Fourier Coefficients of Cusp Forms

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Abstract. We prove the estimate $a_n = O(|n|^{3/8+\epsilon})$ for the Fourier coefficients of a Maaß cusp form of an arbitrary cofinite subgroup of $SL(2,\mathbb{R})$ and provide a uniform way of treating these together with the Fourier coefficients of a holomorphic cusp form.

1. Introduction

Let Γ be a cofinite discrete subgroup of $SL(2,\mathbb{R})$. Let $F(z)$ be either

(a): a holomorphic cusp form of even integral weight $2k$ for Γ, or

(b): a Maaß cusp form for Γ, i.e., an eigenfunction of the noneuclidean Laplace operator which is cuspidal ($k=0$).

The group Γ contains parabolic elements and, more precisely, we assume that the manifold $\Gamma \setminus \mathbb{H}$ has a cusp at infinity with stabilizer the standard parabolic subgroup Γ_∞. That $F(z)$ is automorphic with respect to Γ means

$$F(\gamma z) = (cz+d)^{2k} F(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.1}$$

Since $F(z+1) = F(z)$, $F(z)$ has a Fourier expansion at infinity

$$F(z) = \sum a_n |n|^{k-1/2} W(nz) \tag{1.2}$$

for $\Im z > 0$, where $W(z) = e^{2\pi i z}$ for (a) and $W(z) = |y|^{1/2} K_{i\lambda}(2\pi |y|) e^{2\pi i x}$ for (b). In the case (b) $1/4 + \lambda^2$ is the eigenvalue of Δ corresponding to F. The summation is over positive integers in (a) and over nonzero integers in (b).

The problem of estimating the order of magnitude of the Fourier coefficients a_n has a long history, see Selberg [**11**]. The Hecke bound is $a_n = O(|n|^{1/2})$. The Ramanujan-Petersson conjecture is

$$a_n = O(|n|^{\epsilon}).$$

This was proved by Deligne for the holomorphic case (a) and for certain arithmetic groups like $SL(2,\mathbb{Z})$.

For the case (b) and arithmetic subgroups of $SL(2,\mathbb{R})$ Hecke's bound was improved by Shahidi [**12**]: $a_n = O(|n|^{1/5})$, and, more recently, by Bump-Duke-Hoffstein-Iwaniec [**1**]: $a_n = O(|n|^{5/28+\epsilon})$. However, the Ramanujan conjecture is not proven even for $SL(2,\mathbb{Z})$.

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For a general group, which is not necessarily arithmetic, though, the known results are not so sharp. Anton Good [**4**] proved that

$$a_n = O(n^{1/3+\epsilon}) \tag{1.3}$$

for the case (a) and weight $2k > 2$. This was proved using Rankin–Selberg convolutions. Let $D(s) = \sum |a_n|^2 |n|^{-s}$ be the Rankin–Selberg convolution of F with itself. Then we have the integral representation

$$D(s) = \frac{4\pi^{s+2k-1}}{\Gamma(s+2k-1)} \int_{\Gamma\backslash\mathbb{H}} y^{2k}|F|^2 E(z,s)\, dxdy/y^2 \tag{1.4}$$

in the case (a) and

$$D(s) = \frac{2\pi^s\Gamma(s)}{\Gamma(s/2)^2\Gamma(s/2+i\lambda)\Gamma(s/2-i\lambda)} \int_{\Gamma\backslash\mathbb{H}} |F|^2 E(z,s)\, dxdy/y^2 \tag{1.5}$$

in the case (b), where $E(z,s)$ is the nonholomorphic Eisenstein series corresponding to the cusp at infinity. The Gamma factors are asymptotic to $|t|^{1-2k}e^{\pi|t|/2}$, as $t \to \pm\infty$ on the critical line $s = 1/2+it$. To estimate effectively the Rankin–Selberg convolution $D(s)$ on its critical line one needs exponential decay of the integrals in equations (1.4) and (1.5) on their critical line. So the issue is to estimate the inner product

