Optimal Stochastic Control for Pairs Trading

Hui Gong, UCL
http://www.homepages.ucl.ac.uk/~ucahgon/

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Introduction

Pairs Trading is an investment strategy used by many Hedge Funds. Consider two co-integrated and correlated stocks which trade at some spread. For a given period, we need to maximise the agent's terminal utility of wealth subject to budget constraints.

Let S_1 and S_2 denote the co-integrated stock price which satisfies the stochastic differential equations

$$dS_1 = (\mu_1 + \delta z(t))\, S_1 dt + \sigma_1 S_1 dB_1 \ , \quad (1)$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 \left(\rho dB_1 + \sqrt{1-\rho^2} dB_2 \right) \ , \quad (2)$$

where B_1 and B_2 are independent Brownian Motions.

The instantaneous co-integrating vector $z(t)$ is defined by

$$z(t) = a + \ln S_1(t) + \beta \ln S_2(t) \ . \tag{3}$$

The dynamics of $z(t)$ is a stationary Ornstein-Uhlenbeck process

$$dz = \alpha\,(\eta - z)\,dt + \sigma_\beta dB_t \ ,$$

where $\alpha = -\delta$ is the speed of mean reversion,
$\sigma_\beta = \sqrt{\sigma_1^2 + \beta^2\sigma_2^2 + 2\beta\sigma_1\sigma_2\rho}$,
$B_t = \frac{\sigma_1+\beta\sigma_2\rho}{\sigma_\beta} B_1 + \beta\frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_\beta} B_2$ is a Brownian motion adapted to $\mathcal{F}_t$ and

$$\eta = -\frac{1}{\delta}\left(\mu_1 - \frac{\sigma_1^2}{2} + \beta\left(\mu_2 - \frac{\sigma_2^2}{2}\right)\right)$$

is the equilibrium level.

The dynamic of wealth value is given by

$$dW = \pi_1(t) dS_1 + \pi_2(t) dS_2 \ . \tag{4}$$

Substitute equation (1) and (2) into the value of wealth (4), then we obtain the SDE as below

$$\begin{aligned} dW &= \pi_1(t)\left(\mu_1 + \delta z(t)\right) S_1 dt + \pi_2(t)\mu_2 S_2 dt \\ &\quad + \pi_1(t)\sigma_1 S_1 dB_1 + \pi_2(t)\sigma_2 S_2 \left(\rho dB_1 + \sqrt{1-\rho^2} dB_2\right) \end{aligned} \tag{5}$$

A pair of controls (π_1, π_2) is said to be admissible if π_1 and π_2 are real-valued, progressively measurable, are such that, (1)(2)(5) define a unique solution (W, S_1, S_2) for any $t \in [0, T]$ and (π_1, π_2, S_1, S_2) satisfy the integrability condition

$$\mathbb{E} \int_t^T (\pi_1 S_1)^2 + (\pi_2 S_2)^2 ds < +\infty \ .$$

Assume that the agent's objective is

$$J(t, W, S_1, S_2) = \max_{(\pi_1,\pi_2)\in\mathcal{A}_t} \mathbb{E}\left[U\left(W_T^{t,W,S_1,S_2}\right)\right] \ , \tag{6}$$

where $J(t, W, S_1, S_2)$ denote the value function, the agent seeks an admissible control pair (π_1, π_2) that maximizes the utility of wealth at time T.

Specifying $U(W)$. Now let us assume the utility function like

$$U(W) = -\exp(-\gamma W) \ , \tag{7}$$

which is the CARA (Constant Absolute Risk Aversion) utility, where $\gamma > 0$ is constant and equal to the absolute risk aversion.

We expect the value function $J(t, W, S_1, S_2)$ satisfy the following HJB partial differential equation:

$$\begin{aligned} J_t + \max_{\pi_1,\pi_2}[\ & (\pi_1(\mu_1 + \delta z)S_1 + \pi_2\mu_2 S_2)\, J_W + (\mu_1 + \delta z)S_1 J_{S_1} + \mu_2 S_2 J_{S_2} \\ & + \pi_1\sigma_1^2 S_1^2 J_{WS_1} + \pi_2\rho\sigma_1\sigma_2 S_1 S_2 J_{WS_1} \\ & + \pi_2\sigma_2^2 S_2^2 J_{WS_2} + \pi_1\rho\sigma_1\sigma_2 S_1 S_2 J_{WS_2} \\ & + \frac{1}{2}\left(\pi_1^2\sigma_1^2 S_1^2 + \rho\pi_1\pi_2\sigma_1\sigma_2 S_1 S_2 + \pi_2^2\sigma_2^2 S_2^2\right) J_{WW} \\ & + \frac{1}{2}\sigma_1^2 S_1^2 J_{S_1 S_1} + \rho\sigma_1\sigma_2 S_1 S_2 J_{S_1 S_2} + \frac{1}{2}\sigma_2^2 S_2^2 J_{S_2 S_2}] = 0\ , \qquad (8) \end{aligned}$$

with the final condition

$$J(T, W, S_1, S_2) = U(W_T) = -\exp(-\gamma W_T)\ . \qquad (9)$$

Ansatz

$$S_1 = e^x \ , \qquad S_2 = e^y \ , \qquad J(t, W, S_1, S_2) = -e^{-\gamma W} g(t, x, y) \ .$$

