DISSOLVING CUSP FORMS: HIGHER ORDER FERMI'S GOLDEN RULES

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Abstract. For a hyperbolic surface embedded eigenvalues of the Laplace operator are unstable and tend to become resonances. A sufficient dissolving condition was identified by Phillips–Sarnak and is elegantly expressed in Fermi's Golden Rule. We prove formulas for higher approximations and obtain necessary and sufficient conditions for dissolving a cusp form with eigenfunction u_j into a resonance. In the framework of perturbations in character varieties, we relate the result to the special values of the L-series $L(u_j \otimes F^n, s)$. This is the Rankin-Selberg convolution of u_j with $F(z)^n$, where $F(z)$ is the antiderivative of a weight 2 cusp form.

1. Introduction

For a hyperbolic surface with cusps the embedded eigenvalues of the Laplace operator Δ in the continuous spectrum are unstable. This is manifested by Fermi's Golden Rule developed in [33]. We describe the result in the simplest case of a surface with one cusp and an eigenvalue of multiplicity one. Let $\lambda_j = 1/4 + r_j^2$ be an embedded eigenvalue with $r_j \in \mathbb{R} \setminus \{0\}$ and the corresponding L^2-normalized eigenfunction (Maaß cusp form) $u_j(z)$. Let $E(z, s)$ be the Eisenstein series, which on the critical line $\Re(s) = 1/2$ is a generalized eigenfunction for Δ, so that $E(z, 1/2 + ir_j)$ corresponds in the same eigenvalue as the Maaß cusp form. We set $s_j = 1/2 + ir_j$. In [32] Phillips and Sarnak identified a condition that turns λ_j into a resonance in Teichmüller space, i.e. dissolving λ_j into a resonance. In [38] Sarnak identified a similar condition for character varieties. Let $\Delta^{(1)}$ denote the infinitesimal variation of the family of Laplacians in either perturbation. Then the dissolving condition – usually called the Phillips–Sarnak condition – is

$$\langle \Delta^{(1)} u_j, E(z, 1/2 + ir_j) \rangle \neq 0. \tag{1.1}$$

In [33] Phillips and Sarnak identified the dissolving condition in terms of the speed that the cuspidal eigenvalue leaves the line $\Re(s) = 1/2$ to become a resonance to the left half-plane. If $s_j(\epsilon)$ denotes the position of the resonance

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or embedded cusp form, with perturbation series

$$s_j(\epsilon) = s_j + s_j^{(1)}(0)\epsilon + \frac{s_j^{(2)}(0)}{2!}\epsilon^2 + \cdots, \tag{1.2}$$

then

$$\Re s_j^{(2)}(0) = -\frac{1}{4r_j^2}\left|\langle \Delta^{(1)}u_j, E(z, 1/2 + ir_j)\rangle\right|^2. \tag{1.3}$$

Our aim in this paper is to investigate what happens when the expression (1.3) vanishes, or equivalently: what happens if the Phillips-Sarnak condition is *not* satisfied.

The proof of (1.3) in [33] uses the Lax-Phillips scattering theory as developed for automorphic functions, see [22]. The crucial ingredient is provided by the cut-off wave operator B. Its spectrum (on appropriate spaces) coincides with the singular set (counting multiplicities). It includes the embedded eigenvalues and the resonances. The motion of an embedded eigenvalue depending on the perturbation parameter ϵ on the complex place $\mathbb{C}$ can be identified as the motion of an eigenvalue of B. Given that Phillips and Sarnak proved that regular perturbation theory applies to this setting, it follows that an embedded eigenvalue moves (with at most algebraic singularities) as function of ϵ, either remaining a cuspidal eigenvalue or becoming a resonance. Eq. (1.3) follows using standard perturbation theory techniques. Balslev provided a different proof of Eq. (1.3) in [2] by introducing the technique of analytic dilations and imitating the setting of Fermi's Golden Rule for the helium atom, see [37]. A slightly modified version of the application of perturbation theory is provided in [28], using the formulas in [21, p. 79].

Once the dissolving condition had been identified, Phillips and Sarnak [32] expressed it as a special value of a Rankin–Selberg convolution of u_j with the holomorphic cusp form f generating the deformation. These special values have been subsequently studied [10, 11, 23] with the aim of showing that a generic surface with cusps has 'few' embedded eigenvalues in the sense of Weyl's law.

A different line of approach has been to develop alternate perturbation settings, where the condition to check is easier to understand. Wolpert, Phillips and Sarnak, and Balslev and Venkov succeded in investigating Weyl's law this way. [39, 34, 3, 4].

A more recent development came through the numerical investigation of the poles of Eisenstein series by Avelin [1]. Working with the Teichmüller space of $\Gamma_0(5)$, she found a fourth order contact of $s_j(\epsilon)$ with the unitary axis $\Re(s) = 1/2$. It is easy to explain why certain directions in the moduli space will not satisfy the Phillips-Sarnak condition (1.1): If the dimension of the moduli space is at least 2, then the map $f \to \langle \Delta^{(1)}u_j, E(z, 1/2 + ir_j)\rangle$ is linear, therefore, is has nontrivial kernel. Avelin also identified numerically the most suitable curve that the singular point follows in the left half-plane. This work (along with the work of Farmer and Lemurell [13]) motivated us to investigate whether one can identify higher order Fermi-type conditions that will explain what happens in this case. We answer affirmatively: we find conditions that guarantee that an embedded eigenvalue becomes a resonance.

For this purpose we introduce the perturbation series of the generalized eigenfunctions $D(z,s,\epsilon)$, with $D(z,s,0)=E(z,s)$:

$$D(z,s,\epsilon)=D(z,s,0)+D^{(1)}(z,s)\epsilon+\frac{D^{(2)}(z,s)}{2!}\epsilon^2+\cdots . \tag{1.4}$$

Theorem 1.1. *Assume that for $k=0,1,\dots,n-1$ the functions $D^{(k)}(z,s)$ are regular at a simple cuspidal eigenvalue $s_j=1/2+ir_j$. Then $D^{(n)}(z,s)$ has at most a first order pole at s_j.*

(1) *If $D^{(n)}(z,s)$ has a pole at s_j, then the embedded eigenvalue becomes a resonance.*
(2) *Moreover, with $\|\cdot\|$ the standard L^2-norm,*

$$\Re s_j^{(2n)}(0)=-\frac{1}{2}\binom{2n}{n}\left\|\operatorname*{res}_{s=s_j}D^{(n)}(z,s)\right\|^2,$$

and this is the leading term in the expansion of $\Re s_j(\epsilon)$, i.e. $\Re s_j^{(j)}(0)=0$ for $j<2n$.

Corollary 1.2. *An embedded simple eigenvalue s_j becomes a resonance if and only if for some $m\in\mathbb{N}$ the function $D^{(m)}(z,s)$ has a pole at s_j.*

Remark 1.3. For $n=1$ the condition in the theorem is the classical Fermi's Golden Rule, see (3.8) with $n=1$. Our method provides a new proof of this well-known result without using energy inner products, see [33] but assuming Theorem 2.2.

Remark 1.4. The assumptions of the theorem may equivalently be stated as $\Re(s^{(j)}(0))=0$ for $j=1,\dots,2n-1$. So in the theorem we are really assuming that the embedded eigenvalue does not become a resonance to order *less* than $2n$.

Remark 1.5. At first glance it may seem that the condition identifies one perturbation object with another, equally unknown. However, the condition can surprisingly also be expressed as the nonvanishing at a special point of a Dirichlet series. The relevant series is more complicated than the standard Rankin–Selberg convolution. In the case of character varieties and $n=2$ this Dirichlet series is

$$\sum_{n=1}^{\infty}\left(\sum_{k_1+k_2=n}\frac{a_{k_1}}{k_1}\frac{a_{k_2}}{k_2}b_{-n}\right)\frac{1}{n^s}, \tag{1.5}$$

where a_n are the Fourier coefficients of f, and b_n are the coefficients of u_j.

Even more $D^{(n)}(z,s)$ has been the object of intense investigation by Goldfeld, O'Sullivan, Chinta, Diamantis, the authors, Jorgenson et. al. [14, 15, 6, 12, 26, 29, 30, 20]. It can be defined for $\Re(s)>1$ as

$$D^{(n)}(z,s)=\sum_{\Gamma_\infty\backslash\Gamma}\left(2\pi i\int_{i\infty}^{\gamma z}\Re f(w)\,dw\right)^n\Im(\gamma z)^s. \tag{1.6}$$

In fact, in [27, 29] it was proved that

$$\mathrm{Res}_{s=s_j}D^{(1)}(z,s)=\langle\Delta^{(1)}u_j,E(z,1/2+ir_j)\rangle u_j(z),$$

which gives the Phillips–Sarnak condition when one takes the L^2-norm. This motivated us to investigate the residues of $D^{(n)}(z,s)$ and derive Theorem 1.1. The character perturbation setup is analyzed in section 4.

Remark 1.6. The simplicity of s_j is not important. We state the theorem for any multiplicity of s_j as Theorem 3.1 in Section 2.

Remark 1.7. This theorem gives an algorithmic method of checking whether in a particular direction of moduli space an embedded eigenvalue becomes a resonance. If $D^{(1)}(z,s)$ is regular at s_j, which is equivalent to the vanishing of the Phillips–Sarnak condition, then the embedded eigenvalue stays an eigenvalue to second order and we need to check the higher order condition $D^{(2)}(z,s)$. If this is regular one looks at the next term in the perturbation series of $D(z,s,\epsilon)$ etc.

Remark 1.8. There is an easy argument that explains why a pole of $D^{(n)}(z,s)$ at s_j forces the embedded eigenvalue to become a resonance. The argument is sketched in section 3.2.

In this article (Section 4) we investigate the the case of character varieties. The application of Theorem 1.1 to the analysis of Teichmüller deformations will appear in [31].

