Math 7502

Homework 7

Due: March 6, 2008

1. * Solve the games with payoff matrices

   (a) \[
   \begin{pmatrix}
   1 & 4 \\
   7 & 2
   \end{pmatrix},
   \] (b) \[
   \begin{pmatrix}
   3 & 6 \\
   2 & 4
   \end{pmatrix}.
   \]

(a) The matrix \[
\begin{pmatrix}
1 & 4 \\
7 & 2
\end{pmatrix}
\] has no saddle point as row min \[
1 \frac{\rightarrow}{4}
\] and minimax = 1 \( \neq \) maximin = 2. We look for mixed (randomized) strategies. Assume that Player I plays row 1 with probability \( p \) and row 2 with probability \( 1 - p \), where \( 0 \leq p \leq 1 \). If Player II plays column 1, Player I expects a payoff

\[1 \cdot p + 7(1 - p) = 7 - 6p.\]

If Player II plays column 2, Player I expects a payoff

\[4p + 2(1 - p) = 2 + 2p.\]

Player I wants to guarantee wins of at least \( \min(7 - 6p, 2 + 2p) \). And he wants to make sure this is as large as possible. So his strategy is to maximize (over \( p \in [0, 1] \)) the \( \min(7 - 6p, 2 + 2p) \). The figure shows the minimum and we see that this is maximized at the intersection of the two lines \( v = 7 - 6p \) and \( v = 2 + 2p \) (\( v \) represents the value of the game). We solve

\[7 - 6p = 2 + 2p \implies p = \frac{5}{8}.\]

Then the value of the game is \( 2 + 2 \cdot 5/8 = 13/4 \). Player I plays row 1 with probability \( 5/8 \) and row 2 with probability \( 1 - 5/8 = 3/8 \). If Player II plays column 1 with probability \( q \) and column 2 with probability \( 1 - q \), the his expected payoff, assuming Player I plays row 1, is

\[1 \cdot q + 4(1 - q) = 4 - 3q.\]

The slack variable corresponding to this constraint has to be 0, as \( p \neq 0 \). So \( 4 - 3q = 13/4 \implies q = 1/4 \). Player II will play column 1 with probability \( 1/4 \) and column 2 with probability \( 3/4 \).
Figure 1: Player I's viewpoint

Figure 2: Player II's viewpoint
Second method: If Player I plays row 1, the expected payoff for Player II is
\[ 1 \cdot q + 4(1 - q) = 4 - 3q. \]
If Player I plays row 2, the expected payoff for Player II is
\[ 7q + 2(1 - q) = 5q + 2. \]
Player II can make sure that Player I wins at most \( \max(4 - 3q, 5q + 2) \). And he wants to make sure this is as small as possible. So his strategy is to minimize (over \( q \in [0, 1] \)) the \( \max(4 - 3q, 5q + 2) \). The figure shows the maximum and we see that this is minimized at the intersection of the two lines \( v = 4 - 3q \) and \( v = 5q + 2 \) (\( v \) represents the value of the game). We solve
\[ 4 - 3q = 5q + 2 \implies q = \frac{1}{4}. \]

(b) The matrix
\[
\begin{pmatrix}
3 & 6 \\
2 & 4
\end{pmatrix}
\]
has a saddle point \( a_{11} = 3 \). So the players will settle for strategy (row) 1 for Player I and strategy (column) 1 for Player II. They will play pure strategies.

Second method: Assume that Player I plays row 1 with probability \( p \) and row 2 with probability \( 1 - p \), where \( 0 \leq p \leq 1 \). If Player II plays column 1, Player I expects a payoff
\[ 3p + 2(1 - p) = p + 2. \]
If Player II plays column 2, Player I expects a payoff
\[ 6p + 4(1 - p) = 2p + 4. \]
Player I wants to guarantee wins of at least \( \min(p + 2, 2p + 4) \). And he wants to make sure this is as large as possible. So his strategy is to maximize (over \( p \in [0, 1] \)) the \( \min(p + 2, 2p + 4) \). For all \( p \in [0, 1] \) we have \( p + 2 < 2p + 4 \). So he wants to maximize \( p + 2 \). This is achieved for \( p = 1 \). So he will play row 1 all the time. His expected payoff is \( p + 2 = 3 = a_{11} \).

