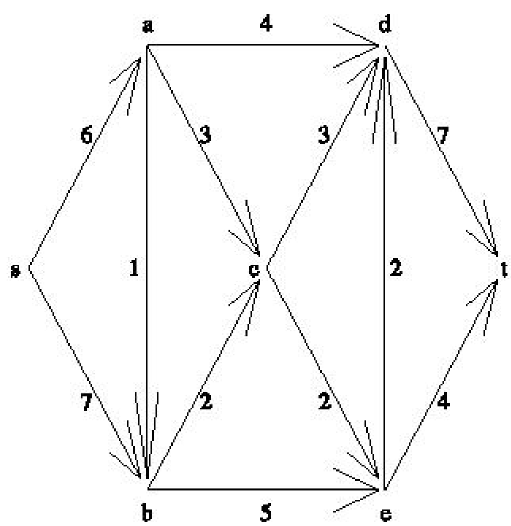
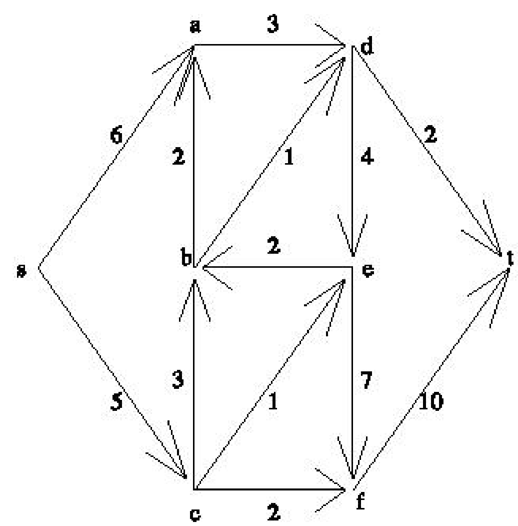
# Math 7502

## Homework 8

**Due: March 13, 2008**

1. ∗ Find a maximum flow and a minimum cut for the following networks. Explain the steps you take. Explain why the flow and the cut you found are optimal.




2. (Hard) Show that any basic feasible solution to the dual of the maximal flow problem corresponds to a certain cut with the dual objective value equal to the capacity of the cut. Explain why this gives another (not easier) proof of the maximal flow–minimal cut theorem.

   Label the nodes by $1, 2, \ldots, m$, so that $s = 1$ and $t = m$. Let $A$ be the node-arc incidence matrix, i.e. its rows are labeled after the rows and the columns after the arcs in the network. For each arc $(i, j)$ we assign $+1$ are the intersection of the row $i$ and the column for $(i, j)$ and $-1$ for the intersection of the row $j$ with the column for $(i, j)$. In the primal problem we have are variables $x_{ij}$ and $f$, the flow. The constraints are $x_{ij} \geq 0$ and $x_{ij} \leq c_{ij}$, where $c_{ij}$ is the capacity of the arc $(i, j)$, and

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} f, & i = 1 \\ 0 & i \neq 1, m \\ -f, & i = m \end{cases} \tag{1}$$