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2. Background and preliminaries

An admissible surface, see [25, 24], is a two dimensional non-compact Riemannian manifold M of finite area with hyperbolic ends, i.e. there is a compact set M_0 such that M has a decomposition

$$M=M_0\cup\bigcup_{\mathfrak{a}=1}^{\mathfrak{k}}Z_{\mathfrak{a}}$$

and

$$Z_{\mathfrak{a}}\cong S^1\times[c_{\mathfrak{a}},\infty),\quad c_{\mathfrak{a}}>0$$

carries coordinates $(x_{\mathfrak{a}},y_{\mathfrak{a}})\in S^1\times[c_{\mathfrak{a}},\infty)$ and is equipped with the hyperbolic metric

$$\frac{dx_{\mathfrak{a}}^2+dy_{\mathfrak{a}}^2}{y_{\mathfrak{a}}^2}.$$

The end $Z_{\mathfrak{a}}$ is called a cusp. Müller [25, 24] has worked out the spectral theory of admissible surfaces. The Laplace operator Δ, defined originally on compactly supported smooth functions, has a unique self-adjoint extension on $L^2(M)$, which we denote by L. The spectrum of L consists of discrete spectrum (eigenvalues $\lambda_j = s_j(1-s_j)$) with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

(a finite or infinite set accumulating at ∞) and continuous spectrum $[1/4, \infty)$ of multiplicity $\mathfrak{k}$, provided by generalized eigenfunctions $E_{\mathfrak{a}}(z,s)$. These can to be constructed as in [24, 9] and, only in the special case of hyperbolic surfaces, are given by series of the type

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s.$$

We will call the generalized eigenfunctions Eisenstein series.

Each $E_{\mathfrak{a}}(z,s)$

(1) admits meromorphic continuation to $\mathbb{C}$ with poles in $\Re(s) < 1/2$ or on the interval $(1/2, 1]$,
(2) satisfies the eigenvalue equation
$$\Delta E_{\mathfrak{a}}(z,s) + s(1-s)E_{\mathfrak{a}}(z,s) = 0, \quad \text{and}$$
(3) satisfies the functional equation
$$E_{\mathfrak{a}}(z,s) = \sum_{\mathfrak{b}=1}^{\mathfrak{k}} \phi_{\mathfrak{a}\mathfrak{b}}(s) E_{\mathfrak{b}}(z, 1-s)$$

for some functions $\phi_{\mathfrak{a}\mathfrak{b}}(s)$. The determinant of the scattering matrix $\Phi(s) = (\phi_{\mathfrak{a}\mathfrak{b}}(s))_{\mathfrak{a},\mathfrak{b}=1}^{\mathfrak{k}}$ is denoted $\phi(s)$. The poles of $\phi(s)$ are called resonances. The scattering matrix satisfies a functional equation $\Phi(s)\Phi(1-s) = I_{\mathfrak{k}\times\mathfrak{k}}$, and, moreover,

$$\Phi(\bar{s}) = \overline{\Phi(s)}, \quad \Phi(s)^* = \Phi(\bar{s}). \tag{2.1}$$

The resolvent of the Laplace operator $R(s) = (\Delta + s(1-s))^{-1}$ defined on $L^2(M)$ for $\Re(s) > 1/2$, $s \notin \operatorname{spec}(L)$, admits a meromorphic continuation to $\mathbb{C}$, if we restrict the domain to a smaller space, e.g. $C_c^\infty(M)$, compactly supported functions on M. The limiting absorption principle holds: i.e. the resolvent kernel (Green's function) $r(z, z', s)$ satisfies

$$r(z,z',s) - r(z,z',1-s) = \frac{1}{1-2s} \sum_{\mathfrak{a}=1}^{\mathfrak{k}} E_{\mathfrak{a}}(z,s) E_{\mathfrak{a}}(z', 1-s). \tag{2.2}$$

At a spectral point s_j with eigenvalue $s_j(1-s_j) > 1/4$ the resolvent kernel has a pole described by the Laurent expansion

$$r(z,z',s) = \frac{P}{s(1-s) - s_j(1-s_j)} + \cdots, \tag{2.3}$$

where P is the spectral projection to the eigenspace with eigenvalue $s_j(1-s_j)$.

An admissible surface has generically finitely many discrete eigenvalues, a result due to Colin de Verdière [9], and a consequence of the infinite dimensionality of the admissible metrics i.e. arbitrary metrics on M_0. Determining the number of eigenvalues is much trickier if we demand that M is hyperbolic, as the Teichmüller space is finite dimensional. However, the perturbation setup works in the case of admissible surfaces and they appear as a technical devise in Teichmüller perturbations.

We are interested in perturbations of the Laplace operator L on M. The simplest kind of such arises from a perturbation of the Riemannian metric inside M_0 (compact perturbations). Let $\epsilon \in (-\epsilon_0, \epsilon_0)$. Let $g(\epsilon)$ be a real analytic family of metrics on M, with $g(\epsilon) = g(0)$ on $M \setminus M_0$. The Laplacian then admits a real analytic expansion

$$L(\epsilon) = L(0) + \epsilon L^{(1)} + \frac{\epsilon^2}{2} L^{(2)} + \cdots .$$

As the family of metrics agree with $g(0)$ up in the cusps, the Laplacian does not change up in the cusps and the operators $L^{(i)}$ are compactly supported operators, since $L^{(i)} f$ has support in M_0 for every (smooth) function f. We denote by $D(z, s, \epsilon)$ any of the generalized eigenfunctions $E_{\mathfrak{a}}(z, s, \epsilon)$ of $L(\epsilon)$.

Theorem 2.1. *The family $D(z, s, \epsilon)$ is real analytic in ϵ for $\epsilon \in (-\epsilon_0, \epsilon_0)$ and meromorphic in $s \in \mathbb{C} \setminus \{1/2\}$. The n-th derivative in ϵ is given by*

$$D^{(n)}(z, s) = -R(s) \sum_{i=1}^{n} \binom{n}{i} L^{(i)} D^{(n-i)}(z, s). \tag{2.4}$$

Sketch of proof. The real analyticity follows from the construction of the generalized eigenfunctions $D(z, s, \epsilon)$ using pseudo-Laplacians and the fact that the construction can be differentiated at every step in ϵ. This is explained (for the first derivative at least) in [29], see also [5]. The formula for $D^{(n)}(z, s)$ can be proved by differentiating $(L(\epsilon) + s(1-s))D(z, s, \epsilon) = 0$ to get

$$\sum_{i=0}^{n} \binom{n}{i} \frac{d^i}{d\epsilon^i} (L(\epsilon) + s(1-s)) \Big|_{\epsilon=0} D^{(n-i)}(z, s) = 0.$$

Then we isolate the term $D^{(n)}(z, s)$ using $R(s)$ for $\Re(s) > 1/2$. Since the operators $L^{(i)}$ are compactly supported, the resolvent is applied to a compactly supported function and the right-hand side of (2.4) can be meromorphically continued to $\mathbb{C}$. The identity (2.4) holds on $\mathbb{C}$ by the principle of analytic continuation. □

If we expand $E_{\mathfrak{a}}(z, s)$ in the cusp $Z_{\mathfrak{b}}$, the zero Fourier coefficient takes the form

$$\delta_{\mathfrak{a}\mathfrak{b}} y_{\mathfrak{b}}^s + \phi_{\mathfrak{a}\mathfrak{b}}(s) y_{\mathfrak{b}}^{1-s}.$$

It follows from Theorem 2.1, that, for $s \neq 1/2$, $\phi_{\mathfrak{a}\mathfrak{b}}(s,\epsilon)$ is also real analytic in ϵ, since

$$\phi_{\mathfrak{a}\mathfrak{b}}(s,\epsilon) = \frac{1}{y_{\mathfrak{b}}^{1-s}} \left(\int_0^1 E_{\mathfrak{a}}(z_{\mathfrak{b}}, s, \epsilon)\, dx_{\mathfrak{b}} - \delta_{\mathfrak{a}\mathfrak{b}} y_{\mathfrak{b}}^s \right).$$

The singular set σ includes the embedded eigenvalues and resonances at the same time. The only points in the singular set σ off the real axis are

(1) s_j with $s_j(1-s_j)$ an embedded eigenvalues counted with its multiplicity, and
(2) resonances s_j counted with multiplicity the order of the pole of the scattering determinant at s_j.

For the points in $[0,1]\cap\sigma$, see [33]. A point in the singular set is called singular.

The important theorem about the singular set needed is the following theorem:

Theorem 2.2. [33, Corollary 5.2] *If $s_j(0)$ is in the singular set $\sigma(0)$ for $\epsilon = 0$ and has multiplicity 1, then it moves real analytically in ϵ for $|\epsilon|$ sufficiently small. If the multiplicity is greater than one, then the singular points decompose into a finite system of real analytic functions having at most algebraic singularities.*

In the setting of compact perturbations of admissible surfaces, Müller [24] proved the same statement. These results use the family of cut-off wave operators $B(\epsilon)$ and follow from standard perturbation theory, once it is proved that the resolvent $R_{B(\epsilon)}(s)$ is real analytic for $|\epsilon|$ sufficiently small. The technically difficult aspect of [33] is the identification of the spectrum of $B(\epsilon)$ with the singular set $\sigma(\epsilon)$.

3. Dissolving conditions of higher order

3.1. Main statements.

From this section onwards we restrict ourselves, for simplicity, to the case that M has one cusp, i.e. $\mathfrak{k} = 1$. Let $s_j = s_j(0)$ be a singular point of multiplicity m. Let $\hat{s}_j(\epsilon)$ be the weighted mean of the branches of the singular points generated by splitting $s_j(0)$ under perturbation, i.e.

$$\hat{s}_j(\epsilon) = \frac{1}{m}\sum_{l=1}^{m} s_{j,l}(\epsilon).$$

We are now ready to state and prove the more precise version of Theorem 1.1:

Theorem 3.1. *Assume that for $k = 0, 1, \ldots, n-1$ the functions $D^{(k)}(z,s)$ are regular at a cuspidal eigenvalue $s_j = 1/2 + ir_j$. Then $D^{(n)}(z,s)$ has at most a first order pole at s_j.*

(1) *If $D^{(n)}(z,s)$ has a pole at s_j, then the embedded eigenvalue becomes a resonance.*

(2) *Moreover, with $\|\cdot\|$ the standard L^2-norm,*

$$\Re \hat{s}_j^{(2n)}(0) = -\frac{1}{2m}\binom{2n}{n}\left\| \operatorname*{res}_{s=s_j} D^{(n)}(z,s)\right\|^2. \tag{3.1}$$

Corollary 3.2. *At least one of the cusp forms with given s_j becomes a resonance if and only if for some $m \in \mathbb{N}$ the function $D^{(m)}(z,s)$ has a pole at s_j.*

3.2. **Poles of $D^{(n)}(z,s)$, and dissolving cusp forms.** Before we prove Theorem 3.1, we indicate an argument that explains why a singularity of $D^{(n)}(z,s)$ at s_j is connected to dissolving cusp forms. For simplicity we consider hyperbolic surfaces so that the generalized eigenfunction $D(z,s,\epsilon)$ is an Eisenstein series. Let us assume that $D^{(k)}(z,s)$ is regular at s_j for $k = 1, \dots, n-1$. Assume u_j is a simple cusp form and that $u_j(\epsilon)$ remains a cusp form with $u_j(0) = u_j$. It is known that cusp forms are perpendicular to the Eisenstein series $D(z,s,\epsilon)$ for all s. This gives

$$\langle u_j(\epsilon), D(z,s,\epsilon)\rangle = 0. \tag{3.2}$$

Phillips and Sarnak [33] proved the real analyticity of $u_j(\epsilon)$. We differentiate (3.2) to get

$$\sum_{k=0}^{n}\binom{n}{k}\left\langle u_j^{(n-k)}, D^{(k)}(z,s)\right\rangle = 0$$

for s close to s_j. By the assumptions the term with $k = n$ should be a regular function at s_j. Under the same assumptions, using (2.4) and (2.3) we see that $D^{(n)}(z,s)$ has at most a first order pole at s_j with residue a multiple of $u_j(0)$. By regularity of $\left\langle u_j, D^{(n)}(z,s)\right\rangle$ this residue has to vanish. This approach does not prove Corollary 3.2 but shows the sufficiency of the condition that some $D^{(n)}(z,s)$ has a pole to conclude that s_j becomes a resonance. Corollary 3.2 shows that this is also necessary.

