Math 7502

Homework 1

Due: January 17, 2008

1. Describe and graph the region in the first quadrant of the x_1x_2-plain determined by the linear inequalities:

$$5x_1 + 10x_2 \leq 50, \quad x_1 + x_2 \leq 6, \quad 10x_1 + 5x_2 \leq 50.$$

We plot the lines $5x_1 + 10x_2 = 50$, $x_1 + x_2 = 6$ and $10x_1 + 5x_2 = 50$. Each one separates the plane into two half-planes. On one of them we get $<$ and on the other $>$. We need to find on which half-plane this happens and for this we check the origin $(0, 0)$, which does not belong to the three lines. Clearly $5 \cdot 0 + 10 \cdot 0 < 50$, $0 + 0 < 6$ and $10 \cdot 0 + 5 \cdot 0 < 50$. This means that $(0, 0)$ is included in the half-planes allowed and is in the feasible region. This gives the graphs: We find the intersection of the

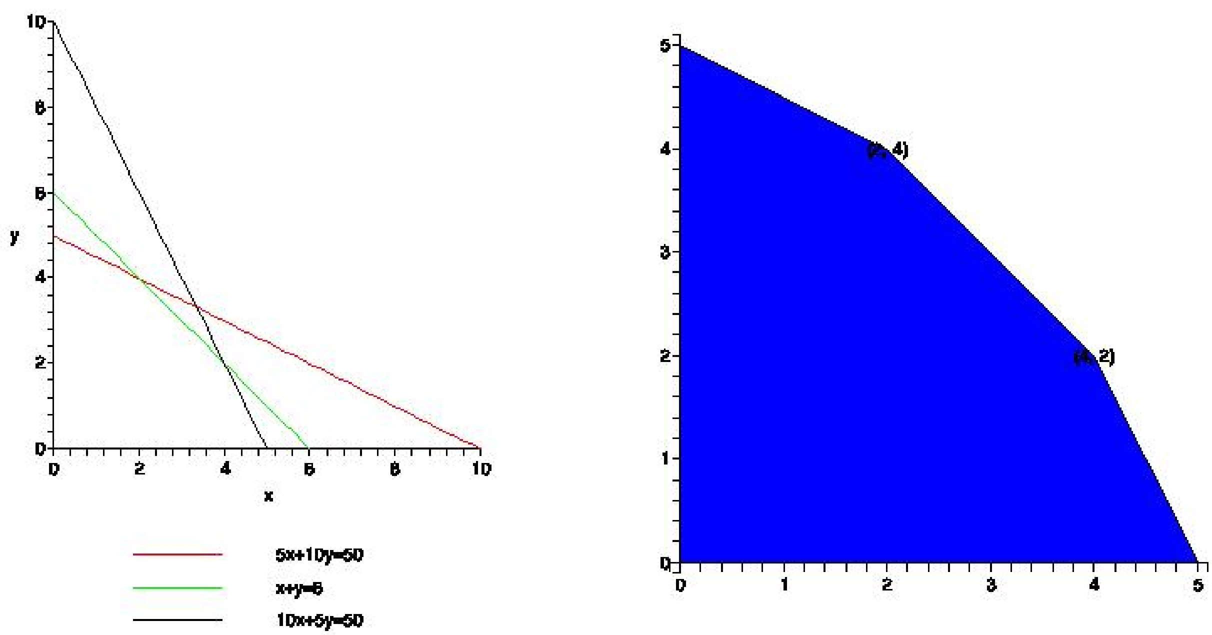

Figure 1: The feasible region for problem 1

lines $5x_1 + 10x_2 = 50$ and $x_1 + x_2 = 6$ by solving the system. We get $x_1 = 2, x_2 = 4$. We find the intersection of the lines $10x_1 + 5x_2 = 50$ and $x_1 + x_2 = 6$ by solving the system. We get $x_1 = 4, x_2 = 2$. We do not need to find the intersection of the lines $5x_1 + 10x_2 = 50$ and $10x_1 + 5x_2 = 50$, as it is not in the feasible region.

2. Maximize the daily profit in manufacturing two alloys A_1 and A_2 which are different mixtures of two metals M_1 and M_2 as shown:

Metal	Proportion of metal In Alloy A_1	Proportion of metal In Alloy A_2	Daily supply in tons
M_1	0.5	0.25	10
M_2	0.5	0.75	15
Net Profit per ton	30	25	

Solve the program graphically in the xy-plane. Write it in matrix form. Write this program in canonical form. What is the meaning of the two slack variables needed? Find all basic solutions in canonical form. Which ones are feasible? Find the basic feasible solutions in standard form.