Assume that Player II plays column 1 with probability \( q \) and column 2 with probability \( 1 - q \), where \( 0 \leq q \leq 1 \). If Player I plays row 1, Player II expects a payoff
\[ 3q + 6(1 - q) = -3q + 6. \]
If Player I plays row 2, Player II expects a payoff
\[ 2q + 4(1 - q) = 4 - 2q. \]
Player II can make sure that Player I does not win more than \( \max(-3q + 6, 4 - 2q) \). And he wants to make sure this is as small as possible. So his strategy is to minimize (over \( q \in [0, 1] \)) the \( \max(-3q + 6, 4 - 2q) \). For all \( q \in [0, 1] \) we have \( 6 - 3q > 4 - 2q \). So he wants to minimize \( 6 - 3q \). This is achieved for \( q = 1 \). So he will play column 1 all the time. His expected payoff (loss) is \( 6 - 3 \cdot 1 = 3 = a_{11} \).
2. What happens if you solve a linear program to find the equilibrium for Paper-Scissors-Rock using the payoff matrix

\[ A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \]

without adding a number to make all entries positive.

If we set up the program

\[
\begin{align*}
\text{maximize} & \quad y_1 + y_2 + y_3 \\
\text{subject to} & \quad Ay \leq (1, 1, 1)^t \\
& \quad y \geq 0,
\end{align*}
\]

we get the simplex tableau

\[
\begin{array}{ccccccc|c}
0 & 1 & -1 & 1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 1 & 1 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
\]

We pivot on \(a_{31} = 1\) to get

\[
\begin{array}{ccccccc|c}
0 & 1 & -1 & 1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 1 & 1 & 2 \\
1 & -1 & 0 & 0 & 0 & 1 & 1 \\
\hline
0 & 2 & 1 & 0 & 0 & -1 & -1
\end{array}
\]

We now pivot on \(a_{12} = 1\) as the largest element in the last row is 2. This gives

\[
\begin{array}{ccccccc|c}
0 & 1 & -1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 \\
1 & 0 & -1 & 1 & 0 & 1 & 2 \\
\hline
0 & 0 & 3 & -2 & 0 & -1 & -3
\end{array}
\]

Now we have a positive entry \((a_{43} = 3)\) on the last row but the entries above it (in the same column) are negative or zero. This means that the program is unbounded.

3. * Let \(A\) be the payoff matrix for a two person zero-sum game. Show that, if \(A = -A^t\), then the value of the game is 0.

Set

\[ v = \min_q \max_p p^t A q \]
the value of the game. Here \( p, q \in \mathbb{R}^n \) are probability vectors, i.e. \( p, q \geq 0 \), and \( \sum_j p_j = \sum_j q_j = 1 \). The dimensions of the vectors are the same, as the matrix \( A \) is antisymmetric, i.e. a square matrix.

Then

\[
\begin{align*}
v &= \min_q \max_p (p^t A q) \quad \text{as it is a } 1 \times 1 \text{ matrix-number} \\
&= \min_q \max_p q^t A^t p \quad \text{as } (BC)^t = C^t B^t \\
&= \min_q \max_p (-q^t Ap) \quad \text{as } A^t = -A \\
&= \min_q (\min_p q^t Ap) \quad \text{as } \max(-S) = -\min S \text{ for any set } S \\
&= -\max_q \min_p q^t Ap \quad \text{as } \min(-S) = -\max S \text{ for any set } S \\
&= -\max_p \min_q p^t A q \quad \text{we relabel } p \text{ into } q \text{ and vice versa} \\
&= -v \quad \text{by the von Neumann minimax theorem}
\end{align*}
\]

The result is \( v = -v \) i.e. \( v = 0 \).