$$\langle f(z), E(z, 1/2+it)\rangle, \tag{1.6}$$

where $f(z) = y^{2k}|F|^2$. Closely related are the inner products

$$\langle f(z), \phi_j(z)\rangle, \tag{1.7}$$

where the ϕ_j's form an orthonornal basis for the discrete spectrum of the Laplace operator on $L^2(\Gamma \setminus \mathbb{H})$ with corresponding eigenvalues $\lambda_j = s_j(1-s_j) = 1/4 + t_j^2$. Here $s_j = 1/2 + t_j$ and either $1/2 \le s_j \le 1$ or $\Re s_j = 1/2$. The study of the inner products (1.6) and (1.7) plays a role in proving that a positive proportion of the zeros of the L-series of F lie on its critical line, see Hafner [**5**].

The estimates one would like to have are of the form

$$\int_T^{T+1} |\langle f(z), E(z,1/2+it)\rangle|^2 \, dt \ll e^{-\pi T}T^m, \tag{1.8}$$

and

$$\langle f(z), \phi_j(z)\rangle \ll e^{-\pi t_j/2} t_j^{m/2}, \tag{1.9}$$

and, if possible,

$$\sum_{0<t_j\le T} |\langle f, \phi_j\rangle|^2 e^{\pi t_j} + \sum_{j=1}^{\kappa} \int_{-T}^{T} |\langle f(z), E_j(z,1/2+it)\rangle|^2 e^{\pi|t|}\, dt \ll T^m \tag{1.10}$$

for some nonnegative constant m and $\ll$ means that the left hand side is bounded by a constant multiple of the right hand side as $T \to \infty$ (or $j \to \infty$ in (1.9)). Here $E_j(z,s)$, $j = 1, \dots, \kappa$, is the Eisenstein series corresponding to the j-cusp of Γ, where Γ has κ inequivalent cusps. We have

THEOREM 1.1 (Good [**4**]). *If Γ is general cofinite subgroup of $SL(2,\mathbb{R})$ and F is a holomorphic cusp form of weight $2k > 2$ then* (1.10) *holds with $m = 4k$.*

THEOREM 1.2 (Jutila [**7**] and [**8**]). *For $\Gamma = SL(2,\mathbb{Z})$ the bound* (1.10) *holds with $m = 4k + \epsilon$ for both cases* (a) *and* (b).

Good's proof uses heavily Kloosterman sums and the fact that any holomorphic cusp form is a linear combination of Poincaré series. This is why it does not apply to the case (b). Jutila's proof unifies both cases by using the similarity of the estimate (1.10) with the additive divisor problem.

THEOREM 1.3 (Sarnak [**10**]). *For general subgroup* Γ *and* F *a Maaß cusp form the estimates* (1.8) *and* (1.9) *hold with* $m = 2 + \epsilon$.

This was the first theorem for general group Γ and the case (b). Subsequently I improved on it using the same method:

THEOREM 1.4 (Petridis [**9**]). *For general* Γ *and* F *a Maaß cusp form with* $\lambda \neq 0$ *the estimates* (1.8) *and* (1.9) *hold with* $m = 1$.

In the last two theorems one can also assume that Γ is cocompact and the estimate (1.9) holds with the same m. Good's and Jutila's methods do not apply in the cocompact case.

The last two theorems give the following estimates for the Fourier coefficients of a Maaß cusp form: $a_n = O(|n|^{5/12+\epsilon})$ (Sarnak [**10**]), which I improved to $a_n = O(|n|^{2/5+\epsilon})$.

Hejhal [**6**] has heuristic arguments showing that the Ramanujan conjecture $a_n = O(|n|^{\epsilon})$ holds for groups with no exceptional eigenvalues, i.e., eigenvalues less than 1/4. He uses known and conjectural properties of the horocyclic flow.

In this work I prove

THEOREM 1.5. *For a general group* Γ *we can take in* (1.10) $m = 4k+1$ *in both cases* (a) *and* (b), *provided that* $\lambda \neq 0$.

COROLLARY 1. We have the following bound for the Fourier coefficients

$$a_n = O(|n|^{3/8+\epsilon}). \tag{1.11}$$

This bound is worse than Good's for the holomorphic case but is new in the Maaß case (b).