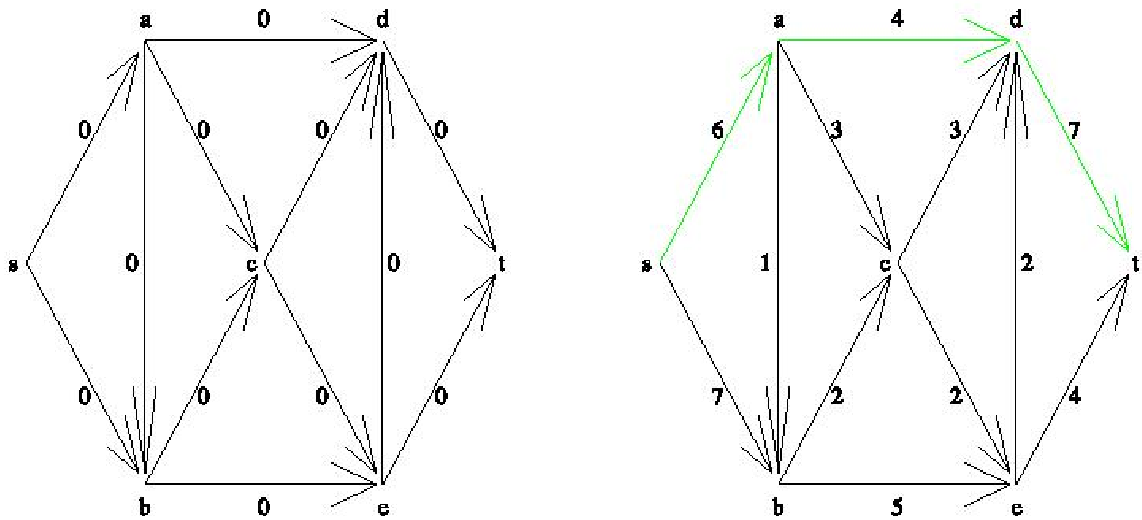



Figure 1: Initialization and auxiliary graph $G'$. Path to follow is in green and $\Delta = 4$

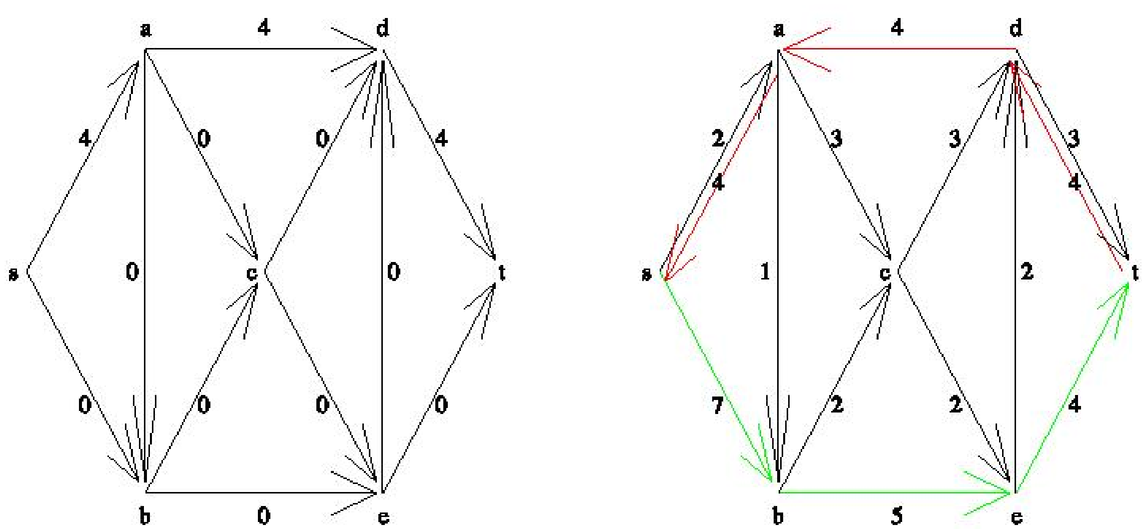



Figure 2: New flow and auxiliary graph $G'$. Path to follow is in green and $\Delta = 4$

