Weyl's estimate for $\zeta(\frac{1}{2}+it)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \mathrm{Re}(s) > 1$$

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(\frac{1-s}{2})}\Gamma(\tfrac{1-s}{2})\zeta(1-s)$$

Critical strip $0 \le \mathrm{Re}\, s \le 1$

$$\sum_{n\le X} \frac{1}{n^s} \underset{\text{by parts}}{\overset{\text{summation}}{=}} s\int_1^X \frac{[x]}{x^{s+1}}\,dx + \frac{[X]}{X^s}$$

$$x = [x] + \underbrace{\{x\}}_{\text{fractional part}}$$

$$= \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s\int_1^X \frac{\{x\}}{x^{s+1}}\,dx$$

$$+ \frac{1}{X^{s-1}} - \frac{\{X\}}{X^s}$$

$X \to \infty$

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\,dx$$

$$\boxed{\zeta(s) = \sum_{n\le X}\frac{1}{n^s} + \frac{1}{(s-1)X^{s-1}} + \frac{\{X\}}{X^s} - \int_X^{\infty}\frac{\{x\}\,dx}{x^{s+1}}}$$

$$X = t^2 \qquad s = \tfrac{1}{2}+it$$

$$\boxed{\zeta\left(\tfrac{1}{2}+it\right) = \sum_{n \le t^2} \frac{1}{n^{1/2+it}} + O(1)}$$

Approximate functional equations

$0 < \sigma < 1$

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \underset{\substack{\uparrow \\ \pi^{\cdots} \frac{\Gamma(1-s/2)}{\Gamma(s)}}}{\chi(s)} \sum_{n \le y} \frac{1}{n^{1-s}}$$

with

$s = \sigma + it$

$2\pi \cdot x \cdot y = t > 0$

$x = y = \sqrt{t/2\pi}$

$$+ O(x^{-\sigma} \log t) + O(|t|^{1/2-\sigma} y^{\sigma-1})$$

$$\sum_{|t_j| \le T} \int_{\Gamma \backslash \mathbb{H}} u_j \, |E(z, \tfrac{1}{2}+it)|^2 \frac{dx\,dy}{y^2}$$

If $\mathcal{I}$ use φ Maaß cusp forms

$$\sum_{|t_j| \le T} \int_{\Gamma \backslash \mathbb{H}} u_j \, |\varphi(z)|^2 \frac{dx\,dy}{y^2}$$

$$+ \frac{1}{4\pi} \int^{T} \int E(z, \tfrac{1}{2}+it) \, |\varphi(z)|^2 \frac{dx\,dy}{y^2} \, dt$$

$-T$

$\ll T^{\varepsilon}$

Jutila

Weyl $\qquad \zeta(\tfrac{1}{2}+it) \ll t^{1/6}\log t$

$$e(x) = e^{2\pi i x} \qquad I = (a,b]$$

$$a, b \in \mathbb{Z}$$

$$\|x\| = \operatorname{dist}\{x, \mathbb{Z}\} = \min_{n\in\mathbb{Z}} |x-n|$$

$$\sum_n n^{-\frac{1}{2}-it} = \sum_n n^{-\frac{1}{2}} \underbrace{e^{-it\log n}}_{e(f(n))}$$

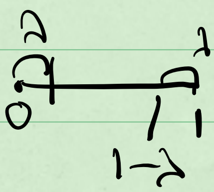
$$f(x) = \frac{-t\log x}{2\pi}$$

$$\Big|\sum_{n\in I} e(f(n))\Big| \le |I| \qquad \text{trivial}$$

<u>Th 2.2</u> (Kuzmin–Landau) If f cont. diff. f' monotonic $\|f'\| \ge \lambda > 0$

<u>Then</u> $\displaystyle\sum_{n\in I} e(f(n)) \ll \lambda^{-1}$

<u>Proof</u> $\quad f'$ increasing $\exists k$



$$\cancel{k}+\lambda \le f'(x) \le \cancel{k}+1-\lambda$$

$e(kn)=1$

$$g(n) = f(n+1) - f(n) = f'(\xi)$$
$$g(n+1) = f(n+2) - f(n+1) = f'(\xi')$$
g is increasing as well

$$e(f(n)) = \frac{e(f(n)) - e(f(n+1))}{1 - e(g(n))}$$
$$= \big(e(f(n)) - e(f(n+1))\big)\, c_n$$

$$c_n = \frac{1}{2}\left(1 + i \cot \pi g(n)\right)$$

$$\left| \sum_{a < n \le b} e(f(n)) \right| = \sum_n \big(e(f(n)) - e(f(n+1))\big) c_n$$

$$\underset{\text{by parts}}{\overset{\text{summation}}{=}} \sum_{n=a+1}^{b-1} e(f(n)) \underbrace{(c_n - c_{n-1})} + e(f(a+1))\, c_{a+1} \ldots$$

$$= \left| \sum_{a+1}^{b-1} e(f(n)) \big(\cot \pi g(n) - \cot \pi g(n+1)\big) \right| + \ldots$$