3.3. **Polar structure of the Taylor coefficients of $\phi(s,\epsilon)$.** Since the singular set is defined partly though the poles of $\phi(s)$, we now investigate the perturbation series of $\phi(s,\epsilon)$, in order to track the singular points as ϵ varies.

The functional equation for $D(z,s,\epsilon)$ is

$$D(z,s,\epsilon) = \phi(s,\epsilon)D(z,1-s,\epsilon). \tag{3.3}$$

Since $D(z,s,\epsilon)$ is real analytic in ϵ the same is true for $\phi(s,\epsilon)$ and we may introduce the perturbation series of the scattering matrix $\phi(s,\epsilon)$:

$$\phi(s,\epsilon) = \phi(s,0) + \phi^{(1)}(s)\epsilon + \frac{\phi^{(2)}(s)}{2!}\epsilon^2 + \cdots .$$

We differentiate (3.3) to identify the perturbation coefficients of $\phi(s,\epsilon)$:

$$D^{(n)}(z,s) = \sum_{i=0}^{n}\binom{n}{i}\phi^{(i)}(s)D^{(n-i)}(z,1-s). \tag{3.4}$$

Proposition 3.3. The perturbation coefficients of the scattering matrix are given by

$$\phi^{(n)}(s) = \frac{1}{2s-1}\int_M E(z,s)\sum_{i=1}^{n}\binom{n}{i}L^{(i)}D^{(n-i)}(z,s)\,d\mu(z), \quad n \geq 1.$$

Proof. The proof is already in [35]. We include the argument here. We proceed by induction. By using first (2.4) and then (2.2) we find that

$$\begin{aligned}
D^{(1)}(z,s) &= -R(s)L^{(1)}E(z,s)\\
&= -R(1-s)L^{(1)}E(z,s) + \frac{1}{2s-1}\int_M E(z',s)L^{(1)}E(z',s)\,d\mu(z')E(z,1-s)\\
&= \phi(s)(-R(1-s)L^{(1)}E(z,1-s)) + \frac{1}{2s-1}\int_M E(z',s)L^{(1)}E(z',s)\,d\mu(z')E(z,1-s)\\
&= \phi(s)D^{(1)}(z,1-s) + \frac{1}{2s-1}\int_M E(z',s)L^{(1)}E(z',s)\,d\mu(z')E(z,1-s).
\end{aligned}$$

From (3.4) we know that

$$D^{(1)}(z,s) = \phi^{(1)}(s)E(z,1-s) + \phi(s)D^{(1)}(z,1-s),$$

and since $E(z,1-s)$ does not vanish identically, we get the result for $n=1$.

Assume the formula has been proved for $m < n$. Using (2.4) and (2.2) we get

$$\begin{aligned}
D^{(n)}(z,s) &= -R(1-s)\sum_{i=1}^{n}\binom{n}{i}L^{(i)}D^{(n-i)}(z,s)\\
&\quad +\frac{1}{2s-1}\left(\int_M E(z,s)\sum_{i=1}^{n}\binom{n}{i}L^{(i)}D^{(n-i)}(z,s)\,d\mu(z)\right)E(z,1-s)\\
&= -R(1-s)\sum_{i=1}^{n}\binom{n}{i}L^{(i)}\sum_{k=0}^{n-i}\binom{n-i}{k}\phi^{(k)}(s)D^{(n-i-k)}(z,1-s) + Q(z,s)\\
&= -R(1-s)\sum_{k=0}^{n-1}\phi^{(k)}(s)\sum_{i=1}^{n-k}\binom{n}{k}\binom{n-k}{i}L^{(i)}D^{(n-i-k)}(z,1-s) + Q(z,s)\\
&= \sum_{k=0}^{n-1}\binom{n}{k}\phi^{(k)}(s)\left(-R(1-s)\sum_{i=1}^{n-k}\binom{n-k}{i}L^{(i)}D^{(n-k-i)}(z,1-s)\right) + Q(z,s)\\
&= \sum_{k=0}^{n-1}\binom{n}{k}\phi^{(k)}(s)D^{(n-k)}(z,1-s) + Q(z,s),
\end{aligned}$$

where

$$Q(z,s) = \frac{1}{2s-1}\int_M E(z',s)\sum_{i=1}^{n}\binom{n}{i}L^{(i)}D^{(n-i)}(z',s)\,d\mu(z')E(z,1-s).$$

Comparing with (3.4), we get that

$$\phi^{(n)}(s) = \frac{1}{2s-1}\int_M E(z,s)\sum_{i=1}^{n}\binom{n}{i}L^{(i)}D^{(n-i)}(z,s)\,d\mu(z),$$

which finishes the proof. □

Proposition 3.3 allows to recover all the scattering terms in terms of the perturbed Eisenstein series. However, for $\phi^{(n)}(s)$ one uses information for $D^{(j)}(z,s)$ with j up to n. For our purposes this is *not* good enough. The following technical yet important proposition allows to use fewer $D^{(j)}(z,s)$.

Proposition 3.4. The perturbed terms of the scattering function $\phi^{(n)}(s)$ are given for $i = 1, 2, \dots, n-1$ by

$$\begin{aligned}(2s-1)\phi^{(n)}(s) = {}& \binom{n}{i}\left\langle \sum_{k=1}^{n-i}\binom{n-i}{k}L^{(k)}D^{(n-i-k)}(z,s), D^{(i)}(z,\bar{s})\right\rangle \\ &+ \sum_{k=i+1}^{n}\binom{n}{k}\left\langle D^{(n-k)}(z,s), \sum_{m=0}^{i-1}\binom{k}{m}L^{(k-m)}D^{(m)}(z,\bar{s})\right\rangle. \end{aligned} \tag{3.5}$$

Proof. To simplify the notation we suppress z and s and $\bar{s}$ in the inner products. It will be understood that the terms on the left of $\langle\cdot,\cdot\rangle$ should carry s and the one on the right should have $\bar{s}$. Note that for any functions f, g we have $\langle R(s)f, g\rangle = \langle f, R(\bar{s})g\rangle$, since $R(s)^* = R(\bar{s})$. Moreover, since $L(\epsilon)$ are self-adjoint, the same applies to $L^{(j)}$. Even if the Eisenstein series are not in L^2, since $L^{(j)}$ are compactly supported, we can easily justify the integration by parts in the following calculation. We have from Proposition 3.3 and (2.4)

$$\begin{aligned}(2s-1)\phi^{(n)}(s) &= \left\langle \sum_{i=1}^{n}\binom{n}{i}L^{(i)}D^{(n-i)}, E\right\rangle \\ &= \binom{n}{1}\left\langle D^{(n-1)}, L^{(1)}E\right\rangle + \sum_{i=2}^{n}\binom{n}{i}\left\langle D^{(n-i)}, L^{(i)}E\right\rangle \\ &= \binom{n}{1}\left\langle \sum_{k=1}^{n-1}\binom{n-1}{k}(-R)L^{(k)}D^{(n-1-k)}, L^{(1)}E\right\rangle + \sum_{i=2}^{n}\binom{n}{i}\left\langle D^{(n-i)}, L^{(i)}E\right\rangle \\ &= \binom{n}{1}\left\langle \sum_{k=1}^{n-1}\binom{n-1}{k}L^{(k)}D^{(n-1-k)}, -RL^{(1)}E\right\rangle + \sum_{i=2}^{n}\binom{n}{i}\left\langle D^{(n-i)}, L^{(i)}E\right\rangle \\ &= \binom{n}{1}\left\langle \sum_{k=1}^{n-1}\binom{n-1}{k}L^{(k)}D^{(n-1-k)}, D^{(1)}\right\rangle + \sum_{k=2}^{n}\binom{n}{k}\left\langle D^{(n-k)}, L^{(k)}E\right\rangle. \end{aligned}$$

This shows (3.5) for $i = 1$. Assume now that we proved it for a given i. We separate the terms with $k = 1$ and $k = i+1$ in (3.5) and group them together to get

$$\begin{aligned}(2s-1)\phi^{(n)}(s) &= \binom{n}{i}\left\langle \binom{n-i}{1} L^{(1)} D^{(n-i-1)}, D^{(i)} \right\rangle \\ &+ \binom{n}{i+1}\left\langle D^{(n-(i+1))}, \sum_{m=0}^{i-1}\binom{i+1}{m} L^{(i+1-m)} D^{(m)} \right\rangle \\ &+ \binom{n}{i}\left\langle \sum_{k=2}^{n-i}\binom{n-i}{k} L^{(k)} D^{(n-i-k)}, D^{(i)} \right\rangle \\ &+ \sum_{k=i+2}^{n}\binom{n}{k}\left\langle D^{(n-k)}, \sum_{m=0}^{i-1}\binom{k}{m} L^{(k-m)} D^{(m)} \right\rangle .\end{aligned}$$

