1. * Use the two phase simplex algorithm to solve the linear program

\[
\text{maximize} \quad x_1 + x_2 + x_3 \\
\text{subject to} \quad -x_1 - x_2 + x_3 \leq -2 \\
\quad \quad \quad \quad x_1 + 2x_2 + x_3 \leq 5 \\
\quad \quad \quad \quad 3x_1 + x_2 + x_3 \leq 8 \\
\quad x_1, \quad x_2, \quad x_3 \geq 0.
\]

We introduce slack variables \( x_4, x_5, x_6 \) to write the program in canonical form:

\[
\text{maximize} \quad x_1 + x_2 + x_3 \\
\text{subject to} \quad -x_1 - x_2 + x_3 + x_4 = -2 \\
\quad \quad \quad \quad x_1 + 2x_2 + x_3 + x_5 = 5 \\
\quad \quad \quad \quad 3x_1 + x_2 + x_3 + x_6 = 8 \\
\quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6 \geq 0.
\]

Although we identify the identity matrix in the last three columns of the system, we do not get automatically a basic feasible solution, as we have a negative coefficient \(-2\) in the first equation. We rewrite the first equation as

\[ x_1 + x_2 - x_3 - x_4 = +2. \]

Now we realize that we need to add an artificial variable \( x_7 \) and first minimize \( x_7 \). This leads to the first phase of the program to be

\[
\text{minimize} \quad x_7 \\
\text{subject to} \quad x_1 + x_2 - x_3 - x_4 + x_7 = +2 \\
\quad x_1 + 2x_2 + x_3 + x_5 = 5 \\
\quad 3x_1 + x_2 + x_3 + x_6 = 8 \\
\quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6, \quad x_7 \geq 0.
\]

In tableau format we get

\[
\begin{array}{cccccccc}
1 & 1 & -1 & -1 & 0 & 0 & 1 & 2 \\
1 & 2 & 1 & 0 & 1 & 0 & 0 & 5 \\
3 & 1 & 1 & 0 & 0 & 1 & 0 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}
\]
This is not a valid simplex tableau, because there is a nonzero number below the identity matrix. We add the first row to the last row to get

\[
\begin{array}{cccccc|c}
1 & 1 & -1 & -1 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 1 & 0 & 5 \\
3 & 1 & 1 & 0 & 0 & 1 & 8 \\
1 & 1 & -1 & -1 & 0 & 0 & 2
\end{array}
\]

The basic feasible solution is now \((0, 0, 0, 0, 5, 8, 2)\) and is not optimal (for the minimum of \(x_7\)) as we have two positive entries on the last row. We include \(x_1\) in the basic variables and check that the smallest quotient is \(2/1\), which is \(< 8/3 < 5/1\). So we pivot on the \(a_{11}\) entry. We subtract the first row from the second and the fourth, and we subtract three times the first row from the third row to get

\[
\begin{array}{cccccc|c}
1 & 1 & -1 & -1 & 0 & 0 & 1 \\
0 & 1 & 2 & 1 & 1 & 0 & -1 \\
0 & -2 & 4 & 3 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}
\]

The basic feasible solution is now \((2, 0, 0, 0, 3, 2, 0)\) and is optimal, since the entries on the last row are nonpositive. In fact the last row says that the minimum \(x_7\) is 0, so there exists a feasible solution to the original system \((2, 0, 0, 0, 3, 2)\). We ignore the artificial variable and the seventh column and go back to the maximization of \(x_1 + x_2 + x_3\). This gives the tableau

\[
\begin{array}{cccccc|c}
1 & 1 & -1 & -1 & 0 & 0 & 2 \\
0 & 1 & 2 & 1 & 1 & 0 & 3 \\
0 & -2 & 4 & 3 & 0 & 1 & 2 \\
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
\]

This is not a valid simplex tableau, as below the first column i.e. the vector \(e_1\) we have a nonzero number. We subtract the first row from the last to get the simplex tableau

\[
\begin{array}{cccccc|c}
1 & 1 & -1 & -1 & 0 & 0 & 2 \\
0 & 1 & 2 & 1 & 1 & 0 & 3 \\
0 & -2 & 4 & 3 & 0 & 1 & 2 \\
0 & 0 & 2 & 1 & 0 & 0 & -2
\end{array}
\]

