Math 7502

Homework 3: solutions

Due: January 31, 2008

1. $*$ Consider the region defined by the following constraints:

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ -x_1 + 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(i) Maximize $-4x_1 + x_2$ subject to the constraints above.

(ii) Minimize $3x_1 - 4x_2$ subject to the constraints above.

(iii) Maximize $-x_1 + 3x_2$ subject to the constraints above. You should find the maximal solution value is unbounded. Explain this carefully by (a) exhibiting feasible points with objective value increasing to infinity (b) showing the situation on the graph of the feasible region in the x_1, x_2-plane.

We write the program in canonical form by introducing slack variables x_3, x_4:

$$\begin{aligned} -x_1 + x_2 + \quad x_3 \qquad &= 2 \\ -x_1 + 2x_2 + \qquad\quad x_4 &= 6 \end{aligned}$$

(i) We write the program with the objective function $-4x_1 + x_2$ in tableau format

-1	1	1	0	2
-1	2	0	1	6
-4	1	0	0	0

The basic variables are x_3, x_4 and the nonbasic x_1, x_2. The basic feasible solution is $(0, 0, 2, 6)$ and the current value of f is 0. This is not optimal, since $1 > 0$ in the last row. So we include x_2 in our basic variables. We have to choose which row to use. The first row allows for an increase of 2 and the second an increase of 3 for x_2. We choose the first row, so we pivot on the 1 in the first row and second column. We multiply the first row by 2 and subtract from the second row and we also subtract it from the third row. We get the new tableau

-1	1	1	0	2
0	0	-2	1	2
-3	0	-1	0	-2

Now the basic variables are x_2, x_4 and the nonbasic are x_1 and x_3. The basic feasible solution is $(0, 2, 0, 2)$ and the value of f is 2. This solution is optimal, since the coefficients -3 and -1 on the last row are negative.

(ii) We want to minimize $3x_1 - 4x_2$. We maximize first instead $-3x_1 + 4x_2$ and for this we introduce the simplex tableau

-1	1	1	0	2
-1	2	0	1	6
-3	4	0	0	0

The basic solution we start with here is $(0, 0, 2, 6)$ with value of the objective function 0. Since the entry 4 on the last row is positive this is not an optimal solution. We include x_2 in the basic variables. The first row allows an increase of 2 and the second an increase of 3 for x_2. We choose the minimum, i.e. we pivot on the first row-second column entry. We subtract twice the first row from the second row and we subtract 4 times the first row from the third row to get the tableau

-1	1	1	0	2
1	0	-2	1	2
1	0	-4	0	-8

Now the basic variables are x_2, x_4 and the basic solution is $(0, 2, 0, 2)$. The value of the objective function is 8. The solution is not optimal, since we have the positive coefficient 1 on the last row. So we want to include x_1 in the basic variables. The first equation has negative coefficient for x_1, so it does not block (restrict) the increase of the variable x_1. However, the second equation only allows an increase of 2. We pivot on the second row-first column entry. We add the second row to the first and subtract it from the third. This gives the new tableau

0	1	-1	1	4
1	0	-2	1	2
0	0	-2	-1	-10

The basic variables are x_1 and x_2. The basic feasible solution is $(2, 4, 0, 0)$ and the value of the objective function is 10. Since the coefficients -2 and -1 are negative on the last row, this solution is optimal. This means that the maximum of $-3x_1 + 4x_2$ is 10 and the minimum of the objective $3x_1 - 4x_2$ is -10.

(iii) We introduce the simplex tableau

-1	1	1	0	2
-1	2	0	1	6
-1	3	0	0	0

The basic solution we start with here is $(0, 0, 2, 6)$ with value of the objective function 0. Since the entry 3 on the last row is positive this is not an optimal solution. We include x_2 in the basic variables. The first row allows an increase of 2 and the second an increase of 3 for x_2. We choose the minimum, i.e. we pivot on the first row-second column entry. We subtract twice the first row from the second row and we subtract 3 times the first row from the third row to get the tableau

-1	1	1	0	2
1	0	-2	1	2
2	0	-3	0	-6

Now the basic variables are x_2, x_4 and the basic solution is $(0, 2, 0, 2)$. The value of the objective function is 6. The solution is not optimal, since we have the positive coefficient 2 on the last row. So we want to include x_1 in the basic variables. The first equation has negative coefficient for x_1, so it does not block (restrict) the increase of the variable x_1. However, the second equation only allows an increase of 2. We pivot on the second row-first column entry. We add the second row to the first and subtract twice the second row from the third. This gives the new tableau