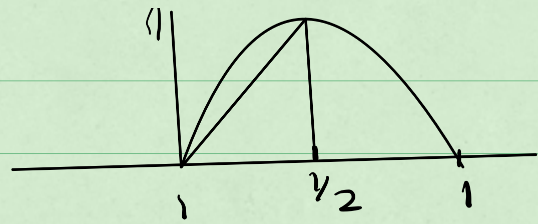
$$\le \sum_{n=a+1}^{b-1} \cancel{|} \cot \pi g(n) - \cot \pi g(n+1) \cancel{|} + \ldots$$

$$0 < \lambda \le g(n) \le 1 - \lambda \le 1$$

$$= \cot \pi g(b-1) - \cot \pi g(a+1) + \ldots$$

$$\|g(n)\| \ge \lambda \qquad \sin \pi x$$

$$0 \le x \le 1$$



$$\sin \pi x \geq 2x$$

$$\frac{\sin \pi x}{x} \geq 2$$

$$|\sin \pi x| \geq 2|x| \qquad -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}$$

$$|\cot \pi x| \leq \frac{1}{2\|x\|}$$

$$|\cot \pi g(n)| \ll \frac{1}{\|g(n)\|} \leq \frac{1}{\lambda}$$

<u>Van der Corput's result</u>

$$a \geq 1 \qquad \lambda \geq 0$$

$$\lambda \leq |f''(x)| \leq a\lambda \qquad \text{on } x \in I$$

$$\sum_{a < n \leq b} e(f(n)) \ll a|I|\lambda^{1/2} + \lambda^{-1/2}$$

Lemma 2.1 Weyl differencing

q integer $\leq b-a$

$$|S| = \Big|\sum_{a < n \leq b} e(f(n))\Big| \leq \frac{b-a}{\sqrt{q}} + \left(\frac{b-a}{q}\sum_{r=1}^{q-1}\Big|\underbrace{\sum(\quad)}_{\downarrow}\Big|\right)^{1/2}$$

$$\sum_{a < n \le b-r} e(f(n+r) - f(n))$$

Proof $\quad e(f(n)) = 0$ for $n \le a$, $n > b$

$$S = \frac{1}{q} \sum_{\underline{n}} \sum_{m=1}^{q} e(f(m+n))$$

$$a < m+n \le b$$

n cannot be $< a-q$

or $> b-1$

$$\#\{n\} \le b-1-a+q \le b-a+q$$
$$\le 2(b-a)$$

$$|S| \le \frac{1}{q} \sum_n 1 \cdot \Big| \sum_{m=1}^{q} e(f(n+m)) \Big|$$

$$\overset{C\text{-}S}{\le} \frac{1}{q} \Big(\cancel{\sum_n} \, 2(b-a) \cdot \sum_n \Big| \sum_{m=1}^{q} e(f(n+m)) \Big|^2 \Big)^{1/2}$$

$$\Big| \sum_{m=1}^{q} e(f(m+n)) \Big|^2 = \sum_{m,k} e(f(m+n) - f(k+n))$$

$$= \overset{\text{diagonal}}{q} + 2 \sum_{k<m} e(f(m+n) - f(k+n))$$

$$\sum_n \Big| \sum \Big|^2 \le q \cdot 2(b-a)$$
$$+ 2 \Big| \sum_n \sum_{\underline{k<m}} e(f(m+n) - f(k+n)) \Big|$$

how often $\quad m+n = k+n$

$r = m-k \quad d = k+n$

$$| \qquad | \le \Big| \sum^{q-1} (q-r) e(f(d+r) - f(d)) \Big|$$

$q-r$ times

$$S \le \frac{1}{q}\left(4(b-a)^2 q + 4(b-a) q \sum_{r=1} \left| \sum_{d} e(\cdot) \right) \right)^{1/2}$$