The basic feasible solution \((2, 0, 0, 0, 3, 2)\) is not optimal (for \(x_1 + x_2 + x_3\)), as we have positive entries on the last row. We include \(x_3\) in the basic variables. We check that \(2/4 < 3/2\), so we pivot on the entry \(a_{33} = 4\). We first divide the third row by 4 to get
1  1  -1  -1  0  0  2
0  1  2  1  1  0  3
0  -1/2  1  3/4  0  1/4  1/2
0  0  2  1  0  0  -2

We add the third row to the first and subtract twice the third row from the second and fourth rows to get

1  1/2  0  -1/4  0  1/4  5/2
0  2  0  -1/2  1  -1/2  2
0  -1/2  1  3/4  0  1/4  1/2
0  1  0  -1/2  0  -1/2  -3

The basic feasible solution is now (5/2, 0, 1/2, 0, 2, 0) and is not optimal, as we still have positive coefficients in the last row. We include $x_2$ in the basic variables. We check that $2/2 < (5/2)/(1/2)$, so we pivot on the $a_{22} = 2$ entry. We divide the second row by 2 to get

1  1/2  0  -1/4  0  1/4  5/2
0  1  0  -1/4  1/2  -1/4  1
0  -1/2  1  3/4  0  1/4  1/2
0  1  0  -1/2  0  -1/2  -3

We now subtract the second row from the fourth, subtract half the second row from the first and add half the second row to the third to get

1  0  0  -1/8  -1/4  3/8  2
0  1  0  -1/4  1/2  -1/4  1
0  0  1  5/8  1/4  1/8  1
0  0  0  -1/4  -1/2  -1/4  -4

The basic feasible solution is now (2, 1, 1, 0, 0, 0) and is optimal, as the coefficients on the last row are nonpositive. The maximum of $x_1 + x_2 + x_3$ is achieved at this basic feasible solution and is 4.

Graphing in this problem is difficult, as we have 3 variables and the equations represent planes in $\mathbb{R}^3$. So we avoid a graphical approach.

2. * (a) Use the two phase simplex algorithm to solve the linear program

minimize $x_1 + x_2$
subject to $4x_1 + x_2 \geq 4$
       $x_1 + 6x_2 \geq 6$
       $6x_1 + 10x_2 \geq 23$
       $x_1 , x_2 \geq 0.$
We use slack variables to write the system in canonical form

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 \\
\text{subject to} & \quad 4x_1 + x_2 - x_3 = 4 \\
& \quad x_1 + 6x_2 - x_4 = 6 \\
& \quad 6x_1 + 10x_2 - x_5 = 23 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{align*}
\]

In tableau format we have

\[
\begin{array}{ccccc|c}
4 & 1 & -1 & 0 & 0 & 4 \\
1 & 6 & 0 & -1 & 0 & 6 \\
6 & 10 & 0 & 0 & -1 & 23 \\
\hline
-1 & -1 & 0 & 0 & 0 & 0
\end{array}
\]

As we do not have at the same time the identity matrix and positive constants on the right column we introduce artificial variables \(x_6, x_7, x_8\). The first phase of the two-phase simplex is

\[
\begin{align*}
\text{minimize} & \quad x_6 + x_7 + x_8 \\
\text{subject to} & \quad 4x_1 + x_2 - x_3 + x_6 = 4 \\
& \quad x_1 + 6x_2 - x_4 + x_7 = 6 \\
& \quad 6x_1 + 10x_2 - x_5 + x_8 = 23 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.
\end{align*}
\]

In tableau format we have

\[
\begin{array}{ccccc|c}
4 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 4 \\
1 & 6 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 6 \\
6 & 10 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 23 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0
\end{array}
\]

This is not a valid simplex tableau, as we have nonzero entries below the identity matrix. We add the first three rows to the last one.

\[
\begin{array}{ccccc|c}
4 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 4 \\
1 & 6 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 6 \\
6 & 10 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 23 \\
\hline
11 & 17 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 33
\end{array}
\]

The basic feasible solution is \((0,0,0,0,0,4,6,23)\) and is not optimal, due to the positive entries on the last row. We decide to include \(x_2\) in the basic variables, since \(17 > 11\) (Dantzig’s rule). Since \(6/6 < 23/10 < 4/1\) we pivot on the \(a_{22} = 6\) entry.
The basic feasible solution is \((0, 1, 0, 0, 0, 4, 0, 23)\) and is not optimal (for the minimum of \(x_6 + x_7 + x_8\)), since we have positive entries on the last row. Because \(49/6 > 11/6\) we choose to include \(x_1\) in the basic variables (Dantzig’s rule). Since 

\[
\frac{3}{23/6} < \frac{13}{13/3} < \frac{1}{1/6}
\]

we pivot on the \(a_{11} = 23/6\) entry.