We use the obvious identity for binomial coefficients

$$\binom{n}{i}\binom{n-i}{1} = \binom{n}{i+1}\cdot(i+1)$$

and bump the summation variable by i in the third sum to get

$$\begin{aligned}&\binom{n}{i+1}\left\langle D^{(n-(i+1))}, (i+1)L^{(1)}D^{(i)} + \sum_{m=0}^{i-1}\binom{i+1}{m} L^{(i+1-m)} D^{(m)} \right\rangle \\ &+ \sum_{k=i+2}^{n}\binom{n}{i}\binom{n-i}{k-i}\left\langle L^{(k-i)}D^{(n-k)}, D^{(i)}\right\rangle + \sum_{k=i+2}^{n}\binom{n}{k}\left\langle D^{(n-k)}, \sum_{m=0}^{i-1}\binom{k}{m} L^{(k-m)} D^{(m)} \right\rangle .\end{aligned}$$

We use

$$\binom{n}{i}\binom{n-i}{k-i} = \binom{n}{k}\binom{k}{i}$$

and (2.4) to see that the expression is now

$$\begin{aligned}&\binom{n}{i+1}\left\langle \sum_{k=1}^{n-i-1}\binom{n-i-1}{k} L^{(k)} D^{(n-i-1-k)}, -R\left(\sum_{m=0}^{i}\binom{i+1}{m} L^{(i+1-m)} D^{(m)}\right) \right\rangle \\ &+ \sum_{k=i+2}^{n}\binom{n}{k}\binom{k}{i}\left\langle L^{(k-i)}D^{(n-k)}, D^{(i)}\right\rangle + \sum_{k=i+2}^{n}\binom{n}{k}\left\langle D^{(n-k)}, \sum_{m=0}^{i-1}\binom{k}{m} L^{(k-m)} D^{(m)} \right\rangle .\end{aligned}$$

We use (2.4) again to get

$$\begin{aligned}&\binom{n}{i+1}\left\langle \sum_{k=1}^{n-i-1}\binom{n-i-1}{k} L^{(k)} D^{(n-i-1-k)}, D^{(i+1)} \right\rangle \\ &+ \sum_{k=i+2}^{n}\binom{n}{k}\left(\binom{k}{i}\left\langle L^{(k-i)}D^{(n-k)}, D^{(i)}\right\rangle + \left\langle D^{(n-k)}, \sum_{m=0}^{i-1}\binom{k}{m} L^{(k-m)} D^{(m)} \right\rangle\right) .\end{aligned}$$

Finally we get

$$\begin{aligned}(2s-1)\phi^{(n)}(s) =&\binom{n}{i+1}\left\langle \sum_{k=1}^{n-i-1}\binom{n-i-1}{k}L^{(k)}D^{(n-i-1-k)}, D^{(i+1)}\right\rangle\\ &+\sum_{k=i+2}^{n}\binom{n}{k}\left\langle D^{(n-k)}, \sum_{m=0}^{i}\binom{k}{m}L^{(k-m)}D^{(m)}\right\rangle.\end{aligned}$$

□

We can now use Proposition 3.5 to translate information about $D^{(i)}(z,s)$ at s_j into information about $\phi^{(k)}(s)$ at s_j:

Theorem 3.5. *Assume that $D^{(q)}(z,s)$ is regular at $s_j = 1/2 + ir_j$ for $q = 0, 1, \dots, n-1$. Then*

1. *the function $\phi^{(l)}(s)$ is regular at s_j for $l = 0, 1, \dots 2n-1$.*
2. *the function $\phi^{(2n)}(s)$ has at most a simple pole at s_j. Furthermore the residue at s_j is given by*

$$\operatorname*{res}_{s=s_j}\phi^{(2n)}(s) = -\phi(s_j)\binom{2n}{n}\left\|\operatorname*{res}_{s=s_j}D^{(n)}(z,s)\right\|^2.$$

Proof. We take $n = l$ for the various values of $l \leq 2n$ in Proposition 3.4. Assume first that $l < 2n$ and let in Proposition 3.4 the integer i be the integral part of $l/2$. Then $i < n$ and $l - i - 1 < n$ and therefore, by the assumption on $D^{(q)}(z,s)$ for $q = 0, 1, \dots, n-1$, we see immediately - using the expression on the right of (3.5) - that $\phi^{(l)}(s)$ is regular at s_j.

To prove the claim about $\phi^{(2n)}(s)$ we choose the integer i in Proposition 3.4 to equal n. By (3.5) and the assumptions on $D^{(q)}(z,s)$ we see that $\phi^{(2n)}(s)$ has at most a simple pole at s_j and that the residue is given by

$$\frac{1}{2s_j-1}\binom{2n}{n}\int_M\sum_{k=1}^{n}\binom{n}{k}L^{(k)}D^{(n-k)}(z,s_j)\operatorname*{res}_{s=s_j}D^{(n)}(z,s)d\mu(z). \tag{3.6}$$

From (3.4) and the assumptions on $D^{(q)}(z,s)$ we get that

$$\operatorname*{res}_{s=s_j}D^{(n)}(z,s_j) = \phi(s_j)\operatorname*{res}_{s=s_j}D^{(n)}(z,1-s).$$

Since $D^{(n)}(z,s) = \overline{D^{(n)}(z,\overline{s})}$ we have also, since $1-\overline{s_j} = s_j$, that

$$\operatorname*{res}_{s=s_j}D^{(n)}(z,1-s) = -\overline{\operatorname*{res}_{s=s_j}D^{(n)}(z,s)},$$

and therefore

$$\operatorname*{res}_{s=s_j}D^{(n)}(z,s) = -\phi(s_j)\overline{\operatorname*{res}_{s=s_j}D^{(n)}(z,s)}. \tag{3.7}$$

From (2.4) and (2.3) we see that

$$\text{(3.8)}\quad \operatorname*{res}_{s=s_j} D^{(n)}(z,s) = -\frac{1}{2s_j-1}\sum_{i=1}^{m}\left\langle \sum_{k=1}^{n}\binom{n}{k} L^{(k)}D^{(n-k)}(z,s_j), u_{j,i}\right\rangle u_{j,i},$$

where $\{u_{j,i}\}$ is an orthonormal basis for the eigenspace for the eigenvalue $s_j(1-s_j)$. Inserting this in (3.6) (after first using (3.7)) we find that $\operatorname{res}_{s=s_j}\phi^{(2n)}(s)$ is given by

$$\text{(3.9)}\qquad -\frac{\phi(s_j)}{|2s_j-1|^2}\binom{2n}{n}\sum_{i=1}^{m}\left|\left\langle \sum_{k=1}^{n}\binom{n}{k} L^{(k)}D^{(n-k)}(z,s_j), u_{j,i}\right\rangle\right|^2$$

which is easily seen to be the claimed result comparing (3.8). □

Remark 3.6. We need one more ingredient about $\phi(s,\epsilon)$ before proving Theorem 3.1. Since $\phi(s,\epsilon)=\overline{\phi(\bar{s},\epsilon)}$ we deduce that

$$\text{(3.10)}\qquad \frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)} = \overline{\left(\frac{\phi'(\bar{s},\epsilon)}{\phi(\bar{s},\epsilon)}\right)},$$

where $'$ denotes derivative in the s variable, as is standard in the Selberg theory of the trace formula. This follows from the fact that for an analytic function f we have

$$\frac{d}{ds}\overline{f(\bar{s})} = \overline{f'(\bar{s})}.$$

Proof of Theorem 3.1. We want to track the movement of the embedded eigenvalue/resonance in the left half-plane. We define Γ to be the semicircular contour $\gamma_1(t)=ue^{it}+s_j$, $\pi/2\le t\le 3\pi/2$ followed by the vertical segment $\gamma_2(t)=s_j+it$, $-u\le t\le u$. Here u is chosen small enough, so that the only singular point for $\epsilon=0$ inside the ball $B(s_j,u)$ is s_j with multiplicity $m=m(s_j)$. This contour is traversed counterclockwise. For small enough ϵ the total multiplicities of the singular points $s_j(\epsilon)$ inside $B(s_j,u)$ is $m(s_j)$. Perturbation theory allows to study the weighted mean $\hat{s}(\epsilon)$ of the branches of eigenvalues of $B(\epsilon)$. We have

$$\text{(3.11)}\qquad m(\hat{s}(\epsilon)-s_j) = -\frac{1}{2\pi i}\int_{\Gamma}(s-s_j)\frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)}\,ds + \sum_{j\in C}(s_j(\epsilon)-s_j),$$

where C is indexing the cusp forms eigenbranches inside $B(s_j,u)$, i.e. the cusp forms that remain cusp forms. Let the last sum be denoted by $p(\epsilon)$. The reason for using Γ and not the whole $\partial B(s_j,u)$ is that on the right half-disc $\phi(\epsilon)$ has zeros, which we do not want to count. Notice that $\overline{\int_\gamma f(s)\,ds} = \int_{\bar{\gamma}}\bar{f}(\bar{s})\,ds$ and, therefore, by (3.10)

$$m(\overline{\hat{s}(\epsilon)-s_j}) = \frac{1}{2\pi i}\int_{\bar{\Gamma}}(s-\bar{s}_j)\overline{\left(\frac{\phi'(\bar{s},\epsilon)}{\phi(\bar{s},\epsilon)}\right)}\,ds+\overline{p(\epsilon)} = \frac{1}{2\pi i}\int_{\bar{\Gamma}}(s-\bar{s}_j)\frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)}\,ds+\overline{p(\epsilon)}.$$

Denoting by $-\gamma$ the contour γ traversed in the opposite direction, we get

$$\begin{aligned} m(\overline{\hat{s}(\epsilon)-s_j}) &= -\frac{1}{2\pi i}\int_{-\bar{\Gamma}}(s-\bar{s}_j)\frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)}\,ds+\overline{p(\epsilon)} \\ &= -\frac{1}{2\pi i}\int_{T^{-1}(-\bar{\Gamma})}(1-w-\bar{s}_j)\frac{\phi'(1-w,\epsilon)}{\phi(1-w,\epsilon)}\,(-dw)+\overline{p(\epsilon)}, \end{aligned}$$

where $s=T(w)=1-w$ is a conformal map. By the functional equation $\phi(s,\epsilon)\phi(1-s,\epsilon)=1$, see (2.1), we get

$$\phi'(s,\epsilon)\phi(s,\epsilon)-\phi(s,\epsilon)\phi'(1-s,\epsilon)=0,$$

which implies

$$\frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)}=\frac{\phi'(1-s,\epsilon)}{\phi(1-s,\epsilon)}.$$