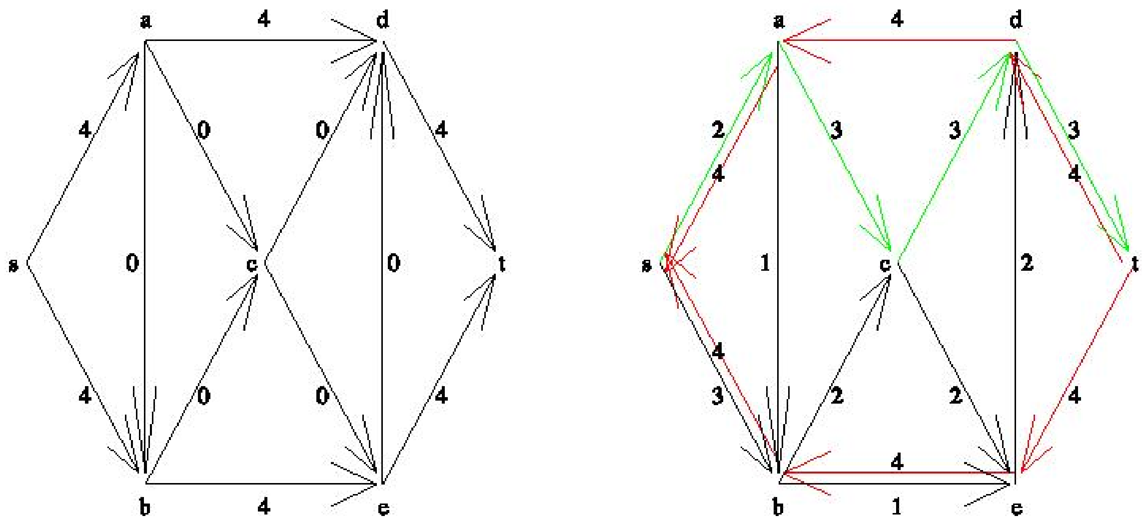



Figure 3: New flow and auxiliary graph $G'$. Path to follow is in green and $\Delta = 2$

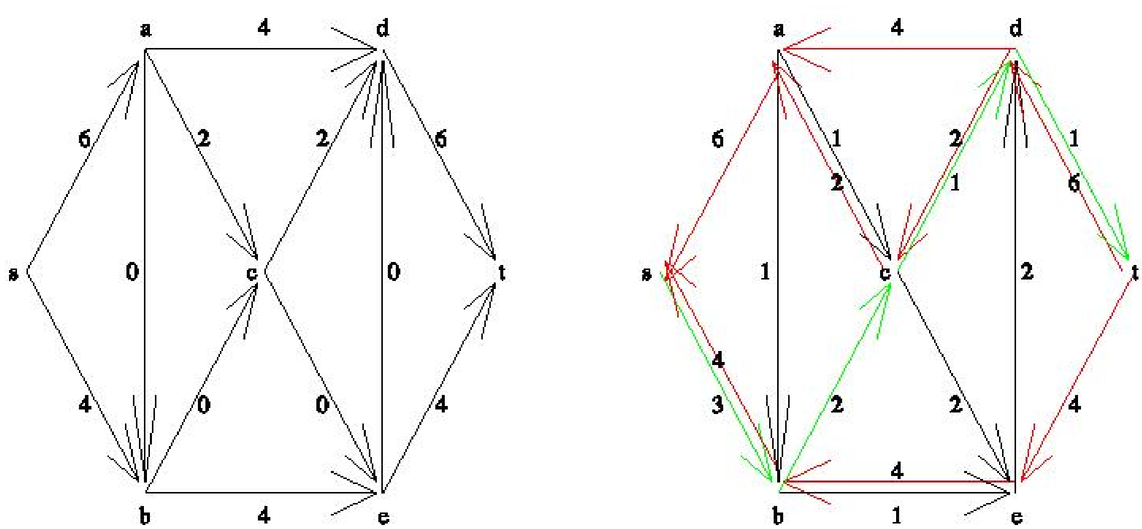
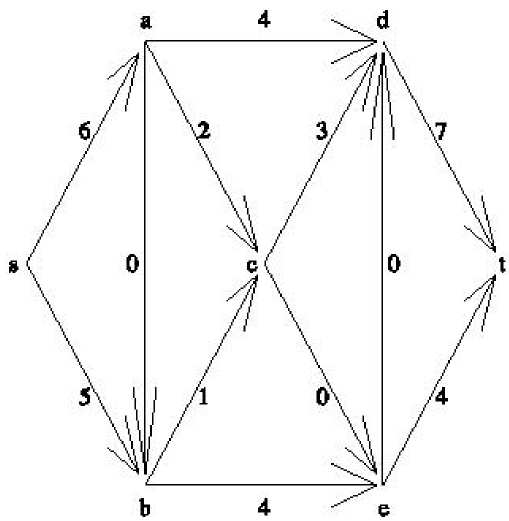