The basic feasible solution is \((18/23, 20/23, 0, 0, 0, 0, 0, 221/23)\) and is not optimal (for the minimum of \(x_6 + x_7 + x_8\)), since we have positive entries on the last row. Since \(34/23 > 26/23\), we include \(x_4\) in the basic variables (Dantzig’s rule). Since

\[
\frac{221/23}{34/23} < \frac{20/23}{1/23}
\]

we pivot on the \(a_{34} = 26/23\) entry. We get
The basic feasible solution is $(1/2, 2, 0, 13/2, 0, 0, 0, 0)$ and is optimal, as the entries on the last row are nonpositive. The maximum of $-x_6 - x_7 - x_8$ is 0, i.e. the minimum of $x_6 + x_7 + x_8$ is 0. Consequently the original program has a feasible point and we can proceed to the second phase of the two-phase simplex algorithm by erasing the columns of the artificial variables. We get

\[
\begin{array}{cccccc}
1 & 0 & -5/17 & 0 & 1/34 & 1/2 \\
0 & 1 & 3/17 & 0 & -2/17 & 2 \\
0 & 0 & 13/17 & 1 & -23/34 & 13/2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

This is not a valid simplex tableau, because we have nonzero entries below the identity matrix. We add the first two rows to the last one to get

\[
\begin{array}{cccccc}
1 & 0 & -5/17 & 0 & 1/34 & 1/2 \\
0 & 1 & 3/17 & 0 & -2/17 & 2 \\
0 & 0 & 13/17 & 1 & -23/34 & 13/2 \\
0 & 0 & -2/17 & 0 & -3/34 & 5/2
\end{array}
\]

The basic feasible solution is now $(1/2, 2, 0, 13/2, 0)$ and is optimal, since the entries in the last row are nonpositive. The maximum of $-x_1 - x_2$ is $-5/2$, i.e. the minimum of $x_1 + x_2$ is $5/2 = 1/2 + 2$.

* (b) Solve the same problem graphically and explain what the two phase simplex algorithm does geometrically (on the graph).

The arrows show that we started at $(0, 0)$ which was NOT feasible for the original problem. Then we moved to two more non feasible points for the original problem $(0, 1)$ and $(18/23, 20/23)$. Finally we reached a basic feasible solution for the original problem at $(1/2, 2)$ at the end of the first phase of the method. This point is now in the feasible region and is optimal, so we do not need to continue to find a better basic feasible solution to the original program.

3. Let $x_1, x_2, \ldots, x_k$ be points in $\mathbb{R}^n$. We say that $y \in \mathbb{R}^n$ is a convex combination of $x_1, \ldots, x_k$ if we can find scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$y = \sum_{j=1}^k \lambda_j x_j, \quad \lambda_j \geq 0, \quad j = 1, \ldots, k, \quad \sum_{j=1}^k \lambda_j = 1.$$
Figure 1: The unbounded feasible region for problem 2

Figure 2: Graphical solution for problem 2 and the path of the two phase simplex
(a) Let $S$ be the set of convex combinations of $x_1, \ldots, x_k$. Prove that $S$ is a convex set. The set $S$ is called the convex hull of $x_1, \ldots, x_k$.

Let $y$ and $z$ be in $S$, i.e. are convex combinations of $x_1, \ldots, x_k$. Then we can find scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mu_1, \mu_2, \ldots, \mu_k$ such that

$$y = \sum_{j=1}^{k} \lambda_j x_j, \quad z = \sum_{j=1}^{k} \mu_j x_j, \quad \lambda_j, \mu_j \geq 0, \quad j = 1, \ldots, k, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad \sum_{j=1}^{k} \mu_j = 1.$$ 

Let $t \in [0, 1]$. We need to show that $(1 - t)y + tz \in S$. We have

$$(1 - t)y + tz = \sum_{j=1}^{k} ((1 - t)\lambda_j + t\mu_j)x_j.$$ 

The coefficients $(1 - t)\lambda_j + t\mu_j$ are nonnegative, as linear combinations of nonnegative numbers. Moreover,

$$\sum_{j=1}^{k} (1 - t)\lambda_j + t\mu_j = (1 - t) \sum_{j=1}^{k} \lambda_j + t \sum_{j=1}^{k} \mu_j = (1 - t) \cdot 1 + t \cdot 1 = 1 - t + t = 1.$$ 

This proves that $(1 - t)y + tz$ is a convex combination of $x$ and $y$.