0	1	-1	1	4
1	0	-2	1	2
0	0	1	-2	-10

The new basic solution is $(2, 4, 0, 0)$. It is not optimal because of the entry 1 in the last row. However, we have above 1 two negatives entries -1 and -2. This should mean that there is no maximum for the objective function. Explanations:

(a) We rewrite the objective function out of the last row as

$$f = 10 + x_3 - 2x_4.$$

If we can increase x_3 without bound, the objective function also increases to infinity. The other two equations are

$$x_2 - x_3 + x_4 = 4 \quad x_1 - 2x_3 + x_4 = 2$$

If we keep $x_4 = 0$, we solve to get $x_2 = 4 + x_3$, $x_1 = 2x_3 + 2$. So, as we increase x_3 we remain in the feasible region. The feasible points are $(2x_3 + 2, 4 + x_3, x_3, 0)$, $x_3 \geq 0$ and the objective values for these is $f = 10 + x_3$.

(b) We plot the feasible region and the family of lines $-x_1 + 3x_2 = k$, with increasing k. We get the figures

2. $*$ A nut packager has on hand 150 kg of peanuts, 100 kg of cashews, and 50 kg of almonds. The packager can sell three kinds of mixtures of these nuts: a cheap

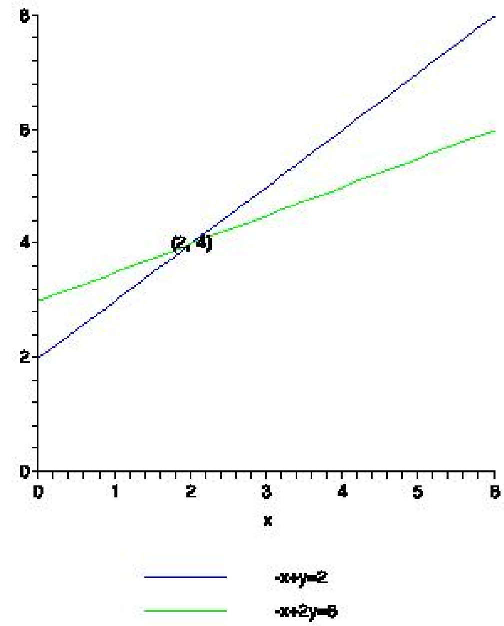
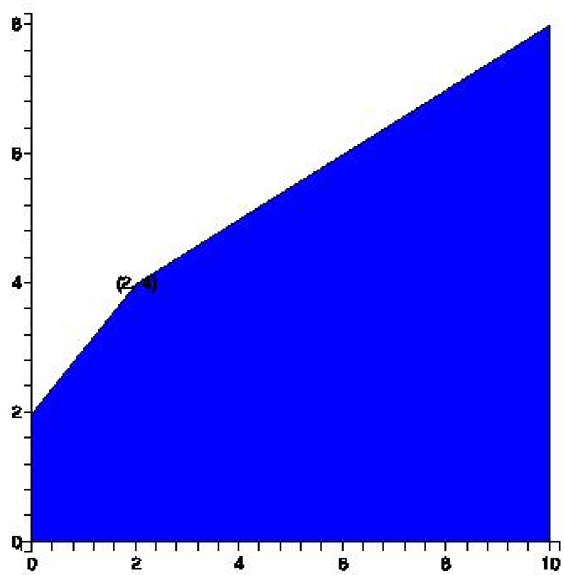


Figure 1: The constraints for problem 1 and the unbounded feasible region



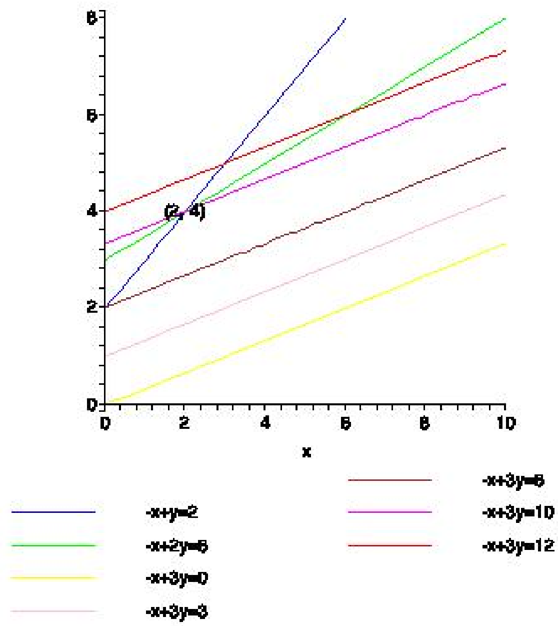


Figure 2: The objective functions gives us parallel lines $-x_1 + 3x_2 = k$ that meet the feasible region in further away points with unbounded k

mix consisting of 80% peanuts and 20% cashews; a party mix with 50% peanuts, 30% cashews, and 20% almonds; and a deluxe mix with 20% percent peanuts, 50% cashews, and 30% almonds. If the 1 kg can of the cheap mix, the party mix and the deluxe mix can be sold for 0.9, 1.1 and 1.3 pounds respectively, how many cans of each type would the packager produce in order to maximize the return? Use the simplex method in tableau format and a hand-held calculator for the computations.