In case Γ is cocompact we can ignore the term involving the Eisenstein series and we get

COROLLARY 2. If $F(z)(dz)^k$ is a holomorphic k-differential, then

$$\sum_{0<t_j\leq T} |\langle f, \phi_j\rangle|^2 e^{\pi t_j} \ll T^{4k+1}. \tag{1.12}$$

We see that Theorem 1.5 is worse than Theorem 1.1 for the holomorphic case and worse than Theorem 1.2 for $SL(2,\mathbb{Z})$.

A further question relevant to the case (b) is the dependance of the estimate on the eigenvalue $1/4+\lambda^2$. All the estimates depend on the supremum norm of F, so we get $a_n = O(|n|^{3/8+\epsilon}e^{\pi\lambda/2}\lambda^{1/2})$, if we assume that $||F||_2 = 1$.

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2. The holomorphic case

Let D_k denote the noneuclidean Laplacian for functions of weight $2k$

$$D_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iky\frac{\partial}{\partial x}. \tag{2.1}$$

Let $g(z)=y^kF(z)$, so that $|g|^2=f$. Then

$$D_k g = k(k-1)g. \tag{2.2}$$

We use polar coordinates around any point z_0 in the upper half-plane $\mathbb{H}$ given by

$$\frac{z-z_0}{z-\bar{z}_0}=\tanh(r/2)e^{i\theta}, \tag{2.3}$$

where r is the hyperbolic distance. Let g be any function which satisfies (2.2) and which is bounded in a neighborhood of z_0. Then we have a Fourier expansion

$$g(z)\left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^k=\sum_{n=-\infty}^{\infty}\varphi_n(r)e^{in\theta}. \tag{2.4}$$

Let

$$\breve{D}_k=\frac{\partial^2}{\partial r^2}+\frac{\cosh r}{\sinh r}\frac{\partial}{\partial r}+\frac{1}{\sinh^2 r}\frac{\partial^2}{\partial\theta^2}+\frac{2}{1+\cosh r}\left(k^2-ik\frac{\partial}{\partial\theta}\right) \tag{2.5}$$

be D_k in polar coordinates. Then $\breve{D}_k\varphi_n(r)e^{in\theta}=k(k-1)\varphi_n(r)e^{in\theta}$. The operator $\breve{D}_k$ and the solutions to this equation are discussed in Fay [**3**]. Here is a summary of the results. More details can be found in the Appendix.

We change variable by setting $u=\cosh r$. So $\varphi_n(r)$ satisfies

$$\frac{d^2\varphi_n}{du^2}+\frac{2u}{u^2-1}\frac{d\varphi_n}{du}=\left\{\frac{k(k-1)}{u^2-1}+\frac{n^2}{(u^2-1)^2}-\frac{2(k^2+kn)}{(u+1)(u^2-1)}\right\}\varphi_n. \tag{2.6}$$

The solution to this equation which is regular at $u=1$ (corresponding to $r=0$) is

$$C_n(r)=\left(\frac{u-1}{u+1}\right)^{n/2}\left(\frac{2}{1+u}\right)^k=\tanh^n(r/2)\left(\cosh^2(r/2)\right)^{-k} \tag{2.7}$$

for $n\geq 0$ and

$$\begin{aligned} C_n(r) &= \left(\frac{u-1}{u+1}\right)^{|n|/2}\left(\frac{2}{1+u}\right)^k F(2k,|n|,|n|+1,(u-1)/(u+1)) \\ &= \tanh^{|n|}(r/2)\left(\cosh^2(r/2)\right)^{-k}F(2k,|n|,|n|+1,\tanh^2(r/2)) \end{aligned} \tag{2.8}$$

for $n<0$. Here $F(a,b,c,z)$ is the Gauss hypergeometric function. Therefore

$$g(z)\left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^k=\sum_{n=-\infty}^{\infty}b_nC_n(r)e^{in\theta}$$

for some constants b_n.

