Math 7502

Homework 6

Due: February 28, 2008

1. Consider the two problems

$$\begin{array}{ll} \text{minimize} & c^t \cdot x \\ \text{subject to} & Ax \geq b \\ & x \geq 0, \end{array} \tag{1}$$

and

$$\begin{array}{ll} \text{minimize} & c^t \cdot x \\ \text{subject to} & Ax \geq b + v \\ & x \geq 0, \end{array} \tag{2}$$

where v is a vector is $\mathbf{R}^m$ of small 'size'. We consider the problem (2) as a 'perturbation' of the problem (1). For instance, if (1) is the diet problem with daily need prescribed by the vector b, then in the problem (2) we are allowing a small change in the dietary needs. The question is how does this affect the cost of the diet. Let y_0 be the optimal for the dual problem to (1). Show that the optimal (minimal) cost of the (diet) problem (2) is

$$f = (b + v)^t y_0.$$

Hint: Use duality.

Remark: The vector $-y_0$ appeared in the last row of the last simplex tableau. This problem explains why the last row gives information on the *shadow prices*.

Consider the dual to (1):

$$\begin{array}{ll} \text{maximize} & b^t \cdot y \\ \text{subject to} & A^t y \leq c \\ & y \geq 0, \end{array} \tag{3}$$

Let y_0 be the point where this is achieved. The dual problem of (2) is

$$\begin{array}{ll} \text{maximize} & (b + v)^t \cdot y \\ \text{subject to} & A^t y \leq c \\ & y \geq 0, \end{array} \tag{4}$$

The problems (3) and (4) have the same constraints. Therefore they have the same feasible region. The objective functions are close to each other, as the vector v is small. This means that the optimal point will be the same for both of them. (Think of two lines in $\mathbf{R}^2$ which, although not parallel, they have slopes close to each other. When we move them in a parallel fashion to themselves we will reach the same extreme point, as this is far away from the other extreme points). This optimal point is y_0. So the maximum for (4) is $(b + v)^t y_0$. By the strong duality theorem, this is the minimum for its primal problem, i.e. (2).

2. (a) Suppose that the problem

$$\begin{array}{ll} \text{maximize} & c^t \cdot x \\ \text{subject to} & Ax \le b \\ & x \ge 0 \end{array}$$

has a finite optimal solution. Here A is an $m \times n$ matrix, $b \in \mathbf{R}^m, c \in \mathbf{R}^n$. Show that, no matter what the vector $b' \in \mathbf{R}^m$ might be, the problem

$$\begin{array}{ll} \text{maximize} & c^t \cdot x \\ \text{subject to} & Ax \le b' \\ & x \ge 0 \end{array}$$

cannot be unbounded.

Hint: Use duality.

By the strong duality theorem the minimum for

$$\begin{array}{ll} \text{minimize} & b^t \cdot y \\ \text{subject to} & A^t y \ge c \\ & y \ge 0 \end{array} \tag{5}$$

is equal to the finite maximum of the primal problem. In particular we have a solution y to the constraints of the dual program (5), i.e. a feasible point for the dual program. Consider now the second max problem with an arbitrary b'. Let x be any feasible point for it. By the weak duality theorem,

$$c^t x \le b'^t y \tag{6}$$

for any feasible point of the dual program

$$\begin{array}{ll} \text{minimize} & b'^t \cdot y \\ \text{subject to} & A^t y \ge c \\ & y \ge 0 \end{array} \tag{7}$$

But the constraints in (7) are the same as in (5), so we have a feasible point y for (7). By (6) we have an upper bound for $c^t x$ for the second problem, so the second problem with arbitrary b' cannot be unbounded.

3. (a) Suppose that a two-person zero-sum game with payoff matrix A has a saddle-point. Show that all saddle points have the same value.

Let a_{ij} and a_{kl} be saddle points, i.e. they are minima in their rows and maxima in their columns. Then

$$\begin{array}{lll} a_{ij} & \le a_{il} & \text{as } a_{ij} \text{ is minimum in its row} \\ & \le a_{kl} & \text{as } a_{kl} \text{ is maximum in its column.} \end{array}$$

Similarly

$$\begin{aligned} a_{kl} &\leq a_{kj} \quad \text{as } a_{kl} \text{ is minimum in its row} \\ &\leq a_{ij} \quad \text{as } a_{ij} \text{ is maximum in its column.} \end{aligned}$$

The conclusion is that $a_{ij} = a_{kl}$.

(b) Show that, if a_{ij} is a saddle point then the row i is a maximin row and the column j is a minimax column and

$$\max_k \min_l a_{kl} = \min_l \max_k a_{kl} = a_{ij}. \tag{8}$$

Fix k for the k row. We need to show that $\min_l a_{il} \geq \min_l a_{kl}$. Then the i row is a maximin row. Since a_{ij} is a saddle point it is the minimum of its row, i.e. $\min_l a_{il} = a_{ij} \geq a_{kj}$, as it is also the maximum of its column. Obviously $a_{kj} \geq \min_l a_{kl}$. In fact as i is one of the rows, we have

$$a_{ij} = \max_k \min_l a_{kl}.$$

We also need to show column j is a minimax column. We fix a column l. We need to show that $\max_s a_{sj} \leq \max_s a_{sl}$. Since a_{ij} is maximum in its column, we have $\max_s a_{sj} = a_{ij} \leq a_{il}$, as it is also a minimum in its row. Obviously $a_{il} \leq \max_s a_{sl}$. In fact, since j is one of the columns we have

$$a_{ij} = \min_l \max_k a_{kl}.$$

This proves (8).

(c) If

$$\max_k \min_l a_{kl} = \min_l \max_k a_{kl}$$

then the intersection on the maximin row and the minimax column is a saddle point.

Let i be the k for which the $\max_k$ is achieved and let j be the l for which $\min_l$ is achieved. Then

$$\min_l a_{il} = \max_k a_{kj}.$$

Consider now the intersection of the i row and the j column, i.e. a_{ij}. Then

$$a_{ij} \geq \min_l a_{il} = \max_k a_{kj} \geq a_{ij}.$$

The conclusion is that the inequalities are all equalities, i.e.

$$a_{ij} = \min_l a_{il} = \max_k a_{kj},$$

i.e. a_{ij} is minimum in its row and maximum in its column, i.e. a saddle point.