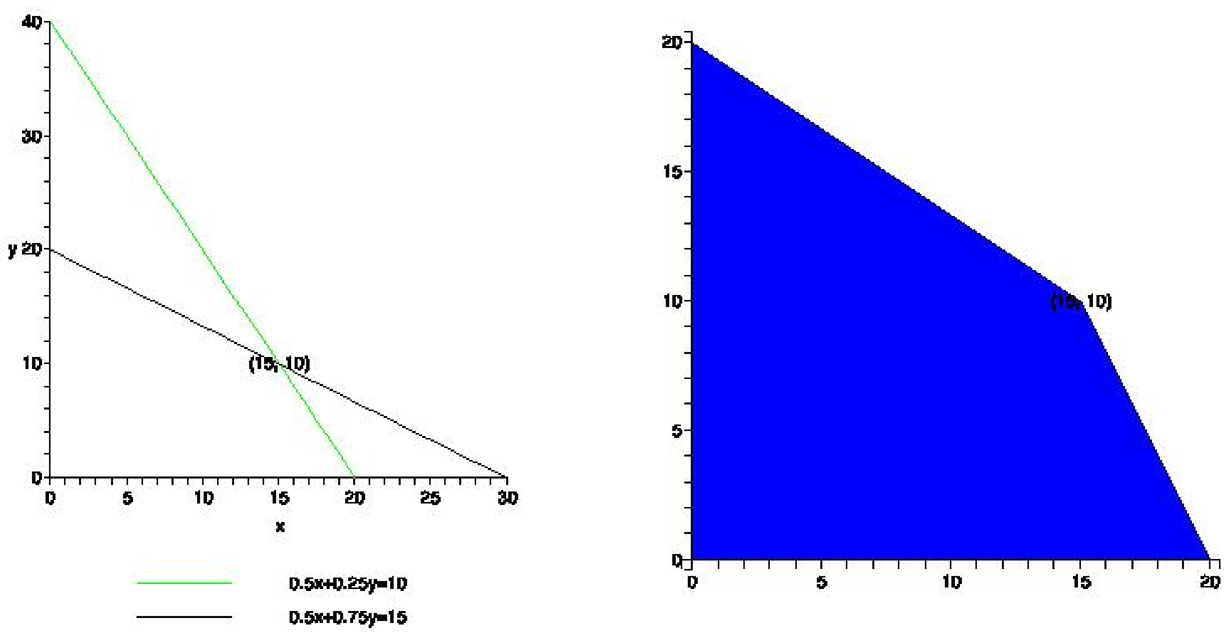

Figure 2: The feasible region for problem 2

Let x_1 and x_2 be the production of alloy A_1 and A_2 in tons per day. Then the daily profit is the objective function

$$f(x_1, x_2) = 30x_1 + 25x_2.$$

The constraints are as follows. Clearly $x_1 \geq 0$ and $x_2 \geq 0$. The x_1 tons of alloy A_1 use $0.5x_1$ tons of metal M_1 and the x_2 tons of alloy A_2 use $0.25x_2$ tons of metal M_1.

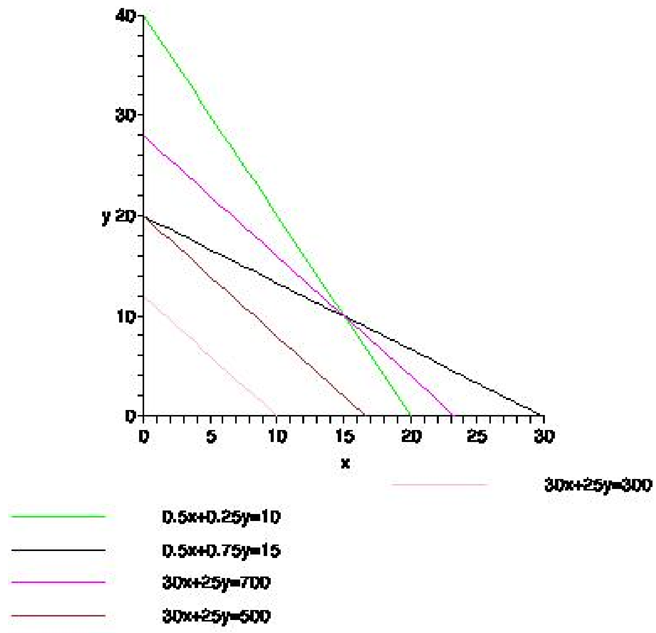

Figure 3: The family of lines $30x_1 + 25x_2 = k$ and the feasible region. The maximum of the objective function is achieved at $(15, 10)$