<u>Th</u> $f(x)$ 3-times diff

$$\lambda \le |f'''(x)| \le a\lambda \qquad a \ge 1, \ \lambda > 0$$

$$\boxed{\left| \sum_{n\in I} e(f(n)) \right| \ll a\,|I|\,\lambda^{1/6} + |I|^{1/2}\lambda^{-1/6}}$$

Weyl difference

$$g(x) = f(x+r) - f(x)$$

$$r\lambda \le |g''(x)| = f''(x+r) - f''(x) \underset{MVT}{=} |r f'''(\xi)| \le ra\lambda$$

$$\sum_{n\in I} e(f(n)) \ll \frac{|I|}{\sqrt{q}} + \left(\frac{|I|}{q} \sum_{r=1}^{q-1} \left(|I|\, a (r\lambda)^{1/2} + (r\lambda)^{-1/2} \right) \right)^{1/2}$$

$$= \frac{|I|}{\sqrt{q}} + \left(\frac{|I|}{q} \left(a\lambda^{1/2} q^{3/2} + \lambda^{-1/2} q^{1/2} \right) \right)^{1/2}$$

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$$

$$\ll |I| q^{-1/2} + a^{1/2} |I| q^{1/4} \lambda^{1/4} + |I| q^{-1/4} \lambda^{-1/4}$$

$$q = [\lambda^{-1/3}] \qquad \lambda \leq 1 \qquad q \leq |I|$$

$$a^{1/2} |I| \lambda^{1/6} + |I|^{1/2} \lambda^{-1/6}$$

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{n \leq t^2} \frac{1}{n^{1/2+it}} + O(1)$$

(A) $n \leq t^{2/3}$ (B) $t^{2/3} \leq n \leq t$

(C) $t \leq n \leq t^2$

$$f(x) = -\frac{1}{2\pi} t \log x$$

$$f'''(x) = -\frac{t}{\pi x^3} \qquad \frac{t}{a^3} \ll |f'''(x)| \leq \frac{t}{a^3}$$

$$\sum_{a < n \leq b} \frac{1}{n^{\cancel{1/2}+it}} \qquad a \ll t^{2/3}$$

$b \leq 2a$

$$\sum_{a < n \leq b} n^{-it} \ll a\left(\frac{t}{a^3}\right)^{1/6} + a^{1/2}\left(\frac{t}{a^3}\right)^{-1/6}$$

$$= a^{1/2} t^{1/6} + a t^{-1/6}$$

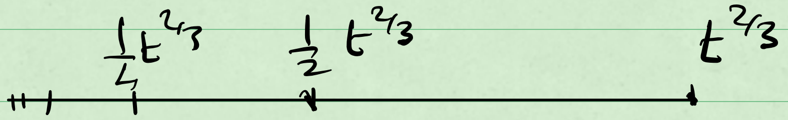
$$t^{1/6} a^{1/2}$$

$$\sum_{a<n\le b} n^{-1/2-it} = a^{-1/2} \quad a^{1/2} t^{1/6}$$

$$\sum_{a<n\le b} \frac{1}{n^{1/2+ti}} << t^{1/6}$$

$b \le 2a$ $\qquad a \lesssim t^{2/3}$

$$\sum_{n\le t^{2/3}} \frac{1}{n^{1/2+it}} << t^{1/6} \log t$$



$\#\text{intervals} << \log_2 (t^{2/3}) << \log t$

(B) $\qquad t^{2/3} < a \le t$

Van der Corput

$a < n \le b \qquad b \le 2a$

$$\sum_{a<n\le b} \frac{1}{n^{it}} \le t^{1/2} + a\, t^{-1/2}$$

$$|f''(x)| = \left|\frac{t}{2\pi x^2}\right| \le \frac{t}{a^2}$$

$\lambda =$

$$\sum_{t^{2/3}\le n\le t} \frac{1}{n^{1/2+it}} << t^{1/6} \log t$$

(c) $t^2 \geq n > t$

$$f'(x) = -\frac{t}{2\pi x}$$

Kuzmin – Landau

$$\sum_{t<n<t^2} n^{-it} \ll 1 \Rightarrow \sum_{t<n<t^2} n^{-1/2-it} = O(1)$$

Cannot apply to $L(f,s)$

$$L(f,\tfrac{1}{2}) \doteq \sum_{n\geq 1} \frac{\lambda(n)}{n^{1/2+it}}$$

$$\sum_{a<n\leq 2a} \frac{\lambda(n)}{n^{it}}$$