Let x_1, x_2, and x_3 be the number of packages of cheap/party/deluxe mix produced. The constraints are $x_1, x_2, x_3 \geq 0$ and we must not use from each kind of nut more than the available amount. This gives for the peanuts

$$0.8x_1 + 0.5x_2 + 0.2x_3 \leq 150,$$

for the cashews

$$0.2x_1 + 0.3x_2 + 0.5x_3 \leq 100$$

and for the almonds

$$0x_1 + 0.2x_2 + 0.3x_3 \leq 50.$$

The objective function to maximize is the return and is

$$f = 0.9x_1 + 1.1x_2 + 1.3x_3.$$

We add slack variables x_4, x_5, x_6 to turn the program into canonical form

$$\begin{array}{llllll} 0.8x_1 + 0.5x_2 + 0.2x_3 + x_4 & & & = 150 \\ 0.2x_1 + 0.3x_2 + 0.5x_3 & +x_5 & & = 100 \\ 0.0x_1 + 0.2x_2 + 0.3x_3 & & +x_6 & = 50 \end{array}$$

with $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$. We write it in simplex tableau.

0.8	0.5	0.2	1	0	0	150
0.2	0.3	0.5	0	1	0	100
0.0	0.2	0.3	0	0	1	50
0.9	1.1	1.3	0	0	0	0

The basic variables are x_4, x_5, x_6 and the nonbasic are x_1, x_2, x_3. The basic feasible solution is $(0, 0, 0, 150, 100, 50)$ and the current value of the return function is $f = 0$. Since the coefficients of f in the last row for the nonbasic variables are positive (f is expressed in terms of the nonbasic variables), we conclude that the current solution is not optimal.

To decide which nonbasic variable to include and which basic to exclude, we compute the quotients of the constants in the right column with the entries in each of the first three rows and choose the miminum for for each nonbasic variable:

$$\begin{array}{llll} \frac{150}{0.8} = 187.5 & \frac{100}{0.2} = 500 & \frac{50}{0} = \text{undefined} & \Longrightarrow \Delta x_1 = 187.5 \quad \text{out of row 1.} \\ \frac{150}{0.5} = 300 & \frac{100}{0.3} = 333\frac{1}{3} & \frac{50}{0.2} = 250 & \Longrightarrow \Delta x_2 = 250 \quad \text{out of row 3.} \\ \frac{150}{0.2} = 750 & \frac{100}{0.5} = 200 & \frac{50}{0.3} = 166\frac{2}{3} & \Longrightarrow \Delta x_3 = 166\frac{2}{3} \quad \text{out of row 3.} \end{array}$$

We have

$\Delta x_1 = 187.5$	$\Delta f = 168.75$
$\Delta x_2 = 250$	$\Delta f = 275$
$\Delta x_3 = 166\frac{2}{3}$	$\Delta f = 216\frac{2}{3}$

It is advantageous to include x_2 out of row 3, and exclude x_6. We perform the row operations with pivot the entry 0.2 in the third row second column. First we divide the third row by 0.2, then we subtract 0.5 multiples of it from the first row and 0.3 multiples of it from the second. We also do the same with the last row: multiple the third row with 1.1 and subtract from the last row.

0.8	0.5	0.2	1	0	0	150
0.2	0.3	0.5	0	1	0	100
0.0	1	1.5	0	0	5	250
0.9	1.1	1.3	0	0	0	0

0.8	0	-0.55	1	0	-2.5	25
0.2	0	0.05	0	1	-1.5	25
0.0	1	1.5	0	0	5	250
0.9	0	-0.35	0	0	-5.5	-275

The bsic variables now are x_2, x_4, x_5 and the nonbasic are x_1, x_3, x_6. The current basic solution is $(0, 250, 0, 25, 25, 0)$. The current return is $f = +275$ and this is not optimal, since on the last row we have the positive entry 0.9 for the nonbasic variable x_1. We compute