We plug this into the expression for $m(\overline{\hat{s}(\epsilon)-s_j})$ to get

$$m(\overline{\hat{s}(\epsilon)-s_j})=-\frac{1}{2\pi i}\int_{T^{-1}(-\bar{\Gamma})}(w-s_j)\frac{\phi'(w,\epsilon)}{\phi(w,\epsilon)}\,dw+\overline{p(\epsilon)}. \tag{3.12}$$

We sum (3.11) and (3.12) and notice that the cuspidal branch contributions cancel, because for a cuspidal branch $s_{j,l}(\epsilon)$ the function $s_{j,l}(\epsilon)-s_j$ is purely imaginary. We deduce

$$2m\Re(\hat{s}(\epsilon)-s_j)=-\frac{1}{2\pi i}\int_{\Gamma+T^{-1}(-\bar{\Gamma})}(s-s_j)\frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)}\,ds.$$

The contour of integration is now the whole circle $\partial B(s_j,u)$ traversed counterclockwise since the contribution from the line segment on $\Re(s)=1/2$ from Γ and $T^{-1}(-\bar{\Gamma})$ cancel. By uniform convergence we can differentiate the last formula in ϵ. We get

(3.13)

$$\begin{aligned} 2m\frac{d^{2n}}{d\epsilon^{2n}}\Re(\hat{s}(\epsilon))\Big|_{\epsilon=0} &= -\frac{1}{2\pi i}\int_{\partial B(s_j,u)}(s-s_j)\frac{d^{2n}}{d\epsilon^{2n}}\left(\frac{\phi'(s,\epsilon)}{\phi(s,\epsilon)}\right)\Big|_{\epsilon=0}ds \\ = -\frac{1}{2\pi i}\int_{\partial B(s_j,u)}(s-s_j)\sum_{k=0}^{2n}\binom{2n}{k}&\frac{d^k\phi'(s,\epsilon)}{d\epsilon^k}\Big|_{\epsilon=0}\frac{d^{2n-k}(\phi(s,\epsilon)^{-1})}{d\epsilon^{2n-k}}\Big|_{\epsilon=0}ds. \end{aligned}$$

We can interchange the order of differentiation

$$\frac{d^k}{d\epsilon^k}\phi'(s,\epsilon)=\frac{d}{ds}\phi^{(k)}(s)$$

and see that this is regular at s_j for $k<2n$ by Theorem 3.5. On the other hand for $k=2n$ it has a double pole at s_j by the same theorem. Concerning

$$\frac{d^{2n-k}}{d\epsilon^{2n-k}}\phi(s,\epsilon)^{-1}$$

we argue as follows: We differentiate m times $\phi(s,\epsilon)^{-1}\phi(s,\epsilon)=1$ to get

$$\sum_{k=0}^{m}\binom{m}{k}\frac{d^k}{d\epsilon^k}\phi(s,\epsilon)^{-1}\Big|_{\epsilon=0}\phi^{(m-k)}(s,0)=0.$$

Let m be less than $2n$. By Theorem 3.5, the fact that $\phi(s)$ is unitary on $\Re(s)=1/2$, and by solving for $\frac{d^m}{d\epsilon^m}\phi(s,\epsilon)^{-1}\big|_{\epsilon=0}$, we see that $\frac{d^m}{d\epsilon^m}\phi(s,\epsilon)^{-1}\big|_{\epsilon=0}$ is regular at s_j. For $m=2n$ we see, by the same argument, that $\frac{d^{2n}}{d\epsilon^{2n}}\phi(s,\epsilon)^{-1}\Big|_{\epsilon=0}$ has at most a simple pole at s_j.

We can now determine the order of the pole of the integrand of the right-hand side in (3.13): By the above considerations we see that the only non-regular term occurs for $k=2n$. This is the term

$$(s-s_j)\frac{d\phi^{(2n)}(s,0)}{ds}\phi^{-1}(s),$$

which has at most a simple pole. By the residue theorem the expression in (3.13) equals minus the residue of

$$(s-s_j)\frac{d\phi^{(2n)}(s,0)}{ds}\phi^{-1}(s).$$

Since the leading term in the Laurent expansion of the derivative in s of $\phi^{(2n)}(s,0)$ equals $-(\operatorname{res}_{s=s_j}\phi^{(2n)}(s,0))/(s-s_j)^2$ we conclude that

$$\begin{aligned}2m\frac{d^{2n}}{d\epsilon^{2n}}\Re(\hat{s}(\epsilon))\Big|_{\epsilon=0} &= \frac{\operatorname{res}_{s=s_j}\phi^{(2n)}(s,0)}{\phi(s_j,0)}\\ &= -\binom{2n}{n}\left\|\operatorname*{res}_{s=s_j}D^{(n)}(z,s)\right\|^2,\end{aligned}$$

where, in the last equality, we used Theorem 3.5 again. This completes the proof of the theorem. □

Proof of Corollary 3.2. The direction that a pole of some $D^{(m)}(z,s)$ at s_j implies that at least one embedded eigenvalue becomes a resonance is proved as follows: If n is the smallest number such that $D^{(n)}(z,s)$ has a pole at s_j, then from Theorem 3.1, we have that $\Re\hat{s}_j^{(2n)}\neq 0$, while $\Re\hat{s}_j^{(2m)}=0$ for $m<n$. If k is the smallest integer with $\Re\hat{s}_j^{(k)}\neq 0$, then $k=2n$, since an odd leading term in the Taylor series of $\Re\hat{s}_j(\epsilon)$ will force $\Re\hat{s}_j(\epsilon)$ to take values larger and smaller than $1/2$. This is impossible, since a singular point cannot move to the right half-plane. Therefore $\hat{s}_j(\epsilon)$ does not have real part equal to $1/2$ for all small ϵ and one of the cuspidal eigenvalues has to dissolve. The opposite direction is obvious: If all embedded eigenvalues remain embedded eigenvalues, then $\Re\hat{s}_j(\epsilon)=1/2$. This implies that $\Re\hat{s}_j^{(2n)}=0$ for all $n\in\mathbb{N}$. □

4. Character varieties

4.1. **Higher order dissolving for character varieties.** We now describe how the above theory can be modified for the twisted spectral problem related to character varieties. Let Γ be a discrete cofinite subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with quotient $M=\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane. For simplicity, we still assume that Γ has precisely one cusp, which we assume is at infinity. Let $f(z)\in S_2(\Gamma)$ be a holomorphic cusp form of weight 2. Then $\omega=\Re(f(z)\,dz)$ and $\omega=\Im(f(z)\,dz)$ are harmonic cuspidal 1-forms. Let α be a compactly supported 1-form in the same cohomology class as one of them. For the exact construction see e.g. [30, Prop. 2.1]. We fix $z_0\in\mathbb{H}$. Define a family of characters

$$\begin{array}{rccc}\chi(\cdot,\epsilon): & \Gamma & \to & S^1\\ & \gamma & \mapsto & \exp(-2\pi i\epsilon\int_{z_0}^{\gamma z_0}\alpha).\end{array}$$

We consider the space

$$\mathrm{L}^2(\Gamma\backslash\mathbb{H},\overline{\chi}(\cdot,\epsilon))$$

of $(\Gamma,\overline{\chi}(\cdot,\epsilon))$-automorphic functions, i.e. functions $f:\mathbb{H}\to\mathbb{C}$ where

$$f(\gamma z)=\overline{\chi}(\gamma,\epsilon)f(z),$$

and

$$\int_{\Gamma\backslash\mathbb{H}}|f(z)|^2\,d\mu(z)<\infty.$$

The automorphic Laplacian $\tilde{L}(\epsilon)$ is the closure of the operator acting on smooth functions in $\mathrm{L}^2(\Gamma\backslash\mathbb{H},\overline{\chi}(\cdot,\epsilon))$ by Δf. We denote its resolvent by $\tilde{R}(s,\epsilon)=(\tilde{L}(\epsilon)+s(1-s))^{-1}$. We introduce unitary operators

$$\begin{array}{rccc}U(\epsilon): & \mathrm{L}^2(\Gamma\backslash\mathbb{H}) & \to & \mathrm{L}^2(\Gamma\backslash\mathbb{H},\overline{\chi}(\cdot,\epsilon))\\ & f & \mapsto & \exp\left(2\pi i\epsilon\int_{z_0}^{z}\alpha\right)f(z).\end{array}\tag{4.1}$$

We then define

$$L(\epsilon)=U^{-1}(\epsilon)\tilde{L}(\epsilon)U(\epsilon)\tag{4.2}$$

$$R(s,\epsilon)=U^{-1}(\epsilon)\tilde{R}(s,\epsilon)U(\epsilon).\tag{4.3}$$

The operators $L(\epsilon)$ on $\mathrm{L}^2(\Gamma\backslash\mathbb{H})$ and $\tilde{L}(\epsilon)$ on $\mathrm{L}^2(\Gamma\backslash\mathbb{H},\overline{\chi}(\cdot,\epsilon))$ are unitarily equivalent. Notice that $L(\epsilon)$ and $R(s,\epsilon)$ act on the fixed space $\mathrm{L}^2(\Gamma\backslash\mathbb{H})$, which allows to apply perturbation theory. It is easy to verify that

$$L(\epsilon)h=\Delta h+4\pi i\epsilon\langle dh,\alpha\rangle-2\pi i\epsilon\delta(\alpha)h-4\pi^2\epsilon^2\langle\alpha,\alpha\rangle\tag{4.4}$$

$$(L(\epsilon)+s(1-s))R(s,\epsilon)=R(s,\epsilon)(L(\epsilon)+s(1-s))=I.\tag{4.5}$$

Here

$$\begin{aligned}\langle f_1dz+f_2d\overline{z},g_1dz+g_2d\overline{z}\rangle &= 2y^2(f_1\overline{g_1}+f_2\overline{g_2})\\ \delta(pdx+qdy) &= -y^2(p_x+q_y).\end{aligned}$$

We notice also that

$$L^{(1)}(\epsilon)h = 4\pi i \langle dh, \alpha \rangle - 2\pi i \delta(\alpha)h - 8\pi^2 \epsilon \langle \alpha, \alpha \rangle, \tag{4.6}$$

$$L^{(2)}(\epsilon)h = -8\pi^2 \langle \alpha, \alpha \rangle, \tag{4.7}$$

$$L^{(i)}(\epsilon)h = 0, \quad \text{when } i \geq 3. \tag{4.8}$$

We notice that $L^{(i)}$ are compactly supported operators and that $\delta(\omega) = 0$ for a harmonic form ω.