Figure 4: New flow and auxiliary graph $G'$. Path to follow is in green and $\Delta = 1$



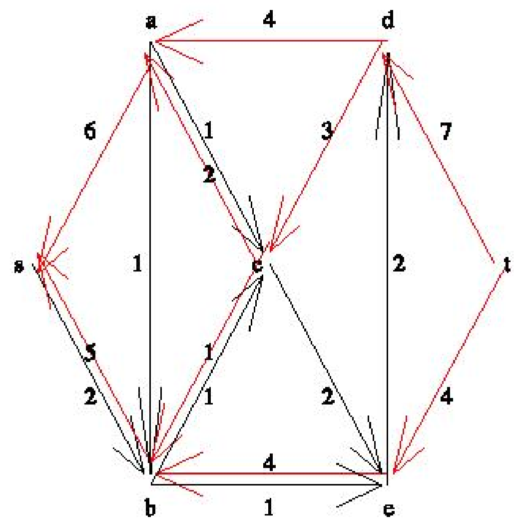




Figure 5: New flow. There is no path in $G'$ joining $s$ and $t$. Optimal solution

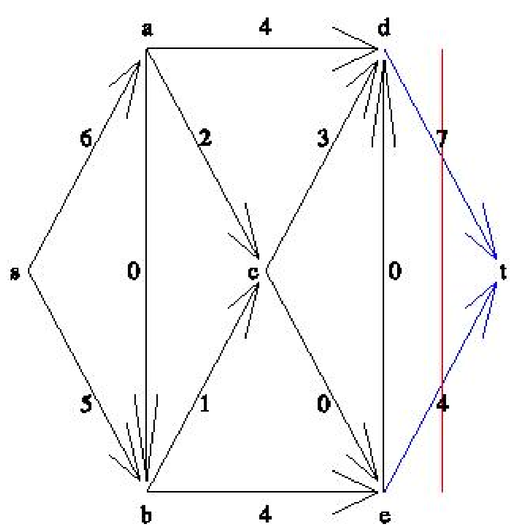



Figure 6: The red line shows the cut with $S = \{s, a, b, c, d, e\}$ and the blue arrows contribute to the capacity of the cut

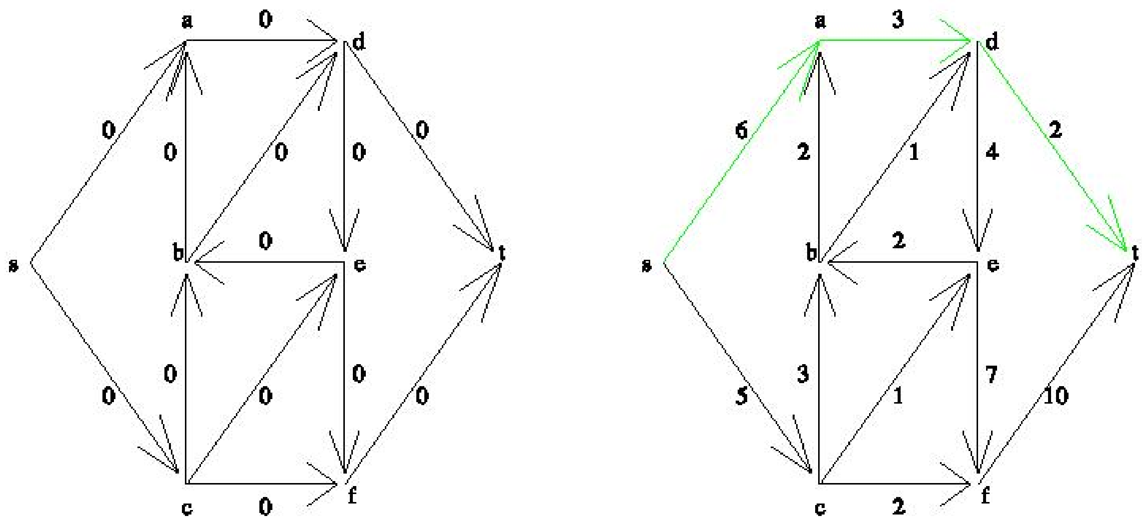



Figure 7: Initialization and auxiliary graph $G'$. Path to follow is in green and $\Delta = 2$