(b) Let $y$ be a convex combination of $a$ and $b \in \mathbb{R}^n$. Assume also that $a, b$ are convex combinations of $x_1, \ldots, x_k$. Prove that $y$ is a convex combination of $x_1, \ldots, x_k$.

We are given that we can find a $t \in [0, 1]$ such that

$$y = (1 - t)a + tb.$$ 

Moreover, since $a$ and $b$ are convex combinations of $x_1, \ldots, x_k$, we can find scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mu_1, \mu_2, \ldots, \mu_k$ such that

$$a = \sum_{j=1}^{k} \lambda_j x_j, \quad b = \sum_{j=1}^{k} \mu_j x_j, \quad \lambda_j, \mu_j \geq 0, \quad j = 1, \ldots, k, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad \sum_{j=1}^{k} \mu_j = 1.$$ 

Then

$$y = (1 - t)a + tb = \sum_{j=1}^{k} ((1 - t)\lambda_j + t\mu_j)x_j.$$ 

The coefficients $(1 - t)\lambda_j + t\mu_j$ are nonnegative, as linear combinations of nonnegative numbers. Moreover,

$$\sum_{j=1}^{k} (1 - t)\lambda_j + t\mu_j = (1 - t) \sum_{j=1}^{k} \lambda_j + t \sum_{j=1}^{k} \mu_j = (1 - t) \cdot 1 + t \cdot 1 = 1 - t + t = 1.$$
(c) Show that the convex hull of \((0, 0), (1, 0), (0, 1)\) and \((1, 1)\) is the square \([0, 1] \times [0, 1]\). The difficult part is to show that every point in the square is a convex combination of the four extreme points of the square. Write \((x, y)\) as a convex combination of \((0, y)\) and \((1, y)\) first and use (b).

Let \(y\) be a convex combination of the four points, i.e. for some scalars \(\lambda_1, \lambda_2, \lambda_3, \lambda_4\) we have

\[y = \lambda_1(0, 0) + \lambda_2(1, 0) + \lambda_3(0, 1) + \lambda_4(1, 1) = (\lambda_2 + \lambda_4, \lambda_3 + \lambda_4),\]

\[\lambda_j \geq 0, \quad j = 1, \ldots, 4, \quad \sum_{j=1}^{4} \lambda_j = 1.\]

To show that \(y\) belongs to the square, we need to show that its coordinates are in the interval \([0, 1]\). As \(\lambda_j \geq 0\), we have

\[\lambda_2 + \lambda_4 \geq 0, \quad \lambda_3 + \lambda_4 \geq 0.\]

On the other hand, the sum of the four coefficients is 1, while they are all nonnegative. This implies that

\[\lambda_2 + \lambda_4 \leq \sum_{j=1}^{4} \lambda_j = 1, \quad \lambda_3 + \lambda_4 \leq \sum_{j=1}^{4} \lambda_j = 1.\]

The converse: Let \(z \in [0, 1] \times [0, 1]\). We need to write \(z = (x, y)\) as a convex combination of the four points. We first notice that, given that \(x \in [0, 1]\), that

\[(x, y) = (1 - x)(0, y) + x(1, y),\]

i.e. \((x, y)\) is a convex combination of \((0, y)\) and \((1, y)\). Now these two points are convex combinations of the vertices:

\[(0, y) = (1 - y)(0, 0) + y(0, 1), \quad (1, y) = (1 - y)(1, 0) + y(1, 1),\]

as \(y \in [0, 1]\). Using (b) we conclude that \((x, y)\) is a convex combination of the four points.

**Remark:** If one insists, one can write the convex combination more explicitly

\[(x, y) = (1-x)(0, y) + x(1, y) = (1-x)(1-y)(0, 0) + (1-x)y(0, 1) + x(1-y)(1, 0) + xy(1, 1).\]