We let $E(z, s, \epsilon)$ be the usual Eisenstein series for the system $(\Gamma, \overline{\chi}(\cdot, \epsilon))$ and define the Γ-invariant function

$$D(z, s, \epsilon) = U^{-1}(\epsilon)E(z, s, \epsilon).$$

The Phillips–Sarnak condition for dissolving cusp forms in this setting is:

$$\left\langle L^{(1)}u_j, E(z, s_j) \right\rangle \neq 0. \tag{4.9}$$

The family of operators $L(\epsilon)$ do not arise from an admissible metric, but all the properties described in Section 2 are well-known, and the proof of the dissolving theorem carries over almost verbatim, so in this case the higher order analogue of Fermi's golden rule also holds:

Theorem 4.1. *Assume that for $k = 0, 1, \dots, n-1$ the functions $D^{(k)}(z, s)$ are regular close to a cuspidal eigenvalue $s_j = 1/2 + ir_j$. Then $D^{(n)}(z, s)$ has at most a first order pole at s_j.*

1. *If $D^{(n)}(z, s)$ has a pole at s_j, then the embedded eigenvalue becomes a resonance.*
2. *Moreover, with $\|\cdot\|$ the standard L^2-norm,*

$$\Re \hat{s}_j^{(2n)}(0) = -\frac{1}{2m} \binom{2n}{n} \left\| \operatorname*{res}_{s=s_j} D^{(n)}(z, s) \right\|^2. \tag{4.10}$$

4.2. **Multiparameter perturbations.** To analyze the higher order dissolving conditions and relate it to a Dirichlet series it is useful to work with families of characters depending on several parameters. To do this we introduce the following notation. Given α_l, $l = 1, \dots, k$ harmonic, compactly supported 1-forms on M, we let $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_k)$ and define

$$D(z, s, \underline{\alpha}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{l=1}^{k} \left(\int_{i\infty}^{\gamma z} \alpha_l \right) \Im(\gamma z)^s. \tag{4.11}$$

Such series have been studied in [30, Lemma 2.4], and we give a quick review of some of their properties:

Let

$$\chi(\gamma, \underline{\epsilon}) = \prod_{l=1}^{k} \exp\left(-2\pi i \epsilon_l \int_{z_0}^{\gamma z_0} \alpha_l \right).$$

be the multiparameter character induced from $\underline{\alpha}$. We know from the theory of Eisenstein series (See e.g. [36, 18, 19]) that

$$E(z,s,\underline{\epsilon}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \chi(\gamma,\underline{\epsilon})\Im(\gamma z)^s, \quad \Re(s)>1.$$

admits meromorphic continuation to $\mathbb{C}$ and that it satisfies a functional equation

$$E(z,s,\underline{\epsilon}) = \phi(s,\underline{\epsilon})E(z,1-s,\underline{\epsilon}). \tag{4.12}$$

If we let

$$U(\underline{\epsilon})f = \prod_{l=1}^{k} \exp\left(2\pi i\epsilon_l\int_{z_0}^{z}\alpha_l\right) f(z) \tag{4.13}$$

we see that when $\Re(s)>1$

$$D(z,s,\underline{\alpha}) = \frac{\partial^k}{\partial\epsilon_1\cdots\partial\epsilon_k}U(-\underline{\epsilon})E(z,s,\underline{\epsilon})\bigg|_{\underline{\epsilon}=\underline{0}}. \tag{4.14}$$

We have – analogous to the 1-parameter situation described in the beginning of this section – that if $L(\underline{\epsilon}) = U^{-1}(\underline{\epsilon})\tilde{L}(\underline{\epsilon})U(\underline{\epsilon})$ then

$$\begin{aligned} L(\underline{\epsilon})h \quad = \quad & \Delta h + 4\pi i\textstyle\sum_{l=1}^{k}\epsilon_l\langle dh,\alpha_k\rangle - 2\pi i\left(\sum_{l=1}^{k}\epsilon_l\delta(\alpha_k)\right)h \\ & -4\pi^2\left(\textstyle\sum_{l,m=1}^{k}\epsilon_l\epsilon_m\langle\alpha_l,\alpha_m\rangle\right)h. \end{aligned} \tag{4.15}$$

Using this we arrive at the following theorem:

Theorem 4.2. *The function $D(z,s,\underline{\alpha})$ admits meromorphic continuation to $\mathbb{C}$. Furthermore it satisfies the following:*

1. *The poles of $D(z,s,\underline{\alpha})$ are included in the singular set for the surface M, and the pole order at a singular point is at most k.*
2. *For $\Re(s)>1/2$, and s not in the singular set, the function $D(z,s,\underline{\alpha})$ is square integrable and satisfies*

$$D(z,s,\underline{\alpha}) = -R(s)\left(\sum_{l=1}^{k}\partial_{\epsilon_l}L(\underline{\epsilon})\big|_{\underline{\epsilon}=\underline{0}}D(z,s,\underline{\alpha}_l)\right),$$

   where $\underline{\alpha}_l$ is $\underline{\alpha}$ with the l-th component removed. This equation provides the analytic continuation of $D(z,s,\underline{\alpha})$ using the meromorphic continuation of the Green's function. The analytically continued function grows at most polynomially as z tends to a cusp.
3. *For $1/2<\sigma_0<\Re(s)<\sigma_1$, and s not in the singular set, the function $D(z,s,\underline{\alpha})$ grows at most polynomially as $|\Im(s)|\to\infty$, and z is in a compact set.*
4. *The function $D(z,s,\underline{\alpha})$ satisfies a functional equation. This is derived by multiplying (4.12) by $U(-\underline{\epsilon})$ and differentiating both sides, using (4.14).*

Proof: (1), (2) and(3) can be found in [30], and (4) follows from diffentiation (4.12). See also [35]. □

Remark 4.3. An example of the functional equation in Theorem 4.2 (4) is

$$\begin{aligned} D(z,s,\alpha_1,\alpha_2) = \phi(s,\underline{0})D(z,1-s,\alpha_1,\alpha_2) \\ + \partial_{\epsilon_1}\phi(s,\underline{\epsilon})\big|_{\underline{\epsilon=0}} D(z,1-s,\alpha_2) + \partial_{\epsilon_2}\phi(s,\underline{\epsilon})\big|_{\underline{\epsilon=0}} D(z,1-s,\alpha_1) \\ + \partial_{\epsilon_2,\epsilon_1}\phi(s,\underline{\epsilon})\big|_{\underline{\epsilon=0}} E(z,1-s). \end{aligned}$$

Remark 4.4. We notice that, although Theorem 4.2 concerns $D(z,s,\underline{\alpha})$, where $\alpha=(\alpha_1,\ldots,\alpha_l)$ with α_l compactly supported, we can also handle non-compact but cuspidal cohomology in the following way: Since Theorem 4.2 immediately gives –through (4.14) – the properties of $\frac{\partial^k}{\partial\epsilon_1\cdots\partial\epsilon_k}E(z,s,\underline{\epsilon})\Big|_{\underline{\epsilon=0}}$, which is invariant under shift of α_l within its cohomology class. Since for every $f(z)\in S_2(\Gamma)$ the harmonic 1-form $\Re(f(z)dz)$ has a compactly supported form in its cohomology class, we see that Theorem 4.2 provides the analytic properties of the series

$$D^{n_1,\ldots,n_k}(z,s,\omega_1,\ldots,\omega_k) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\prod_{l=1}^{k}\left(\int_{i\infty}^{\gamma z}\omega_l\right)^{n_l}\Im(\gamma z)^s, \tag{4.16}$$

where ω_i, $i=1,\ldots,k$, are complex or real harmonic cuspidal 1-forms.

We note also that the 'differentiated scattering matrices'

$$\partial_{\epsilon_k,\ldots,\epsilon_1}\phi(s,\underline{\epsilon})\big|_{\underline{\epsilon=0}}$$

are invariant under shift of α_l within its cohomology class. We shall freely use these connections below.

Remark 4.5. It is well known (see e.g. [18, page 218, Remark 61]) that in the one-cusp case the scattering matrix is *even* in the character (i.e. $\phi(s,\chi)=\phi(s,\overline{\chi})$). It follows that

$$\partial_{\epsilon_k,\ldots,\epsilon_1}\phi(s,\underline{\epsilon})\big|_{\underline{\epsilon=0}} = 0$$

whenever k is odd.

4.3. **Dissolving and special values of Dirichlet series.**

By Theorem 4.1 the Phillips-Sarnak condition for the perturbation induced by ω is equivalent to

$$\operatorname*{res}_{s=s_j} D^1(z,s,\omega)\neq 0.$$

The following lemma identifies situations where the Phillip-Sarnak condition is not satisfied. This is seen as follows:

Lemma 4.6. *Let M be a finite volume hyperbolic surface of genus g. Let λ be an eigenvalue $>1/4$ of multiplicity m. If $g>m$, there exists a holomorphic cusp form $f(z)$ of weight 2 such that for both perturbations induced by the harmonic 1-forms $\omega_1=\Re(f(z)\,dz)$ and $\omega_2=\Im(f(z)\,dz)$ as in (4.4)* *the Phillips–Sarnak condition for dissolving the eigenvalue* λ *is* not *satisfied, i.e.*

$$\operatorname*{res}_{s=s_j} D^1(z,s,\omega_i)=0, \quad i=1,2. \tag{4.17}$$

Note that the condition (4.17) implies that $\operatorname{res}_{s=s_j} D^1(z,s,\omega)=0$ for all ω in the linear complex span of ω_1, ω_2, in particular for $f(z)dz$.

Proof. We have $M=\Gamma\backslash\mathbb{H}$ for some discrete cofinite subgroup Γ. Let E_λ be the eigenspace corresponding to the eigenvalue λ. Consider the linear map $\Lambda : S_2(\Gamma)\to E_\lambda$ sending $f\in S_2(\Gamma)$ to $\operatorname{res}_{s=s_j} D^{(1)}(z,s,f(z)dz)$, which is well-defined by Theorem 4.2 (2). Since $\dim S_2(\Gamma)=g$, the dimension formula implies that Λ has non-trivial kernel when $g-m>0$. For f in the kernel of Λ we have

$$\operatorname*{res}_{s=s_j} D^{(1)}(z,s,\omega_1)+i\operatorname*{res}_{s=s_j} D^{(1)}(z,s,\omega_2)=\operatorname*{res}_{s=s_j} D^{(1)}(z,s,f(z)dz)=0,$$

which gives the required result. □

Remark 4.7. There are numerically many known examples of surfaces of genus $g>1$ that has simple eigenvalues. For these Theorem 4.6 can be applied. Moreover the proof of the lemma shows that the dimension of the relevant f's is at least $g-m$.