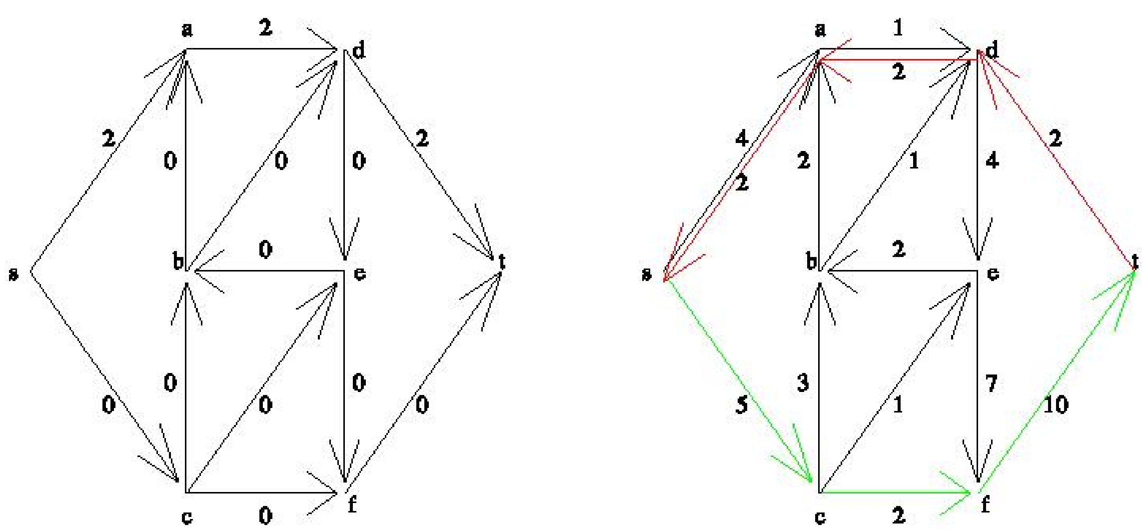



Figure 8: New flow and auxiliary graph $G'$. Red color shows the potential backwards flow, green the path to follow and $\Delta = 2$

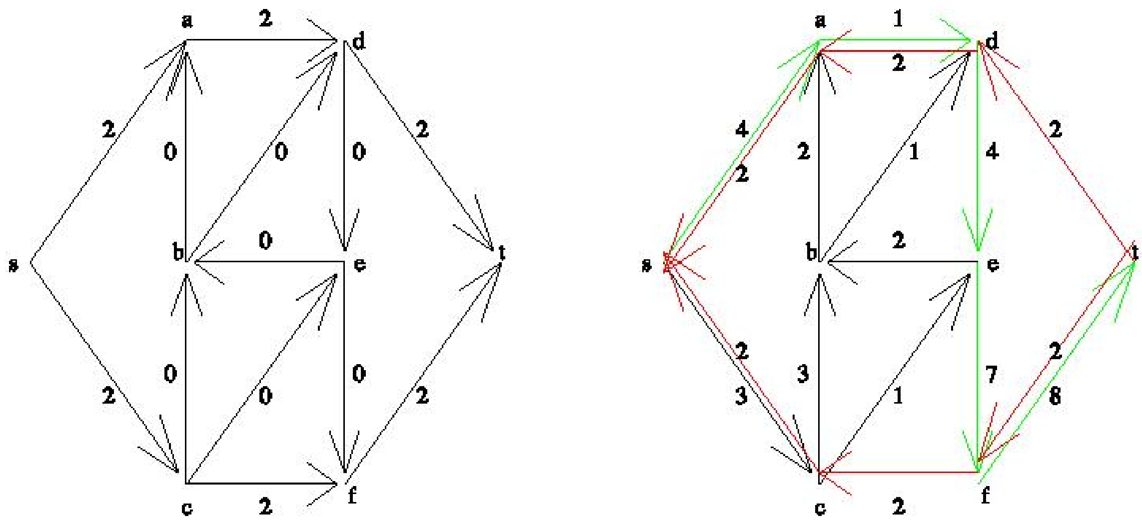



Figure 9: New flow and auxiliary graph $G'$. Red color shows the potential backwards flow, green the path to follow and $\Delta = 1$