Lemma 2.1. *The sequence b_n satisfies*

$$\sum_{n=-N}^{N}|b_n|^2\ll||g||_\infty^2N^{2k}. \tag{2.9}$$

Proof. The function $g(z)=y^kF(z)$ is bounded on the upper half plane and $(z-\bar{z}_0)/(z_0-\bar{z})$ has modulus 1, which implies that

$$B(r)=\int_0^{2\pi}\left|g(z)\left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^k\right|^2d\theta=\int_0^{2\pi}|g(z)|^2\,d\theta \tag{2.10}$$

is bounded by $M = 2\pi||g||_\infty^2$, for $r \geq 0$. Parceval's identity implies

$$B(r) = \sum_{n=-\infty}^{\infty} |b_n|^2 |C_n(r)|^2.$$

We find lower bounds for $C_n(r_N)$ along certain subsequence r_N for $|n| \leq N$, which imply an upper bound on the sequence b_n. We examine separately the positive and the negative terms in the series. For $N > 1$ we choose a sequence $\cosh^2(r_N/2) = N$ so that $C_n(r_N) = (1-1/N)^{n/2} N^{-k} \geq (1-1/N)^{N/2} N^{-k} \geq 0.5 N^{-k}$ for $0 \leq n \leq N$. Then

$$M \geq B(r_N) \geq \sum_{n=0}^{N} |b_n|^2 |C_n(r_N)|^2 \geq \frac{1}{4} \sum_{n=0}^{N} |b_n|^2 N^{-2k}. \tag{2.11}$$

For $n < 0$ we also need a lower bound on the function $F(2k, |n|, |n|+1, \tanh^2(r/2))$. We use the fundamental integral representation for the hypergeometric function, see Erdélyi [**2**, 2.1.3(10), p. 59], to get

$$F(2k, |n|, |n|+1, \tanh^2(r/2)) = \frac{\Gamma(|n|+1)}{\Gamma(|n|)\Gamma(1)} \int_0^1 t^{|n|-1}(1 - t(\tanh^2(r/2))^{-2k}\, dt$$

and, since $1 - t(\tanh^2(r/2)) \leq 1$,

$$F(2k, |n|, |n|+1, \tanh^2(r/2)) \geq |n| \int_0^1 t^{|n|-1}\, dt = 1.$$

Now the argument continues as in (2.11). □

Remark 2.2. Actually much more is true for the coefficients b_n for $n < 0$. Using the identity (23) p. 64 in [**2**], which holds even in the degenerate case of the hypergeometric equation, since it depends only on a change of variables in the fundamental integral representation, we get

$$\begin{aligned} F(a, j, j+1, z) &= (1-z)^{-a+1} F(j+1-a, 1, j+1, z) \\ &= (1-z)^{-a+1} j \int_0^1 (1-t)^{j-1}(1-tz)^{-j-1+a}\, dt \end{aligned}$$

where we take $a = 2k$ and $j = |n|$. As long as $0 \leq z \leq 1$ and $j \geq a-1$ the last integral is greater or equal to $1/j$. Therefore for $j \geq a-1$ we have $F(a, j, j+1, 1-1/N) \geq N^{a-1}$, which in our case gives $F(2k, |n|, |n|+1, 1-1/N) \geq N^{2k-1}$. Following the argument above we get that

$$\sum_{n=-N}^{-k_0} |b_n|^2 (1-1/N)^N N^{-2k} N^{4k-2}$$

is bounded, which implies $\sum_{n=-N}^{-k_0} |b_n|^2 \ll N^{2-2k}$, so the b_n's are zero eventually, if $k > 1$, or they are in l^2 for $k = 1$.

