Math 7502

Homework 7

Due: March 6, 2008

1. $*$ Solve the games with payoff matrices

$$\text{(a)} \begin{pmatrix} 1 & 4 \\ 7 & 2 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}.$$

(a) The matrix $\begin{pmatrix} 1 & 4 \\ 7 & 2 \end{pmatrix}$ has no saddle point as

		row min
1	4	1
7	2	2
1	4	$\leftarrow$ max

and minimax $= 1 \neq$ maximin $= 2$. We look for mixed (randomized) strategies. Assume that Player I plays row 1 with probability p and row 2 with probability $1-p$, where $0 \leq p \leq 1$. If Player II plays column 1, Player I expects a payoff

$$1 \cdot p + 7(1-p) = 7 - 6p.$$

If Player II plays column 2, Player I expects a payoff

$$4p + 2(1-p) = 2 + 2p.$$

Player I wants to guarantee wins of at least $\min(7-6p, 2+2p)$. And he wants to make sure this is as large as possible. So his strategy is to maximize (over $p \in [0,1]$) the $\min(7-6p, 2+2p)$. The figure shows the minimum and we see that this is maximized at the intersection of the two lines $v = 7-6p$ and $v = 2+2p$ (v represents the value of the game). We solve

$$7 - 6p = 2 + 2p \Longrightarrow p = \frac{5}{8}.$$

Then the value of the game is $2 + 2 \cdot 5/8 = 13/4$. Player I plays row 1 with probability $5/8$ and row 2 with probability $1 - 5/8 = 3/8$. If Player II plays column 1 with probability q and column 2 with probability $1-q$, the his expected payoff, assuming Player I plays row 1, is

$$1 \cdot q + 4(1-q) = 4 - 3q.$$

The slack variable corresponding to this constraint has to be 0, as $p \neq 0$. So $4 - 3q = 13/4 \Longrightarrow q = 1/4$. Player II will play column 1 with probability $1/4$ and column 2 with probability $3/4$.