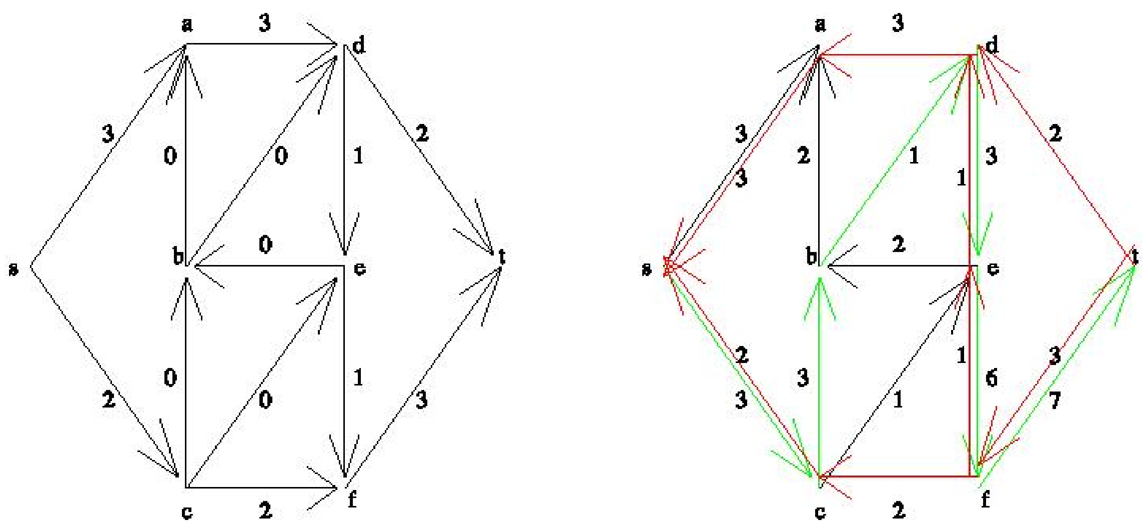



Figure 10: New flow and auxiliary graph $G'$. Red color shows the potential backwards flow, green the path to follow and $\Delta = 1$

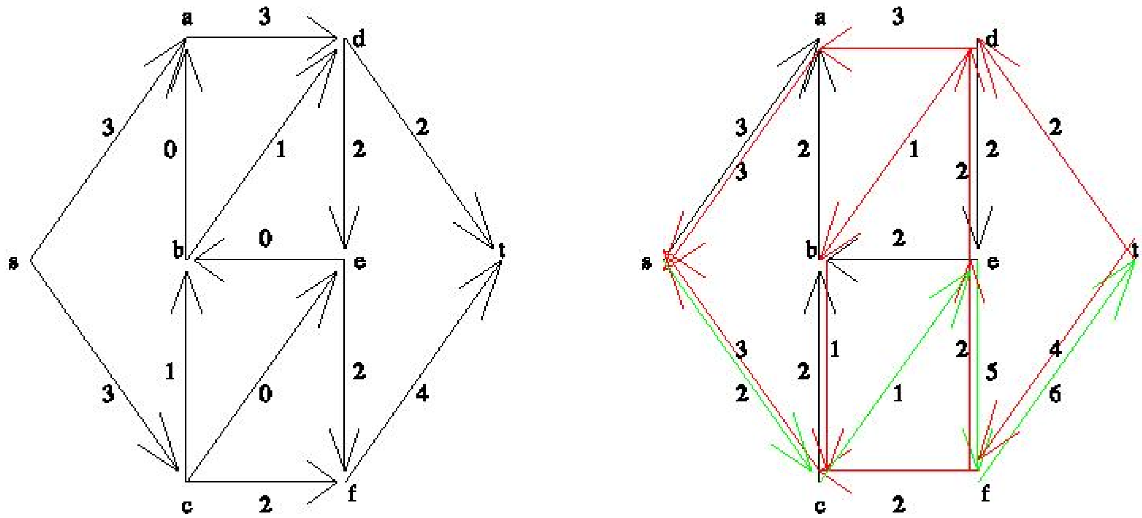



Figure 11: New flow and auxiliary graph $G'$. Red color shows the potential backwards flow, green the path to follow and $\Delta = 1$