Lemma 2.3. *The function $B(r)$ extends to an even analytic function of r in the strip $|\Im r| < \pi/2$ and satisfies the bound*

$$B(r) \ll ||g||_\infty^2 \frac{|\cosh(r/2)|^{4k} + |\cosh(r/2)|^{-4k}}{(1 - |\tanh(r/2)|^2)^{2k}}. \tag{2.12}$$

PROOF. We notice that $|C_n(r)|^2$ extends analytically, since for $n \geq 0$,

$$|C_n(r)|^2 = \tanh^n(r/2)\overline{\tanh^n(\bar{r}/2)}\left(\cosh^2(r/2)\right)^{-k}\left(\overline{\cosh^2(\bar{r}/2)}\right)^{-k}$$

and for $n < 0$

$$|C_n(r)|^2 = |C_{-n}|^2 F(2k, -n, -n+1, \tanh^2(r/2)) F(2k, -n, -n+1, \overline{\tanh^2(\bar{r}/2)})$$

and $F(a,b,c,z)$ is an analytic function on the disc $|z| < 1$ and $z = \tanh(r/2)$ maps the strip $|\Im r| < \pi/2$ conformally onto this disc. We now need an upper bound of the hypergeometric function in $C_n(r)$, $n < 0$, on the strip $|\Im r| < \pi/2$. The fundamental integral representation for the hypergeometric function [**2**, 2.1.3(10), p. 59] gives

$$F(2k, |n|, |n|+1, z) = \frac{\Gamma(|n|+1)}{\Gamma(|n|)\Gamma(1)} \int_0^1 t^{|n|-1}(1-tz)^{-2k}\, dt$$

and, since $|1-z| \leq 2|1-tz|$ for $|z| < 1$ and $0 \leq t \leq 1$,

$$|(1-z)^{2k} F(2k, |n|, |n|+1, z)| \leq 2^{2k}|n| \int_0^1 t^{|n|-1|}\, dt \ = 2^{2k}.$$

This implies that

$$|C_n(r)| \ll |\tanh(r/2)|^{|n|} |\cosh(r/2)|^{2k}$$

for $n < 0$, while for $n \geq 0$ we clearly have

$$|C_n(r)| \ll |\tanh(r/2)|^{|n|} |\cosh(r/2)|^{-2k}$$

We work first with the positive terms of the series $\sum |b_n|^2 |\tanh(r/2)|^{2|n|}$. We set $B_N = \sum_{n=0}^N |b_n|^2$. Summation by parts gives

$$\sum_{n=0}^{\infty} |b_n|^2 |\tanh(r/2)|^{2n} = \sum_{n=0}^{\infty} B_n |\tanh(r/2)|^{2n} (1 - |\tanh(r/2)|^2)$$

and equation (2.9) implies that $B_n \ll ||g||_\infty^2 n^{2k}$. Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} |b_n|^2 |\tanh(r/2)|^{2n} &\ll ||g||_\infty^2 \sum_{n=0}^{\infty} n^{2k} |\tanh(r/2)|^{2n} (1 - |\tanh(r/2)|^2) \\ &\ll ||g||_\infty^2 \frac{1}{(1 - |\tanh(r/2)|^2)^{2k}}, \end{aligned}$$

since

$$\sum_{n=0}^{\infty} n^m w^n = \frac{P_m(w)}{(1-w)^{m+1}},$$

where $P_m(w)$ is a polynomial of degree m that does not vanish at $w = 1$. The same calculation can be applied for the negative part of the series and (2.12) follows. □

REMARK 2.4. We note that on the horizontal lines $r = x \pm i(\pi/2 - 1/t)$, $x \in \mathbb{R}$, we have the bound

$$B(r) \ll e^{(4k)|x|} t^{2k}. \tag{2.13}$$

This is seen as follows. An elementary argument gives

$$\begin{aligned} |\cosh(r/2)|^2 &= \cos^2(\pi/4 - 1/2t) + \sinh^2(x/2) \ll e^{|x|} \\ (1 - |\tanh(r/2)|^2)^{-1} &= \frac{1 + e^{2x} + 2e^x \sin(1/t)}{4e^x \sin(1/t)} \ll te^{|x|}, \end{aligned}$$

since $\sin(1/t) \geq 1/2t$ for t sufficiently large. Moreover the term $|\cosh(r/2)|^{-4k}$ is bounded.

