Math 7502
Homework 1

Due: January 17, 2008

1. Describe and graph the region in the first quadrant of the $x_1x_2$-plain determined by the linear inequalities:

$$5x_1 + 10x_2 \leq 50, \quad x_1 + x_2 \leq 6, \quad 10x_1 + 5x_2 \leq 50.$$ 

We plot the lines $5x_1 + 10x_2 = 50$, $x_1 + x_2 = 6$ and $10x_1 + 5x_2 = 50$. Each one separates the plane into two half-planes. On one of them we get $<$ and on the other $>$. We need to find on which half-plane this happens and for this we check the origin $(0,0)$, which does not belong to the three lines. Clearly $5 \cdot 0 + 10 \cdot 0 < 50$, $0 + 0 < 6$ and $10 \cdot 0 + 5 \cdot 0 < 50$. This means that $(0,0)$ is included in the half-planes allowed and is in the feasible region. This gives the graphs: We find the intersection of the

Figure 1: The feasible region for problem 1

lines $5x_1 + 10x_2 = 50$ and $x_1 + x_2 = 6$ by solving the system. We get $x_1 = 2, x_2 = 4$. We find the intersection of the lines $10x_1 + 5x_2 = 50$ and $x_1 + x_2 = 6$ by solving the system. We get $x_1 = 4, x_2 = 2$. We do not need to find the intersection of the lines $5x_1 + 10x_2 = 50$ and $10x_1 + 5x_2 = 50$, as it is not in the feasible region.
2. Maximize the daily profit in manufacturing two alloys $A_1$ and $A_2$ which are different mixtures of two metals $M_1$ and $M_2$ as shown:

<table>
<thead>
<tr>
<th>Metal</th>
<th>Proportion of metal In Alloy $A_1$</th>
<th>Proportion of metal In Alloy $A_2$</th>
<th>Daily supply in tons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>0.5</td>
<td>0.25</td>
<td>10</td>
</tr>
<tr>
<td>$M_2$</td>
<td>0.5</td>
<td>0.75</td>
<td>15</td>
</tr>
</tbody>
</table>

Net Profit per ton 30 25

Solve the program graphically in the $xy$-plane. Write it in matrix form. Write this program in canonical form. What is the meaning of the two slack variables needed? Find all basic solutions in canonical form. Which ones are feasible? Find the basic feasible solutions in standard form.

Figure 2: The feasible region for problem 2

Let $x_1$ and $x_2$ be the production of alloy $A_1$ and $A_2$ in tons per day. Then the daily profit is the objective function

$$f(x_1, x_2) = 30x_1 + 25x_2.$$ 

The constraints are as follows. Clearly $x_1 \geq 0$ and $x_2 \geq 0$. The $x_1$ tons of alloy $A_1$ use $0.5x_1$ tons of metal $M_1$ and the $x_2$ tons of alloy $A_2$ use $0.25x_2$ tons of metal $M_1$. 

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Since the amount used from metal $M_1$ cannot exceed 10 tons per day, we have

$$0.5x_1 + 0.25x_2 \leq 10$$

while, since the amount used from metal $M_2$ cannot exceed 15 tons per day, we have

$$0.5x_1 + 0.75x_2 \leq 15.$$  

This is because $x_1$ tons of alloy $A_1$ use $0.5x_1$ tons of metal $M_2$ and $x_2$ tons of alloy $A_2$ use $0.75x_2$ tons of metal $M_2$. We plot the lines $0.5x_1 + 0.25x_2 = 10$, $0.5x_1 + 0.75x_2 = 15$. The two lines intersect at $(15, 10)$. You can solve the system to find this solution: subtract them to get $0.5x_2 = 5 \implies x_2 = 10$, etc. We easily determine that $(0, 0)$ is in the feasible region, so the feasible region is the quadrangle in the figure.

We draw also the family of lines $30x_1 + 25x_2 = k$ for various $k$. We see that we increase the value of the objective function, as we move the lines in the direction of the vector $(30, 25)^t$, i.e. up and right. In this move the further point is $(15, 10)$ and this gives the maximum of the objective function: $f(15, 10) = 30 \cdot 15 + 25 \cdot 10 = 450 + 250 = 700$.

We can also check that at the extreme points of the quadrangle (vertices) the values of the objective function are

$$f(0, 0) = 0, f(20, 0) = 30 \cdot 20 = 600, f(0, 20) = 25 \cdot 20 = 500, f(15, 10) = 700.$$
Matrix form: Set \( c = (30, 25)^t \), \( x = (x_1, x_2)^t \), \( b = (10, 15)^t \),
\[
A = \begin{pmatrix}
0.5 & 0.25 \\
0.5 & 0.75
\end{pmatrix}.
\]
The problem is now:
\[
\begin{align*}
\text{maximize} & \quad c^t \cdot x \\
\text{subject to} & \quad x \geq 0, \quad Ax \leq b.
\end{align*}
\]
For the canonical form we introduce slack variables \( x_3 \) and \( x_4 \), which are both \( \geq 0 \) and such that
\[
0.5x_1 + 0.25x_2 + x_3 = 10, \quad 0.5x_1 + 0.75x_2 + x_4 = 15.
\]
Their meaning is as follows: we know that we cannot exceed 10 tons of metal \( M_1 \) per day. The slack variable \( x_3 \) represents the amount of metal \( M_1 \) not used per day. Similarly the slack variable \( x_4 \) represents the amount of metal \( M_2 \) not used per day.
The problem becomes
\[
\begin{align*}
\text{maximize} & \quad 30x_1 + 25x_2 \\
\text{subject to} & \quad x_1, x_2, x_3, x_4 \geq 0 \\
& \quad 0.5x_1 + 0.25x_2 + x_3 = 10 \\
& \quad 0.5x_1 + 0.75x_2 + x_4 = 15
\end{align*}
\]
We have now \( x = (x_1, x_2, x_3, x_4)^t \), \( c = (30, 25, 0, 0)^t \), \( b = (10, 15)^t \) and
\[
A = \begin{pmatrix}
0.5 & 0.25 & 0 & 0 \\
0.5 & 0.75 & 0 & 1
\end{pmatrix}.
\]
The problem is now:
\[
\begin{align*}
\text{maximize} & \quad c^t \cdot x \\
\text{subject to} & \quad x \geq 0, \quad Ax = b.
\end{align*}
\]
This is the canonical form \( (m = 2, n = 4) \).
The basic solutions in canonical form have at least \( n - m = 2 \) variables equal to 0.
This gives 6 solutions:
\[
(0, 0, 10, 15), (0, 40, 0, -15), (0, 20, 5, 0), (20, 0, 0, 5), (30, 0, -5, 0), (15, 10, 0, 0).
\]
Among these we discard the two with negative entries, as they are not feasible. So the basic feasible solutions in canonical form are
\[
(0, 0, 10, 15), (0, 20, 5, 0), (20, 0, 0, 5), (15, 10, 0, 0)
\]
and in standard form (ignoring the slack variables)
\[
(0, 0), (0, 20), (20, 0), (15, 10).
\]
These are the vertices of the quadrangle.