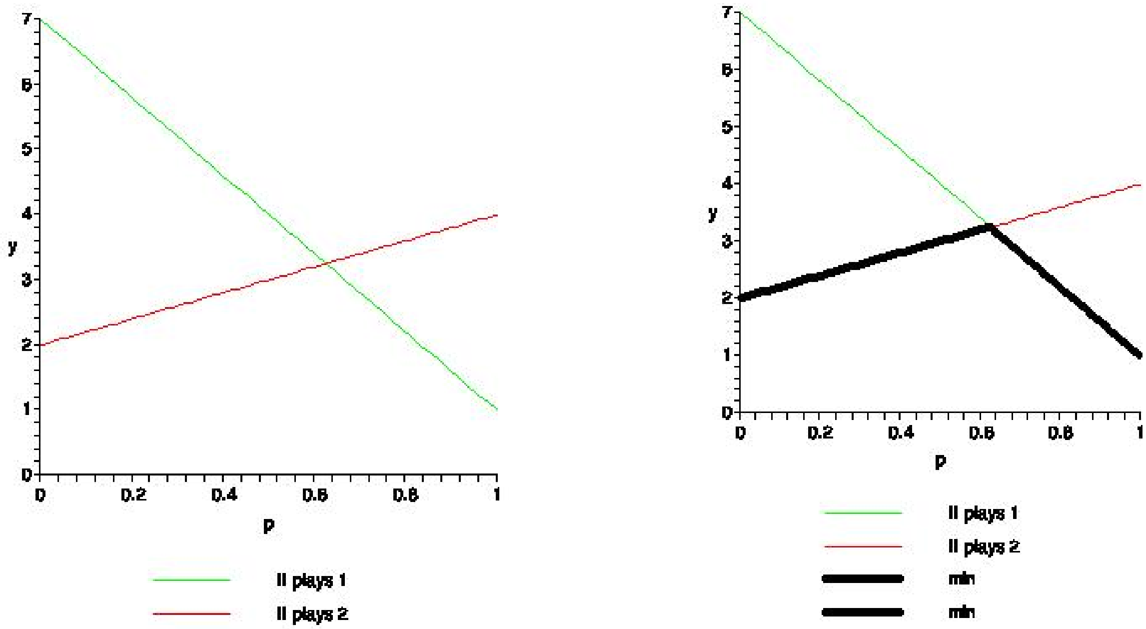

Figure 1: Player I's viewpoint

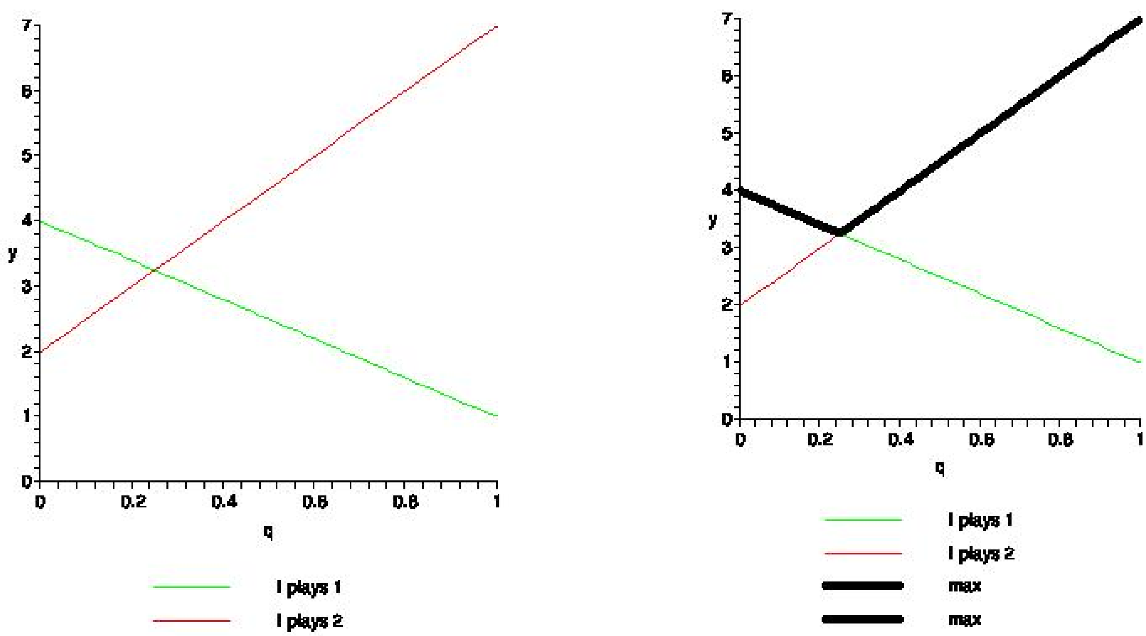

Figure 2: Player II's viewpoint

Second method: If Player I plays row 1, the expected payoff for Player II is

$$1 \cdot q + 4(1 - q) = 4 - 3q.$$

If Player I plays row 2, the expected payoff for Player II is

$$7q + 2(1 - q) = 5q + 2.$$

Player II can make sure that Player I wins at most $\max(4 - 3q, 5q + 2)$. And he wants to make sure this is as small as possible. So his strategy is to minimize (over $q \in [0, 1]$) the $\max(4 - 3q, 5q + 2)$. The figure shows the maximum and we see that this is minimized at the intersection of the two lines $v = 4 - 3q$ and $v = 5q + 2$ (v represents the value of the game). We solve

$$4 - 3q = 5q + 2 \Longrightarrow q = \frac{1}{4}.$$

(b) The matrix $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ has a saddle point $a_{11} = 3$. So the players will settle for strategy (row) 1 for Player I and strategy (column) 1 for Player II. They will play pure strategies.

Second method: Assume that Player I plays row 1 with probability p and row 2 with probability $1 - p$, where $0 \leq p \leq 1$. If Player II plays column 1, Player I expects a payoff

$$3p + 2(1 - p) = p + 2.$$

If Player II plays column 2, Player I expects a payoff

$$6p + 4(1 - p) = 2p + 4.$$

Player I wants to guarantee wins of at least $\min(p+2, 2p+4)$. And he wants to make sure this is as large as possible. So his strategy is to maximize (over $p \in [0, 1]$) the $\min(p + 2, 2p + 4)$. For all $p \in [0, 1]$ we have $p + 2 < 2p + 4$. So he wants to maximize $p + 2$. This is achieved for $p = 1$. So he will play row 1 all the time. His expected payoff is $p + 2 = 3 = a_{11}$.