3. Proof of the theorem

The proof of Theorem follows the exact steps as the proof in [**9**]. However, we need a new family of point-pair invariants. The family of point-pair invariants that are needed are

$$k_T(r) = \int_{T_0}^{T} \frac{tP_{-1/2+it}(\cosh r)\sinh^2 r}{\cosh^{8k+10}(r/2)}\,dt$$

for T_0 sufficiently large. Their Selberg–Harish-Chandra transform $H_T(s)$ is

$$H_T(s) = \int_{T_0}^{T} 2\pi \int_0^\infty P_{-1/2+is}(\cosh r)\tilde{k}_t(r)\sinh r\,dr\,dt$$

where the functions $\tilde{k}_t(r)$ are point-pair invariants of the form treated in [**9**]. If their transform is $\tilde{h}_t(s)$, which is localized at t, for $t > T_0$, then $H_T(s)$ localizes on the whole interval $[T_0, T]$. We need to estimate the supremum norm of $K_T(f)$ which is given by

$$K_T(f) = \int_{T_0}^{T} \tilde{K}_t(f)\,dt.$$

We estimate the integrand pointwise using the results of [**9**] to be

$$||\tilde{K}_t(f)||_\infty \ll e^{-\pi t/2} t^{2k+1/2}$$

An integration by parts in the integral

$$\int_{T_0}^{T} e^{-\pi t/2} t^{2k+1/2}\,dt$$

together with the obvious inequality $e^{\pi t} \leq e^{\pi T}$ for $t \leq T$ gives the estimate (1.10). We notice that the Maaß case is implicit in the above, since the two lemmata above provide the corresponding statements to Lemma 1 and Lemma 2 in [**9**].

The proof of Corollary 1 is the same as the argument in [**4**, p. 546–547].

4. Appendix

In this appendix we prove some standard results about the weight $2k$ Laplacian D_k. We choose the notation as in Fay [**3**, pp. 145–147]. Let

$$K_k = (z - \bar{z})\frac{\partial}{\partial z} + k \tag{4.1}$$

$$L_k = (\bar{z} - z)\frac{\partial}{\partial \bar{z}} - k \tag{4.2}$$

be the Maaß operators. Then

$$D_k = L_{k+1}K_k + k(1+k). \tag{4.3}$$

In order to find the formula for D_k in polar coordinates, we start by showing that

$$\breve{K}_k = \left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^{k+1} \circ K_k \circ \left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^{-k} = e^{-i\theta}\left(\frac{\partial}{\partial r} + \frac{1}{i\sinh r}\frac{\partial}{\partial\theta} - k\tanh(r/2)\right)$$
$$\breve{L}_k = \left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^{k-1} \circ L_k \circ \left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^{-k} = e^{i\theta}\left(\frac{\partial}{\partial r} - \frac{1}{i\sinh r}\frac{\partial}{\partial\theta} + k\tanh(r/2)\right).$$

PROOF. We first work on the formula for $\breve{K}_k$. For any function $g(z)$

$$\begin{aligned}
\breve{K}_k g &= \left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^{k+1}\left((z-\bar{z})\frac{\partial}{\partial z}+k\right)\left(\left(\frac{z-\bar{z}_0}{z_0-\bar{z}}\right)^{-k} g\right)\\
&= (z-\bar{z})\frac{z-\bar{z}_0}{z_0-\bar{z}}\frac{\partial g}{\partial z} + (z-\bar{z})g\frac{-k}{z_0-\bar{z}} + k\frac{z-\bar{z}_0}{z_0-\bar{z}}g\\
&= (z-\bar{z})\frac{z-\bar{z}_0}{z_0-\bar{z}}\frac{\partial g}{\partial z} + \frac{kg}{z_0-\bar{z}}(z-\bar{z}_0-z+\bar{z})\\
&= (z-\bar{z})\frac{z-\bar{z}_0}{z_0-\bar{z}}\frac{\partial g}{\partial z} - \overline{\left(\frac{z-z_0}{z-\bar{z}_0}\right)}kg\\
&= (z-\bar{z})\frac{z-\bar{z}_0}{z_0-\bar{z}}\frac{\partial g}{\partial z} - \tanh(r/2)e^{-i\theta}kg
\end{aligned}$$