4. * Solve using the simplex algorithm the undercut game with payoff matrix

\[
A = \begin{pmatrix}
0 & -1 & 2 & 2 \\
1 & 0 & -1 & 2 \\
-2 & 1 & 0 & -1 \\
-2 & -2 & 1 & 0
\end{pmatrix}.
\]

Let \( V \) be the value of the game and \( p_i, q_i \ i = 1, 2, 3, 4 \) the probabilities of the strategies of the two players. We first add 3 to all the entries of \( A \), so that we get a payoff matrix with positive entries. This gives

\[
B = \begin{pmatrix}
3 & 2 & 5 & 5 \\
4 & 3 & 2 & 5 \\
1 & 4 & 3 & 2 \\
1 & 1 & 4 & 3
\end{pmatrix}.
\]

Let \( v > 0 \) be the value of the game with payoff \( B \). We transform it into a standard linear programming by setting

\[
y_i = \frac{q_i}{v}, \quad i = 1, 2, 3, 4.
\]

The original program was: minimize \( v \) subject to

\[
Bq \leq v \cdot \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

\[
\sum_{j=1}^4 q_j = 1, \quad q \geq 0
\]
Since \( q_1 + q_2 + q_3 + q_4 = 1 \), we have that \( y_1 + y_2 + y_3 + y_4 = 1/v \), so that we try to maximize \( \sum_{j=1}^{4} y_j \) subject to

\[
By \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]
\[
y \geq 0
\]

We add slack variables \( z_1, z_2, z_3, z_4 \) to get the system

\[
\text{maximize } \sum_{j=1}^{4} y_j \\
\text{subject to } By + z = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]
\[
y \geq 0, z \geq 0.
\]

This gives the simplex tableau

\[
\begin{array}{cccccccc|c}
3 & 2 & 5 & 5 & 1 & 0 & 0 & 0 & 1 \\
4 & 3 & 2 & 5 & 0 & 1 & 0 & 0 & 1 \\
1 & 4 & 3 & 2 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 4 & 3 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The basic feasible solution \((0, 0, 0, 0, 1, 1, 1, 1)\) is not optimal. We pivot on \( a_{21} = 4 \). This gives

\[
\begin{array}{cccccccc|c}
3 & 2 & 5 & 5 & 1 & 0 & 0 & 0 & 1 \\
1 & 3/4 & 1/2 & 5/4 & 0 & 1/4 & 0 & 0 & 1/4 \\
1 & 4 & 3 & 2 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 4 & 3 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The basic feasible solution \((1/4, 0, 0, 0, 1/4, 0, 3/4, 3/4)\) is not optimal. we decide to include \( y_3 \) in the basic variables and pivot on \( a_{13} = 7/2 \). We get
The basic feasible solution \((3/14, 0, 1/14, 0, 0, 0, 4/7, 1/2)\) is not optimal. We pivot on \(a_{32} = 24/7\). We get

\[
\begin{array}{cccccccc}
0 & -1/14 & 1 & 5/14 & 2/7 & -3/14 & 0 & 0 & 1/14 \\
1 & 3/4 & 1/2 & 5/4 & 0 & 1/4 & 0 & 0 & 1/4 \\
0 & 13/4 & 5/2 & 3/4 & 0 & -1/4 & 1 & 0 & 3/4 \\
0 & 1/4 & 7/2 & 7/4 & 0 & -1/4 & 0 & 1 & 3/4 \\
0 & 1/4 & 1/2 & -1/4 & 0 & -1/4 & 0 & 0 & -1/4 \\
\end{array}
\]

The basic feasible solution is \((1/12, 1/6, 1/12, 0, 0, 0, 5/12)\) and is optimal, as all the entries on the last row are \(\leq 0\). The maximum of \(\sum_{j=1}^4 y_j = 1/3 = \frac{1}{v}\). So \(v = 3\). The probabilities \(q_j\) are

\[(q_1, q_2, q_3, q_4) = v \cdot y = 3(1/12, 1/6, 1/12, 0) = (1/4, 1/2, 1/4, 0).\]

To find the actual value of the game we take off the 3 added to the entries of the matrix \(A\) to get \(V = v - 3 = 0\). This is naturally expected, since \(A = -A^t\). The probabilities for player I are \((p_1, p_2, p_3, p_4) = (1/4, 1/2, 1/4, 0)\) as the game is symmetric.