Assume that Player II plays column 1 with probability q and column 2 with probability $1 - q$, where $0 \leq q \leq 1$. If Player I plays row 1, Player II expects a payoff

$$3q + 6(1 - q) = -3q + 6.$$

If Player I plays row 2, Player II expects a payoff

$$2q + 4(1 - q) = 4 - 2q.$$

Player II can make sure that Player I does not win more than $\max(-3q + 6, 4 - 2q)$. And he wants to make sure this is as small as possible. So his strategy is to minimize (over $q \in [0, 1]$) the $\max(-3q + 6, 4 - 2q)$. For all $q \in [0, 1]$ we have $6 - 3q > 4 - 2q$. So he wants to minimize $6 - 3q$. This is achieved for $q = 1$. So he will play column 1 all the time. His expected payoff (loss) is $6 - 3 \cdot 1 = 3 = a_{11}$.

2. What happens if you solve a linear program to find the equilibrium for Paper- Scissors-Rock using the payoff matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

without adding a number to make all entries positive.

If we set up the program

$$\begin{array}{ll} \text{maximize} & y_1 + y_2 + y_3 \\ \text{subject to} & Ay \leq (1,1,1)^t \\ & y \geq 0, \end{array} \tag{1}$$

we get the simplex tableau

0	1	-1	1	0	0	1
-1	0	1	0	1	0	1
1	-1	0	0	0	1	1
1	1	1	0	0	0	0

We pivot on $a_{31} = 1$ to get

0	1	-1	1	0	0	1
0	-1	1	0	1	1	2
1	-1	0	0	0	1	1
0	2	1	0	0	-1	-1

We now pivot on $a_{12} = 1$ as the largest element in the last row is 2. This gives

0	1	-1	1	0	0	1
0	0	0	1	1	1	3
1	0	-1	1	0	1	2
0	0	3	-2	0	-1	-3

Now we have a positive entry ($a_{43} = 3$) on the last row but the entries above it (in the same column) are negative or zero. This means that the program is unbounded.

3. $*$ Let A be the payoff matrix for a two person zero-sum game. Show that, if $A = -A^t$, then the value of the game is 0.

Set

$$v = \min_q \max_p p^t A q$$

the value of the game. Here $p, q \in \mathbf{R}^n$ are probability vectors, i.e. $p, q \geq 0$, and $\sum_j p_j = \sum_j q_j = 1$. The dimensions of of the vectors are the same, as the matrix A is antisymmetric, i.e. a square matrix.

Then

$$\begin{array}{rcl} v = & \min_q \max_p (p^t A q)^t & \text{as it is a } 1 \times 1 \text{ matrix-number} \\ = & \min_q \max_p q^t A^t p & \text{as } (BC)^t = C^t B^t \\ = & \min_q \max_p (-q^t A p) & \text{as } A^t = -A \\ = & \min_q (-\min_p q^t A p) & \text{as } \max(-S) = -\min S \text{ for any set } S \\ = & -\max_q \min_p q^t A p & \text{as } \min(-S) = -\max S \text{ for any set } S \\ = & -\max_p \min_q p^t A q & \text{we relabel } p \text{ into } q \text{ and vice versa} \\ = & -v & \text{by the von Neumann minimax theorem} \end{array}$$

The result is $v = -v$ i.e. $v = 0$.

4. $*$ Solve using the simplex algorithm the undercut game with payoff matrix

$$A = \begin{pmatrix} 0 & -1 & 2 & 2 \\ 1 & 0 & -1 & 2 \\ -2 & 1 & 0 & -1 \\ -2 & -2 & 1 & 0 \end{pmatrix}.$$

Let V be the value of the game and p_i, q_i $i = 1, 2, 3, 4$ the probabilities of the strategies of the two players. We first add 3 to all the entries of A, so that we get a payoff matrix with positive entries. This gives

$$B = \begin{pmatrix} 3 & 2 & 5 & 5 \\ 4 & 3 & 2 & 5 \\ 1 & 4 & 3 & 2 \\ 1 & 1 & 4 & 3 \end{pmatrix}.$$

Let $v > 0$ be the value of the game with payoff B. We transform it into a standard linear programming by setting

$$y_i = \frac{q_i}{v}, \quad i = 1, 2, 3, 4.$$

The original program was: minimize v subject to

$$Bq \leq v \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\textstyle\sum_{j=1}^4 q_j = 1, \quad q \geq 0$$