If we show that

$$\frac{\partial r}{\partial z} = \tanh(r/2)\frac{z_0-\bar{z}}{(z-z_0)(z-\bar{z})} \tag{4.4}$$

$$\frac{\partial\theta}{\partial z} = \tanh(r/2)\frac{z_0-\bar{z}}{(z-z_0)(z-\bar{z})}\frac{1}{i\sinh r} = \frac{1}{i\sinh r}\frac{\partial r}{\partial z} \tag{4.5}$$

then

$$\begin{aligned}
(z-\bar{z})\frac{z-\bar{z}_0}{z_0-\bar{z}}\frac{\partial g}{\partial z} &= \frac{z-\bar{z}_0}{z-z_0}\tanh(r/2)\left(\frac{\partial g}{\partial r}+\frac{1}{i\sinh r}\frac{\partial g}{\partial\theta}\right)\\
&= e^{-i\theta}\left(\frac{\partial g}{\partial r}+\frac{1}{i\sinh r}\frac{\partial g}{\partial\theta}\right).
\end{aligned}$$

We have

$$\tanh^2(r/2) = \frac{(z-z_0)(\bar{z}-\bar{z}_0)}{(z-\bar{z}_0)(\bar{z}-z_0)} \tag{4.6}$$

and, in order to show (4.4), we differentiate with respect to z to get

$$\tanh(r/2)\frac{1}{\cosh^2(r/2)}\frac{\partial r}{\partial z} = \frac{\bar{z}-\bar{z}_0}{\bar{z}-z_0}\cdot\frac{z_0-\bar{z}_0}{(z-\bar{z}_0)^2} = \tanh^2(r/2)\frac{z_0-\bar{z}_0}{(z-\bar{z}_0)(z-z_0)}.$$

This implies that

$$\frac{\partial r}{\partial z} = \cosh(r/2)\sinh(r/2)\frac{z_0-\bar{z}_0}{(z-\bar{z}_0)(z-z_0)}. \tag{4.7}$$

So we need to show that

$$\cosh^2(r/2) = \frac{(z-\bar{z}_0)(z_0-\bar{z})}{(z-\bar{z})(z_0-\bar{z}_0)}. \tag{4.8}$$

Since $\cosh^2(r/2) = (1-\tanh^2(r/2))^{-1}$, equation (4.6) implies

$$\cosh^2(r/2) = \frac{(z-\bar z_0)(\bar z - z_0)}{(z-\bar z_0)(\bar z - z_0)-(z-z_0)(\bar z-\bar z_0)} = \frac{(z-\bar z_0)(z_0-\bar z)}{(z-\bar z)(z_0-\bar z_0)}.$$

This completes the proof of (4.4). As far as (4.5) is concerned, we differentiate (2.3) with respect to z and use (4.4) to get

$$\begin{aligned}\frac{z_0-\bar z_0}{(z-\bar z_0)^2} &= \frac{1}{2\cosh^2(r/2)}\cdot\frac{\partial r}{\partial z}\cdot e^{i\theta} + \tanh(r/2) i e^{i\theta}\frac{\partial\theta}{\partial z}\\ &= \frac{1}{2\cosh^2(r/2)}\frac{z_0-\bar z}{(z-\bar z_0)(z-\bar z)} + i\frac{z-z_0}{z-\bar z_0}\frac{\partial\theta}{\partial z}\\ &= \frac{1}{2}\frac{z_0-\bar z_0}{(z-\bar z_0)^2} + i\frac{z-z_0}{z-\bar z_0}\frac{\partial\theta}{\partial z},\end{aligned}$$

where in the last step we used Equation (4.8). So, using (4.7), we get

$$\frac{\partial\theta}{\partial z} = \frac{z_0-\bar z_0}{2i(z-\bar z_0)(z-z_0)} = \frac{1}{i\sinh r}\frac{\partial r}{\partial z},$$

which suffices to show (4.5).