Since the amount used from metal M_1 cannot exceed 10 tons per day, we have

$$0.5x_1 + 0.25x_2 \leq 10$$

while, since the amount used from metal M_2 cannot exceed 15 tons per day, we have

$$0.5x_1 + 0.75x_2 \leq 15.$$

This is because x_1 tons of alloy A_1 use $0.5x_1$ tons of metal M_2 and x_2 tons of alloy A_2 use $0.75x_2$ tons of metal M_2. We plot the lines $0.5x_1+0.25x_2 = 10$, $0.5x_1+0.75x_2 = 15$. The two lines intersect at $(15, 10)$. You can solve the system to find this solution: subtract them to get $0.5x_2 = 5 \Longrightarrow x_2 = 10$, etc. We easily determine that $(0, 0)$ is in the feasible region, so the feasible region is the quadrangle in the figure.

We draw also the family of lines $30x_1+25x_2 = k$ for various k. We see that we increase the value of the objective function, as we move the lines in the direction of the vector $(30, 25)^t$, i.e. up and right. In this move the further point is $(15, 10)$ and this gives the maximum of the objective function: $f(15, 10) = 30 \cdot 15 + 25 \cdot 10 = 450 + 250 = 700$. We can also check that at the extreme points of the quadrangle (vertices) the values of the objective function are

$$f(0, 0) = 0, f(20, 0) = 30 \cdot 20 = 600, f(0, 20) = 25 \cdot 20 = 500, f(15, 10) = 700.$$

Matrix form: Set $c = (30, 25)^t$, $x = (x_1, x_2)^t$, $b = (10, 15)^t$,

$$A = \begin{pmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{pmatrix}.$$

The problem is now:

$$\begin{array}{rl} \text{maximize} & c^t \cdot x \\ \text{subject to} & x \geq 0, \quad Ax \leq b. \end{array}$$

For the canonical form we introduce slack variables x_3 and x_4, which are both ≥ 0 and such that

$$0.5x_1 + 0.25x_2 + x_3 = 10, \quad 0.5x_1 + 0.75x_2 + x_4 = 15.$$

Their meaning is as follows: we know that we cannot exceed 10 tons of metal M_1 per day. The slack variable x_3 represents the amount of metal M_1 not used per day. Similarly the slack variable x_4 represents the amount of metal M_2 not used per day. The problem becomes

$$\begin{array}{rc} \text{maximize} & 30x_1 + 25x_2 \\ \text{subject to} & x_1, x_2, x_3, x_4 \geq 0 \\ & 0.5x_1 + 0.25x_2 + x_3 = 10 \\ & 0.5x_1 + 0.75x_2 + x_4 = 15 \end{array}$$

We have now $x = (x_1, x_2, x_3, x_4)^t$, $c = (30, 25, 0, 0)^t$, $b = (10, 15)^t$ and

$$A = \begin{pmatrix} 0.5 & 0.25 & 1 & 0 \\ 0.5 & 0.75 & 0 & 1 \end{pmatrix}.$$

The problem is now:

$$\begin{array}{rl} \text{maximize} & c^t \cdot x \\ \text{subject to} & x \geq 0, \quad Ax = b. \end{array}$$

This is the canonical form ($m = 2$, $n = 4$).

The basic solutions in canonical form have at least $n - m = 2$ variables equal to 0. This gives 6 solutions:

$$(0, 0, 10, 15), (0, 40, 0, -15), (0, 20, 5, 0), (20, 0, 0, 5), (30, 0, -5, 0), (15, 10, 0, 0).$$

Among these we discard the two with negative entries, as they are not feasible. So the basic feasible solutions in canonical form are

$$(0, 0, 10, 15), (0, 20, 5, 0), (20, 0, 0, 5), (15, 10, 0, 0)$$

and in standard form (ignoring the slack variables)

$$(0, 0), (0, 20), (20, 0), (15, 10).$$

These are the vertices of the quadrangle.