Since $q_1 + q_2 + q_3 + q_4 = 1$, we have that $y_1 + y_2 + y_3 + y_4 = 1/v$, so that we try to maximize $\sum_{j=1}^{4} y_j$ subject to

$$By \le \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
$$y \ge 0$$

We add slack variables z_1, z_2, z_3, z_4 to get the system

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{4} y_j \\ \text{subject to} & \\ & By + z = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ & y \ge 0, z \ge 0. \end{array}$$

This gives the simplex tableau

3	2	5	5	1	0	0	0	1
4	3	2	5	0	1	0	0	1
1	4	3	2	0	0	1	0	1
1	1	4	3	0	0	0	1	1
1	1	1	1	0	0	0	0	0

The basic feasible solution $(0, 0, 0, 0, 1, 1, 1, 1)$ is not optimal. We pivot on $a_{21} = 4$. This gives

3	2	5	5	1	0	0	0	1
1	3/4	1/2	5/4	0	1/4	0	0	1/4
1	4	3	2	0	0	1	0	1
1	1	4	3	0	0	0	1	1
1	1	1	1	0	0	0	0	0

0	-1/4	7/2	5/4	1	-3/4	0	0	1/4
1	3/4	1/2	5/4	0	1/4	0	0	1/4
0	13/4	5/2	3/4	0	-1/4	1	0	3/4
0	1/4	7/2	7/4	0	-1/4	0	1	3/4
0	1/4	1/2	-1/4	0	-1/4	0	0	-1/4

The basic feasible solution $(1/4, 0, 0, 0, 1/4, 0, 3/4, 3/4)$ is not optimal. we decide to include y_3 in the basic variables and pivot on $a_{13} = 7/2$. We get

0	-1/14	1	5/14	2/7	-3/14	0	0	1/14
1	3/4	1/2	5/4	0	1/4	0	0	1/4
0	13/4	5/2	3/4	0	-1/4	1	0	3/4
0	1/4	7/2	7/4	0	-1/4	0	1	3/4
0	1/4	1/2	-1/4	0	-1/4	0	0	-1/4

0	-1/14	1	5/14	2/7	-3/14	0	0	1/14
1	11/14	0	15/14	-1/7	5/14	0	0	3/14
0	24/7	0	-1/7	-5/7	2/7	1	0	4/7
0	1/2	0	1/2	-1	1/2	0	1	1/2
0	2/7	0	-3/7	-1/7	-1/7	0	0	-2/7

The basic feasible solution $(3/14, 0, 1/14, 0, 0, 0, 4/7, 1/2)$ is not optimal. We pivot on $a_{32} = 24/7$. We get

0	-1/14	1	5/14	2/7	-3/14	0	0	1/14
1	11/14	0	15/14	-1/7	5/14	0	0	3/14
0	1	0	-1/24	-5/24	1/12	7/24	0	1/6
0	1/2	0	1/2	-1	1/2	0	1	1/2
0	2/7	0	-3/7	-1/7	-1/7	0	0	-2/7

0	0	1	17/48	13/48	-5/24	1/48	0	1/12
1	0	0	53/48	1/48	7/24	-11/48	0	1/12
0	1	0	-1/24	-5/24	1/12	7/24	0	1/6
0	0	0	25/48	-43/48	11/24	-7/48	1	5/12
0	0	0	-5/12	-1/12	-1/6	-1/12	0	-1/3

The basic feasible solution is $(1/12, 1/6, 1/12, 0, 0, 0, 0, 5/12)$ and is optimal, as all the entries on the last row are ≤ 0. The maximum of $\displaystyle\sum_{j=1}^{4} y_j = 1/3 = \frac{1}{v}$. So $v = 3$. The probabilities q_j are

$$(q_1, q_2, q_3, q_4) = v \cdot y = 3(1/12, 1/6, 1/12, 0) = (1/4, 1/2, 1/4, 0).$$

To find the actual value of the game we take off the 3 added to the entries of the matrix A to get $V = v - 3 = 0$. This is naturally expected, since $A = -A^t$. The probabilities for player I are $(p_1, p_2, p_3, p_4) = (1/4, 1/2, 1/4, 0)$ as the game is symmetric.