To prove the formula for $\breve L_k$, which is L_k in polar coordinates, it is enough to notice that $L_k = \bar K_{-k}$ and use the formula for $\breve K_{-k}$. □

Finally

$$\begin{aligned}\breve D_k - k(k+1) &= \left(\frac{z-\bar z_0}{z_0-\bar z}\right)^k \circ D_k \circ \left(\frac{z-\bar z_0}{z_0-\bar z}\right)^{-k} - k(k+1)\\ &= \left(\frac{z-\bar z_0}{z_0-\bar z}\right)^k L_{k+1}K_k \left(\frac{z-\bar z_0}{z_0-\bar z}\right)^{-k} = \breve L_{k+1}\breve K_k\\ &= e^{i\theta}\left(\frac{\partial}{\partial r} - \frac{1}{i\sinh r}\frac{\partial}{\partial\theta} + (k+1)\tanh(r/2)\right)\\ &\quad\circ\left(e^{-i\theta}\left(\frac{\partial}{\partial r}+\frac{1}{i\sinh r}\frac{\partial}{\partial\theta} - k\tanh(r/2)\right)\right)\\ &= \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{i\sinh r}\right)\frac{\partial}{\partial\theta} + \frac{1}{i\sinh r}\frac{\partial^2}{\partial r\partial\theta} - k\frac{1}{2\cosh^2(r/2)}\\ &\quad - k\tanh(r/2)\frac{\partial}{\partial r} + (k+1)\tanh(r/2)\left(\frac{\partial}{\partial r} + \frac{1}{i\sinh r}\frac{\partial}{\partial\theta} - k\tanh(r/2)\right)\\ &\quad + \frac{1}{\sinh r}\left(\frac{\partial}{\partial r}+\frac{1}{i\sinh r}\frac{\partial}{\partial\theta} - k\tanh(r/2)\right)\\ &\quad - \frac{1}{i\sinh r}\left(\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{i\sinh r}\frac{\partial^2}{\partial\theta^2} - k\tanh(r/2)\frac{\partial}{\partial\theta}\right).\end{aligned}$$

Using the identities $2\cosh^2(r/2) = 1+\cosh r$ and $\tanh(r/2)+1/\sinh r = \cosh r/\sinh r$, we finally get the form of $\breve D_k$ in (2.5).

We now pass to the study of (2.6). The equation (2.6) has three regular singular points at ±1 and ∞ and the roots of the indicial equation at these points are as follows. At $u=1$ the roots are $\pm|n|/2$, at $u=-1$ they are $-k_n-|n|/2$ and $k_n+|n|/2$, where $k_n = kn/|n|$ and $k_0 = k$ and at ∞ they are k and $1-k$. In this case the difference of the roots for the regular singular point $u=1$ is always an integer. There exists exactly one solution that does not blow up as $u\to1$. In fact, if we denote this solution by $y_1(u)$, which is asymptotic to $(u-1)^{|n|/2}$ as $u \to 1$, then the second solution $y_2(u)$ has the form

$$y_2(u) = y_1(u) \ln|u-1| + \sum_{j=1}^{\infty} c_j (u-1)^j$$

for $n = 0$ and

$$y_2(u) = a y_1(u) \ln|u-1| + (u-1)^{-|n|/2} \left(1 + \sum_{j=1}^{\infty} c_j (u-1)^j \right), \quad n \neq 0,$$

for a constant a. Therefore this solution blows up as $u \to 1$. The solution $y_1(u)$ is given in Riemann's notation by

$$P \left\{ \begin{array}{cccc} 1 & -1 & \infty & \\ \frac{|n|}{2} & -k_n - \frac{|n|}{2} & k & u \\ -\frac{|n|}{2} & k_n + \frac{|n|}{2} & 1-k & \end{array} \right\} =$$

$$\left(\frac{u-1}{u+1}\right)^{\frac{|n|}{2}} (u+1)^{-k} P \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & -k_n + k & 0 & \frac{u-1}{u+1} \\ -|n| & k_n + |n| + k & 1-2k & \end{array} \right\}$$

where we used [**2**, (8), pp. 91]. Comparing with the representation of the hypergeometric function in Riemann's notation we get

$$\left(\frac{u-1}{u+1}\right)^{|n|/2} \left(\frac{1}{u+1}\right)^{k} F(k - k_n, |n| + k + k_n, 1 + |n|, (u-1)/(u+1))$$

which gives (2.7) and (2.8).

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