We now introduce a Dirichlet series that plays a major role in investigating movement of an embedded eigenvalue if the Phillips-Sarnak condition is not satisfied. Let $f(z)=\sum_{n=1}^\infty a_n e^{2\pi i n z}$ be the Fourier expansion of $f(z)$ at the cusp $i\infty$. Let

$$u_j(z)=\sum_{n\neq 0} b_n\sqrt{y}K_{s_j-1/2}(2\pi\left|n\right| y)e^{2\pi i n x}$$

be the Fourier expansion of u_j, which for simplicity we may assume to be real-valued. We introduce the antiderivative of $f(z)$ as

$$F(z)=\int_{i\infty}^z f(w)\,dw=\sum_{n=1}^\infty \frac{a_n}{2\pi i n}e^{2\pi i n z}.$$

We define the Dirichlet series

$$L(u_j\otimes F^2,s)=\sum_{n=1}^\infty\left(\sum_{k_1+k_2=n}\frac{a_{k_1}}{k_1}\frac{a_{k_2}}{k_2}b_{-n}\right)\frac{1}{n^{s-1/2}}. \tag{4.18}$$

Since a_n, b_n grow at most polynomially in n we easily see that the above series converges absolutely for $\Re(s)$ sufficiently large.

By unfolding and inserting the relevant Fourier expansions we have, for $\Re(s)$ sufficiently large,

(4.19)

$$\begin{aligned}\left\langle D^2(z,s,f(z)dz),u_j\right\rangle &= \int_0^\infty\int_0^1\left(\int_{i\infty}^z f(w)dw\right)^2 y^s u_j(z)dx\frac{dy}{y^2}\\ &= \int_0^\infty\int_0^1\left(\sum_{n=1}^\infty\frac{a_n}{2\pi in}e(nz)\right)^2 y^s\sum_{n\neq 0}b_n\sqrt{y}K_{s_j-1/2}(2\pi\left|n\right|y)e(nx)dx\frac{dy}{y^2}\\ &= \frac{-1}{4\pi^2}\sum_{n=1}^\infty\int_0^\infty\left(\sum_{k_1+k_2=n}\frac{a_{k_1}}{k_1}\frac{a_{k_2}}{k_2}\right)b_{-n}e^{-2\pi ny}y^{s-1/2}K_{s_j-1/2}(2\pi ny)\frac{dy}{y}\\ &= \frac{-1}{4\pi^2}\sum_{n=1}^\infty\left(\sum_{k_1+k_2=n}\frac{a_{k_1}}{k_1}\frac{a_{k_2}}{k_2}\right)b_{-n}\frac{1}{(2\pi n)^{s-1/2}}\int_0^\infty e^{-t}t^{s-1/2}K_{s_j-1/2}(t)\frac{dt}{t}\\ &= \frac{-1}{2^{2s+1}\pi^{s+1}}L(u_j\otimes F^2,s)\frac{\Gamma(s+s_j-1)\Gamma(s-s_j)}{\Gamma(s)},\end{aligned}$$

where we have used [17, 6.621 3]. Using this we can now prove the basic properties of $L(u_j\otimes F^2,s)$.

Proposition 4.8. The series $L(u_j\otimes F^2,s)$ admits meromorphic continuation to $s\in\mathbb{C}$ with possible poles on the singular set. The poles are at most of first order. Furthermore we have the following functional equation: Let

$$\Lambda(u_j\otimes F^2,s)=\frac{1}{(4\pi)^s}\frac{\Gamma(s+s_j-1)\Gamma(s-s_j)}{\Gamma(s)}L(u_j\otimes F^2,s).$$

Then

$$\Lambda(u_j\otimes F^2,s)=\phi(s)\Lambda(u_j\otimes F^2,1-s)$$

where $\phi(s)$ is the scattering matrix.

Proof. This follows from (4.19) and the properties of $D^2(z,s,f(z)dz)$ as recorded in Theorem 4.2: Since the left hand side of (4.19) is meromorphic for $s\in\mathbb{C}$ this immediately gives meromorphic continuation of $L(u_j\otimes F^2,s)$ to $s\in\mathbb{C}$. Since

$$D^2(z,s,f(z)dz)=D^2(z,s,\omega_1)-D^2(z,s,\omega_2)+2iD^{1,1}(z,s,\omega_1,\omega_2),$$

we have by Theorem 4.2 that $D^2(z,s,f(z)dz)=-R(s)(\psi(z,s))$ where $\psi(z,s)$ has at most a simple pole on the singular set. But then

$$\left\langle D^2(z,s,f(z)dz),u_j\right\rangle=\left\langle\psi(z,s),R(\overline{s})u_j\right\rangle=\frac{1}{s(1-s)-\lambda_j}\left\langle\psi(z,s),u_j\right\rangle$$

which then holds for $s\in\mathbb{C}$ by meromorphic continuation. Comparing with (4.19) and noticing that $\Gamma(s-s_j)$ has a simple pole at $s=s_j$ we prove that the poles of $L(u_j\otimes F^2,s)$ are all at most simple.

The functional equation in Theorem 4.2 (4), reduces, in this case, to

$$D^2(z,s,f(z)dz)=\phi(s)D^2(z,1-s,f(z)dz)+\phi^{(2)}(s,f(z)dz)E(z,1-s).$$

The fact that $\langle E(z,s),u_j\rangle=0$ and Equation (4.19) give

$$\begin{aligned}\frac{1}{4^s\pi^s}L(u_j\otimes F^2,s)\frac{\Gamma(s+s_j-1)\Gamma(s-s_j)}{\Gamma(s)}\\ =\phi(s)[\text{same expression evaluated at } 1-s].\end{aligned}\tag{4.20}$$

□

We note that in the case of multiple cusps the above functional equation becomes more complicated since in general $\phi^{(1)}(s,f(z)dz)$ can be a non-zero matrix (with diagonal entries equal to zero).

Lemma 4.9. *Assume (4.17), i.e. that the Phillips-Sarnak condition is not satisfied for the perturbations induced by both ω_i, $i=1,2$. Then $L(u_j\otimes F^2,s)$ is regular at $s=s_j$.*

Proof. As in the proof of Proposition 4.8 we have that $D^2(z,s,f(z)dz)=-R(s)(\psi(z,s))$ where $\psi(z,s)$ has at most a simple pole at s_j. But assuming (4.17), it follows easily from Theorem 4.2 (2), that $\psi(z,s)$ is in fact regular since the only potential poles would come from $D^1(z,s,\omega_i)$. Therefore, as in the proof of Proposition 4.8, we conclude that $\left\langle D^2(z,s,f(z)dz),u_j\right\rangle$ has at most a simple pole at s_j. Comparing with (4.19) and again using that $\Gamma(s-s_j)$ has a simple pole at $s=s_j$ gives the claim. □

Theorem 4.10. *Assume (4.17), i.e. that the Phillips-Sarnak condition is not satisfied under perturbations induced by both ω_i, $i=1,2$, and that $L(u_j\otimes F^2,s_j)\neq 0$. For all directions ω in the real span of ω_1,ω_2 with at most two exceptions we have*

$$\Re\hat{s}_j^{(4)}(0,\omega)\neq 0.$$

In particular there exists a cusp form with eigenvalue $s_j(1-s_j)$ that is dissolved in this direction.

Proof. The Phillips-Sarnak condition will not be satisfied in the whole span of ω_1, ω_2. Assume that $\Re\hat{s}_j^{(4)}(0,\omega)=0$ for three distinct directions given by $\eta_k=a_k\omega_1+b_k\omega_2$, $k=1,2,3$, i.e. $(a_kb_l-a_lb_k)\neq 0$ for $k\neq l$. We have

$$D^2(z,s,\eta_k)=a_k^2D^2(z,s,\omega_1)+b_k^2D^2(z,s,\omega_2)+2a_kb_kD^{1,1}(z,s,\omega_1,\omega_2).$$

We can solve for $D^2(z,s,\omega_1)$, $D^2(z,s,\omega_2)$, and $D^{1,1}(z,s,\omega_1,\omega_2)$ as long as the following determinant is nonzero:

$$\begin{vmatrix} a_1^2 & b_1^2 & 2a_1b_1\\ a_2^2 & b_2^2 & 2a_2b_2\\ a_3^2 & b_3^2 & 2a_3b_3\end{vmatrix}=-2a_1^2a_2^2a_3^2\begin{vmatrix} 1 & b_1/a_1 & (b_1/a_1)^2\\ 1 & b_2/a_2 & (b_2/a_2)^2\\ 1 & b_3/a_3 & (b_3/a_3)^2\end{vmatrix}=2\prod_{k>l}(a_kb_l-a_lb_k)\neq 0,$$

where we have used the fact that the last matrix is Vandermonde. Since $\Re\hat{s}_j^{(4)}(0,\eta_k)=0$, it follows from Theorem 3.1 that $D^2(z,s,\eta_k)$ is regular at s_j, and by solving the above system that $D^2(z,s,\omega_1)$, $D^2(z,s,\omega_2)$, and $D^{1,1}(z,s,\omega_1,\omega_2)$ are regular. This implies that

$$D^2(z,s,f(z)dz) = D^2(z,s,\omega_1) - D^2(z,s,\omega_2) + 2iD^{1,1}(z,s,\omega_1,\omega_2)$$

is regular also at $s = s_j$. By (4.19) it follows that $L(u_j \otimes F^2, s)$ must have a zero at s_j, since the Gamma factor $\Gamma(s-s_j)$ has a pole at that point. This contradicts the assumption of the theorem.

□

Remark 4.11. We note that the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty}\sum_{j=1}^{n-1} \frac{a_{n-j}a_j}{j}\frac{1}{n^s} = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\frac{a_{n-j}a_j}{(n-j)j}\frac{1}{n^{s-1}}$$

was recently studied by Diamantis, Knopp, Mason, O'Sullivan and Deitmar [7, 8]. The series $L(u_j \otimes F^2, s)$ (See 4.18) has the structure of a Rankin-Selberg convolution between $D(s)$ and the L-function for u_j.