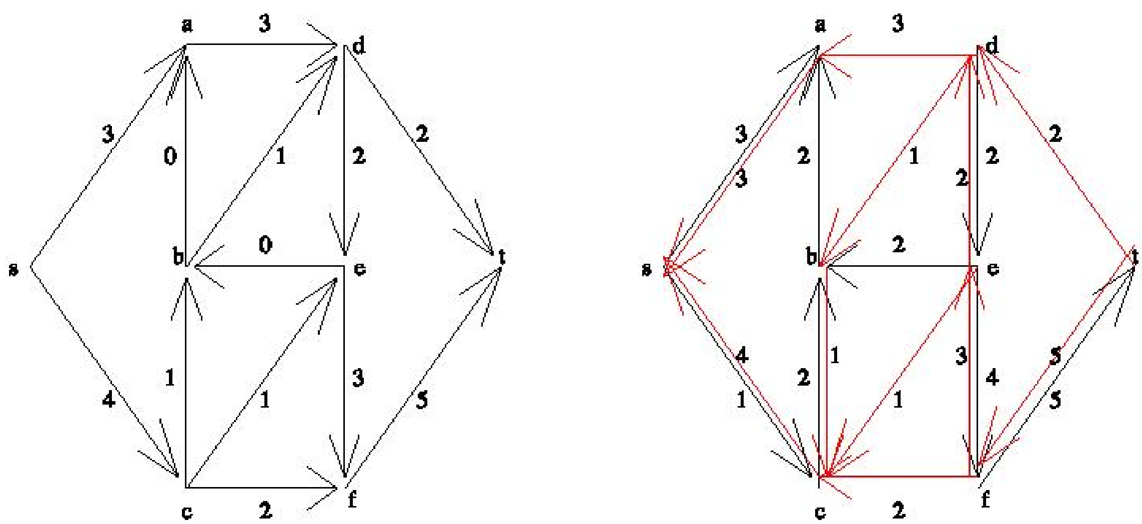



Figure 12: New flow and auxiliary graph $G'$. There is no path in $G'$ from $s$ to $t$. Optimal solution

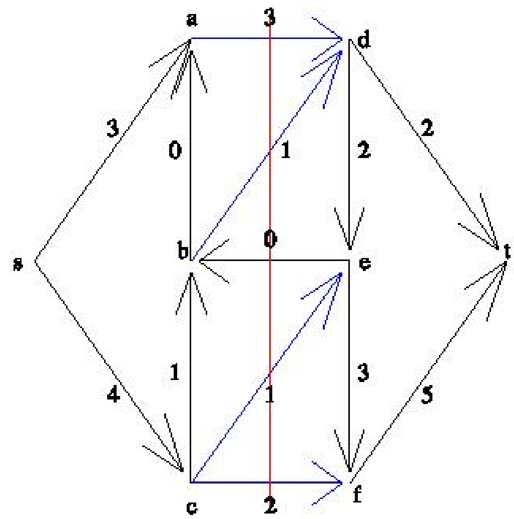



Figure 13: The red line shows the cut with $S = \{s, a, b, c\}$ and the blue arrows contribute to the capacity of the cut

Let $\mathbf{x}$ be the vector of $x_{ij}$ written lexicographically and $\mathbf{c}$ the vector of capacities. Let $\mathbf{e}_i$ be the standard basis vector in $\mathbf{R}^n$. We can rewrite the program as

$$\begin{array}{rc} \text{maximize} & f \\ \text{subject to} & (\mathbf{e}_m - \mathbf{e}_1)f + A\mathbf{x} = \mathbf{0} \\ & \mathbf{x} \leq \mathbf{c} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