Row	Δx_1	Δf
1	25/0.8=31.25	28.125
2	25/0.2=125	112.5
3	250/0 undef	

We have to choose the minimum of Δx_1 so we work with the first row and pivot on the entry in first row and first column. First we divide by 0.8 the first row and then subtract multiples of the new first row from the other rows. More precisely we multiply it with 0.2 and subtract from the second, with 0 and subtract from the third and with 0.9 and subtract your the last. This gives:

1	0	-0.6875	1.25	0	-3.125	31.25
0.2	0.3	0.5	0	1	0	100
0.0	1	1.5	0	0	5	250
0.9	1.1	1.3	0	0	0	0

1	0	-0.6875	1.25	0	-3.125	31.25
0	0	0.1875	-0.25	1	-0.875	18.75
0	1	1.5	0	0	5	250
0	0	0.26875	-1.125	0	-2.6875	-303.125

The current basic solution is $(31.25, 250, 0, 0, 18.75, 0)$ and is not optimal since we have a positive entry 0.26875 in the last row. We can increase x_3 and increase the objective function. Since the first entry in the third column is negative -0.6875 the first row does not restrict or block how much we can increase x_3. The second row gives $\Delta x_3 = 18.75/0.1875 = 100$ and the third row $250/1.5 = 166.66666$. We choose the minimum 100 and pivot along the 0.1875. We first divide the second row with 0.1875, then subtract multiples of it from the other rows. More precisely, we subtract -0.6875 of it from the first row, 1.5 times it from the third and 0.26875 from the fourth. We get

1	0	-0.6875	1.25	0	-3.125	31.25
0	0	1	$-1\frac{1}{3}$	$5\frac{1}{3}$	$-4\frac{2}{3}$	100
0	1	1.5	0	0	5	250
0	0	0.26875	-1.125	0	-2.6875	-303.125

1	0	0	$\frac{1}{3}$	$3\frac{2}{3}$	$-6\frac{1}{3}$	100
0	0	1	$-1\frac{1}{3}$	$5\frac{1}{3}$	$-4\frac{2}{3}$	100
0	1	0	2	-8	12	100
0	0	0	-0.766666	-1.43333	-1.43333	-330

Since the coefficients of the nonbasic variables x_4, x_5, x_6 are negative, the current basic solution is optimal. It is $(100, 100, 100, 0, 0, 0)$ and the corresponding return is $+330$. The nut packager should produce 100 cans from each of the three types of mixes for a profit of 330 pounds.

3. (Only for maths students) In analysis you saw the notion of a convex function. A function $f : [a, b] \to \mathbf{R}$ is called convex if its graph is below the secant segment between any two points of the graph $(x, f(x))$ and $(y, f(y))$, i.e. for all $t \in [0, 1]$ we have
$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$
Consider the set
$$S = \{(x, y) | y \geq f(x), x \in [a, b]\}.$$
Show that f is convex function if and only if S is a convex set in $\mathbf{R}^2$.

$\Longrightarrow$: Assume that f is convex. Let P and Q be two points in S, i.e. $P(x_1, y_1)$ and $Q(x_2, y_2)$ such that
$$y_1 \geq f(x_1), \quad y_2 \geq f(x_2), \tag{1}$$
because P is in S and because Q is in S. The line segment between P and Q consists of points $R(x, y)$ with
$$x = (1-t)x_1 + tx_2, \quad y = (1-t)y_1 + ty_2, \quad t \in [0, 1].$$
To show that $R \in S$, we need to show that $y \geq f(x)$. But
$$f(x) = f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \leq (1-t)f(x_1) + tf(x_2) \leq (1-t)y_1 + ty_2 = y,$$
where in the first inequality we used the fact that f is convex, while in the second we used (**??**). This proves that S is convex.

$\Longleftarrow$: Assume that S is a convex set in $\mathbf{R}^2$. Let $x_1, x_2 \in [a, b]$ and $t \in [0, 1]$. To prove that f is convex, we need to prove that
$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2).$$
The points $P(x_1, f(x_1))$ is in S, as its y-coordinate is $\geq f(x_1)$. Similarly the points $Q(x_2, f(x_2))$ is in S, as its y-coordinate is $\geq f(x_2)$. Then the line segment between P and Q is in S, i.e. for $t \in [0, 1]$ we have
$$(1-t)(x_1, f(x_1)) + t(x_2, f(x_2)) \in S \Leftrightarrow ((1-t)x_1 + tx_2, (1-t)f(x_1) + tf(x_2)) \in S.$$
The points in S have y-coordinate $\geq$ the value of their x-coordinate under f. This means
$$(1-t)f(x_1) + tf(x_2) \geq f((1-t)x_1 + tx_2)$$
for $t \in [0, 1]$. This is exactly the definition of convexity for f.