Remark 4.12. The special value in Theorem 4.10 is on the critical line and at the same height as the trivial zeros of $L(u_j \otimes F^2, s)$.

There is a generalization of Theorem 4.10 which we describe briefly: Define

$$L(u_j \otimes F^l, s) = \sum_{n=1}^{\infty}\left(\sum_{k_1+\cdots+k_l=n}\frac{a_{k_1}}{k_1}\cdots\frac{a_{k_l}}{k_l}b_{-n}\right)\frac{1}{n^{s-1/2}}. \tag{4.21}$$

By essentially the same computation as in (4.19) we have

$$\langle D^n(z,s,f(z)dz), u_j\rangle = \frac{-1}{2^{2s+1}\pi^{s+1}}L(u_j\otimes F^n, s)\frac{\Gamma(s+s_j-1)\Gamma(s-s_j)}{\Gamma(s)}. \tag{4.22}$$

Notice that this computation also proves the meromorphic continuation of $L(u_j \otimes F^n, s)$ to $s \in \mathbb{C}$, and that at s_j the function $L(u_j \otimes F^l, s)$ has a pole of order at most $n-1$.

Theorem 4.13. *Assume that $L(u_j \otimes F^n, s)$ does not have a zero at s_j. For all directions ω in the real span of ω_1, ω_2 with at most n exceptions we have*

$$\Re \hat{s}_j^{(2r)}(0,\omega) \neq 0,$$

for some $r \leq n$. In particular there exists a cusp form with eigenvalue $s_j(1-s_j)$ that is dissolved in this direction.

Proof. Assume that $\Re\hat{s}_j^{(2r)}(0,\eta_k) = 0$, $r = 1,\dots,n$ for $n+1$ distinct directions given by $\eta_k = a_k\omega_1 + b_k\omega_2$, $k = 0,\dots n$, i.e. $(a_kb_l - a_lb_k) \neq 0$ for $k \neq l$. We have

$$D^n(z,s,\eta_k) = \sum_{l=0}^{n}\binom{n}{l}a_k^l b_k^{n-l}D^{l,n-l}(z,s,\omega_1,\omega_2)$$

We can solve for all $D^{l,n-l}(z,s,\omega_1,\omega_2)$ as long as the following determinant is nonzero:

$$\left|\binom{n}{l}a_k^{n-l}b_k^l\right|_{k,l=0}^n = \left(\prod_{l=0}^n\binom{n}{l}\right)\left|a_k^{n-l}b_k^l\right|_{k,l=0}^n = \left(\prod_{l=0}^n\binom{n}{l}a_l^n\right)\left|(b_k/a_k)^l\right|_{k,l=0}^n$$
$$= \left(\prod_{l=0}^n\binom{n}{l}\right)\prod_{k<l}(a_kb_l-a_lb_k)\neq 0,$$

where again we have used the fact that the last matrix is Vandermonde. It follows that all $D^{n-l,l}(z,s)$ are linear combinations of $D^n(z,s,\eta_k)$, $k=0,\dots n$.

Since

$$\Re\hat{s}_j^{(2r)}(0,\eta_k)=0,$$

Theorem 4.1 allows us to conclude that $D^n(z,s,\eta_k)$ is regular at s_j, and therefore also that $D^{n-l,l}(z,s)$ is regular at s_j. Since

$$D^n(z,s,f(z)dz)=\sum_{j=0}^n\binom{n}{j}i^{n-j}D^{j,n-j}(z,s,\omega_1,\omega_2)$$

it follows also that $D^n(z,s,f(z)dz)$ is regular at s_0.

Therefore the left of (4.22) is regular. It follows that the right-hand side of (4.22) is regular, which proves – since $\Gamma(s-s_j)$ has a pole at s_j – that $L(u_j\otimes F^n,s)$ has a zero at s_j. But this contradicts the assumption of the theorem. □

References

[1] Avelin, H. Deformation of $\Gamma_0(5)$-cusp forms. Math. Comp. 76 (2007), no. 257, 361–384

[2] Balslev, E. Spectral deformation of Laplacians on hyperbolic manifolds. Comm. Anal. Geom. 5 (1997), no. 2, 213–247.

[3] Balslev, E.; Venkov, A. Spectral theory of Laplacians for Hecke groups with primitive character. Acta Math. 186 (2001), no. 2, 155–217.

[4] Balslev, E.; Venkov, A. On the relative distribution of eigenvalues of exceptional Hecke operators and automorphic Laplacians. Algebra i Analiz 17 (2005), no. 1, 5–52; translation in St. Petersburg Math. J. 17 (2006), no. 1, 1–37

[5] Bruggeman, R. W. Families of automorphic forms. Monographs in Mathematics, 88. Birkhuser Boston, Inc., Boston, MA, 1994. x+317 pp. ISBN: 3-7643-5046-6

[6] Chinta, G.; Goldfeld, D. Grössencharakter L-functions of real quadratic fields twisted by modular symbols. Invent. Math. 144 (2001), no. 3, 435–449.

[7] A. Deitmar and N. Diamantis. Automorphic forms of higher order. J. London Math. Soc., page jdp015, 2009.

[8] N. Diamantis, M. Knopp, G. Mason, C. O'Sullivan. L-functions of second-order cusp forms. Ramanujan J. 12 (2006), no. 3, 327–347.

[9] Y. Colin de Verdière, Pseudo-laplaciens II. Ann. Inst. Fourier (Grenoble) **33** (1983) no. 2, 87–113.

[10] Deshouillers, J.-M.; Iwaniec, H. The nonvanishing of Rankin–Selberg zeta-functions at special points. The Selberg trace formula and related topics (Brunswick, Maine, 1984), 51–95, Contemp. Math., 53, Amer. Math. Soc., Providence, RI, 1986.

[11] Deshouillers, J.-M.; Iwaniec, H.; Phillips, R. S.; Sarnak, P. Maass cusp forms. Proc. Nat. Acad. Sci. U.S.A. 82 (1985), no. 11, 3533–3534.

[12] N. Diamantis, C. O'Sullivan, Hecke theory of series formed with modular symbols and relations among convolution L-functions. Math. Ann. **318** (2000), no. 1, 85–105.

[13] D. Farmer; S. Lemurell. Deformations of maass forms. Mathematics of Computation, 74:1967–1982, 2005.

[14] D. Goldfeld, Zeta functions formed with modular symbols. Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 111–121, Proc. Sympos. Pure Math., 66, Part 1, Amer. Math. Soc., Providence, RI, 1999.

[15] D. Goldfeld, The distribution of modular symbols. Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), 849–865, de Gruyter, Berlin, 1999.

[16] D. Goldfeld, C. O'Sullivan, Estimating additive character sums for Fuchsian groups, Rankin memorial issues. Ramanujan J. 7 (2003), no. 1-3, 241–267.

[17] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[18] D. A. Hejhal, The Selberg trace formula for PSL(2, R). Vol. 2. Lecture Notes in Mathematics, 1001. Springer-Verlag, Berlin, 1983. viii+806 pp.

[19] Iwaniec, H. Spectral methods of automorphic forms. Second edition. Graduate Studies in Mathematics, 53. American Mathematical Society, Providence, RI; Revista Matemtica Iberoamericana, Madrid, 2002. xii+220 pp.

[20] Jorgenson, J.; O'Sullivan, C. Unipotent vector bundles and higher-order non-holomorphic Eisenstein series. J. Théor. Nombres Bordeaux 20 (2008), no. 1, 131–163.

[21] Kato, T. Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. xxii+619 pp. ISBN: 3-540-58661-X

[22] Lax, P. D.; Phillips, R. S. Scattering theory for automorphic functions. Annals of Mathematics Studies, No. 87. Princeton Univ. Press, Princeton, N.J., 1976. x+300 pp. ISBN: 0-691-08179-4

[23] Luo, W. Nonvanishing of L-values and the Weyl law. Ann. of Math. (2) 154 (2001), no. 2, 477–502.

[24] Müller, W. Spectral geometry and scattering theory for certain complete surfaces of finite volume. Invent. Math. 109 (1992), no. 2, 265–305.

[25] Müller, W. Spectral theory for Riemannian manifolds with cusps and a related trace formula. Math. Nachr. 111 (1983), 197–288.

[26] C. O'Sullivan, Properties of Eisenstein series formed with modular symbols. J. Reine Angew. Math. **518** (2000), 163–186.

[27] C. O'Sullivan, Properties of Eisenstein series formed with modular symbols. Ph. D. Thesis, Columbia University, 1998.

[28] Petridis, Y. N. On the singular set, the resolvent and Fermi's golden rule for finite volume hyperbolic surfaces. Manuscripta Math. 82 (1994), no. 3-4, 331–347.

[29] Y. N. Petridis, Spectral deformations and Eisenstein Series Associated with Modular Symbols. Internat. Math. Res. Notices **2002**, no. 19, 991–1006.

[30] Petridis, Y. N.; Risager, M. S. Modular symbols have a normal distribution. Geom. Funct. Anal. 14 (2004), no. 5, 1013–1043.

[31] Petridis, Y. N.; Risager, M. S., Dissolving cusp forms in Teichmüller spaces, in preparation.

[32] Phillips, R. S.; Sarnak, P. On cusp forms for co-finite subgroups of $\mathrm{PSL}(2,R)$. Invent. Math. 80 (1985), no. 2, 339–364.

[33] Phillips, R.; Sarnak, P. Perturbation theory for the Laplacian on automorphic functions. J. Amer. Math. Soc. 5 (1992), no. 1, 1–32.

[34] Phillips, R.; Sarnak, P. Cusp forms for character varieties. Geom. Funct. Anal. 4 (1994), no. 1, 93–118.
[35] Risager, M. Automorphic forms and modular symbols, Ph.D. Thesis, Aarhus Universitet, 2003.
[36] A. Selberg. Göttingen lecture notes in Collected papers. Vol. I. Springer-Verlag, Berlin, 1989. With a foreword by K. Chandrasekharan.
[37] Simon, B. Resonances in n-body quantum systems with dilatation analytic potentials and the foundations of time-dependent perturbation theory. Ann. of Math. (2) 97 (1973), 247–274.
[38] Sarnak, P. On cusp forms. II. Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), 237–250, Israel Math. Conf. Proc., 3, Weizmann, Jerusalem, 1990.
[39] Wolpert, S. A. Disappearance of cusp forms in special families. Ann. of Math. (2) 139 (1994), no. 2, 239–291.

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