We can assume that the network corresponds to a complete graph with capacity $c_{ij} = 0$, when the arc $(i, j)$ does not appear in the original network. This way we have $m^2+1$ variables. The dual problem has $m^2+1$ constraints. The primal problem has $m^2 + m$ constraints, the dual has $m^2 + m$ variables. We label these as follows: $h_{ij}$ for the constraint $x_{ij} \leq c_{ij}$ and $w_i$ for the constraint $i$ in (1). Since these are equalities, the $w_i$ are unrestricted, while since we have $x_{ij} \leq c_{ij}$, the corresponding variables $h_{ij}$ are $\geq 0$. The dual variables $w_i$ form a vector $\mathbf{w}$ and the dual variables $h_{ij}$ form a vector $\mathbf{h}$. Writing the vector of primal variables as

$$\begin{pmatrix} f \\ \mathbf{x} \end{pmatrix}$$

the matrix of the primal program is

$$\begin{pmatrix} -\mathbf{e}_1 + \mathbf{e}_m & A \\ \mathbf{0} & I \end{pmatrix}$$

where $I$ is the $m^2 \times m^2$ identity matrix and $\mathbf{0}$ is the zero vector in $\mathbf{R}^{m^2}$. The transpose matrix is

$$\begin{pmatrix} (-\mathbf{e}_1 + \mathbf{e}_m)^t & \mathbf{0} \\ A^t & I \end{pmatrix}$$

and the dual constraints come from multiplying this with the vector

$$\begin{pmatrix} \mathbf{w} \\ \mathbf{h} \end{pmatrix}.$$

The first equation is $w_1 - w_m = 1$, since the objective function $f$ of the primal problem had 1 as coefficient of the first variable $f$. The other constraints are $w_i - w_j + h_{ij} \geq 0$. The $\geq 0$ comes from the two facts: (1) we had $\geq$ in the variables of the primal program and (2) the coefficients in the objective function of the primal program were 0 for the variables $x_{ij}$. The objective function for the dual program is $\sum_{ij} c_{ij} h_{ij}$, as the only nonzero coefficients in the constraints of the primal program were in $x_{ij} \leq c_{ij}$.

Now let $(X, \bar{X})$ be a cut of the network. We define

$$w_i = \begin{cases} 0, & i \in X \\ 1, & i \in \bar{X} \end{cases}$$

$$h_{ij} = \begin{cases} 1, & (i,j) \in (X, \bar{X}), \\ 0, & \text{otherwise} \end{cases}.$$

This choice of dual variables is a feasible solution of the dual program:

(1) Since $1 \in X$, $m \in \bar{X}$, $w_1 = 0$, $w_m = 1$, so $w_m - w_1 = 1$ is satisfied.

(2) We study four cases according to whether $i$ and $j$ belong to $X$ or its complement $\bar{X}$.

(i) $i, j \in X$. Then $w_i = w_j = 0$, while $h_{ij} = 0$. So the constraint $w_i - w_j + h_{ij} \geq 0$ is satisfied.

(ii) $i, j \in \bar{X}$. Then $w_i = w_j = 1$, while $h_{ij} = 0$. So the constraint $w_i - w_j + h_{ij} \geq 0$ is satisfied.

(iii) $i \in X$, $j \in \bar{X}$. Then $w_i = 0$, $w_j = 1$, while $h_{ij} = 1$. So the constraint $w_i - w_j + h_{ij} \geq 0$ is satisfied.

(iv) $i \in \bar{X}$, $j \in X$. Then $w_i = 1$, $w_j = 0$, while $h_{ij} = 0$. So the constraint $w_i - w_j + h_{ij} \geq 0$ is satisfied.

The objective function becomes

$$\sum_{ij} c_{ij} h_{ij} = \sum_{h_{ij}=1} c_{ij} = \sum_{i \in X, j \in \bar{X}} c_{ij},$$

which is the capacity of the cut. Now the maximal flow-minimal cut theorem follows from the strong duality theorem (for the equality) and the weak duality theorem (for the statement that every flow is less than equal to the capacity of any cut).