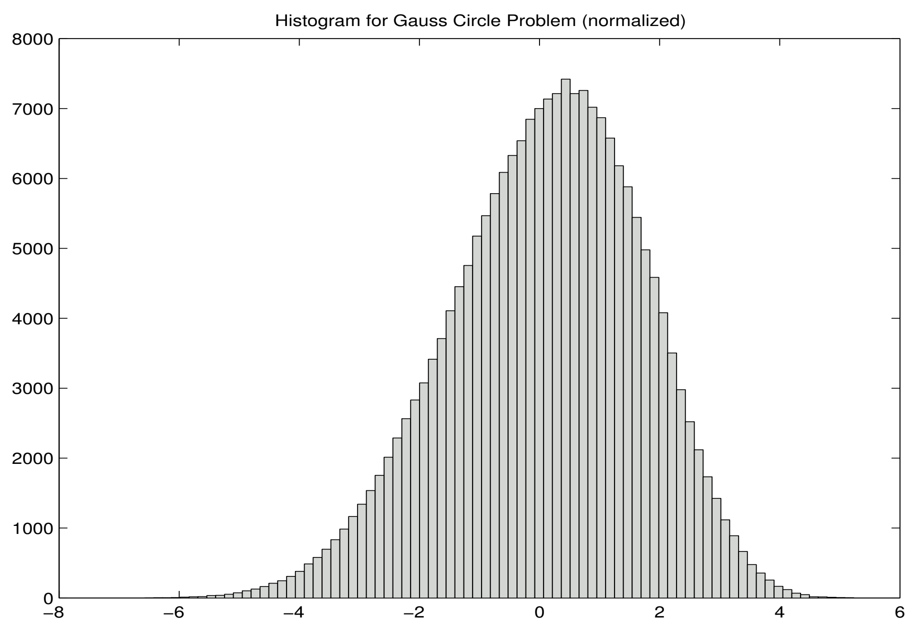

FIGURE 10. Histogram for Gauss' circle problem

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