Then the transformed HJB equation will be

$$\begin{aligned} g_t = \max[& \left(\pi_1(\mu_1 + \delta z)S_1 + \pi_2\mu_2 S_2\right)\gamma g - (\mu_1 + \delta z)g_x - \mu_2 g_y \\ & + \pi_1\sigma_1^2 S_1 \gamma g_x + \pi_2\rho\sigma_1\sigma_2 S_2\gamma g_x + \pi_2\sigma_2^2 S_2\gamma g_y + \pi_1\rho\sigma_1\sigma_2 S_1\gamma g_y \\ & - \frac{1}{2}\left(\pi_1^2\sigma_1^2 S_1^2 + 2\pi_1\pi_2\rho\sigma_1\sigma_2 S_1 S_2 + \pi_2^2\sigma_2^2 S_2^2\right)\gamma^2 g \\ & - \frac{1}{2}\sigma_1^2(g_{xx} - g_x) - \frac{1}{2}\sigma_2^2(g_{yy} - g_y) - \rho\sigma_1\sigma_2 g_{xy}] \ , \end{aligned} \tag{10}$$

subject to

$$g(T, x, y) = 1 \ . \tag{11}$$

Optimal Control Weights (initial)

$$\pi_1^* = \frac{(\mu_1 + \delta z)}{\gamma(1-\rho^2)\sigma_1^2 S_1} + \frac{g_x}{\gamma g S_1} - \rho \frac{\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2 S_1}, \quad (12)$$

$$\pi_2^* = \frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2 S_2} + \frac{g_y}{\gamma g S_2} - \rho \frac{(\mu_1 + \delta z)}{\gamma(1-\rho^2)\sigma_1\sigma_2 S_2}. \quad (13)$$

Particular Case $\delta = 0$

If we set $\delta = 0$, the value function is independent of the stocks and the amount invested in each stock are constant. Then the closed form for the value function and the amount corresponding to $\delta = 0$ will be

$$
\begin{aligned}
g(t,x,y) &= \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\mu_1^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}-2\frac{\rho\mu_1\mu_2}{\sigma_1\sigma_2}\right)(T-t)\right], \\
\pi_1^* S_1 &= \frac{\mu_1}{\gamma(1-\rho^2)\sigma_1^2}-\frac{\rho\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2}, \\
\pi_2^* S_2 &= \frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2}-\frac{\rho\mu_1}{\gamma(1-\rho^2)\sigma_1\sigma_2}.
\end{aligned}
$$

Reduce the HJB equation into a one-dimensional equation by letting

$$X = \mu_1 + \delta z = \mu_1 + \delta(a + x + \beta y) \ ,$$

and by using the exponential change of variable

$$\Phi(t, X) = -\ln(g(t, x, y))$$

i.e. $g = \exp(-\Phi)$.Then we will obtain the linear parabolic PDE

$$\begin{aligned} \Phi_t \ = \ & -\frac{1}{1-\rho^2}\left(\frac{1}{2}(\frac{X^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2}) - \rho\frac{\mu_2 X}{\sigma_1\sigma_2}\right) + \frac{1}{2}(\sigma_1^2 + \beta\sigma_2^2)(\delta\Phi_X) \\ & -\frac{1}{2}(\sigma_1^2 + \beta^2\sigma_2^2 + 2\sigma_1\sigma_2\beta\rho)(\delta^2\Phi_{XX}) \ . \end{aligned} \qquad (14)$$

for any real number X and time $0 \leqslant t < T$ and is subject to the terminal condition

$$\Phi(T, X) = 0 \ . \qquad (15)$$

Assume $\Phi(t, X) = a(t)X^2 + b(t)X + c(t)$ is an explicit solution of the PDE in (14), where the coefficient a, b, c are given by

$$
\begin{aligned}
a(t) &= \frac{1}{2}\frac{(T-t)}{(1-\rho^2)\sigma_1^2} \,, \\
b(t) &= -\frac{1}{4}\frac{(\sigma_1^2+\beta\sigma_2^2)\delta}{(1-\rho^2)\sigma_1^2}(T-t)^2 - \frac{\rho\mu_2}{(1-\rho^2)\sigma_1\sigma_2}(T-t) \,, \\
c(t) &= \frac{1}{2}\frac{\mu_2^2}{(1-\rho^2)\sigma_2^2}(T-t) + \frac{1}{4}\frac{(\sigma_1^2+\beta\sigma_2^2+2\sigma_1\sigma_2\beta\rho)\delta^2}{(1-\rho^2)\sigma_1^2}(T-t)^2 \\
&\quad + \frac{1}{4}\frac{\mu_2(\sigma_1^2+\beta\sigma_2^2)\delta\rho}{(1-\rho^2)\sigma_1\sigma_2}(T-t)^2 + \frac{1}{24}\frac{(\sigma_1^2+\beta\sigma_2^2)^2\delta^2}{(1-\rho^2)\sigma_1^2}(T-t)^3 \,.
\end{aligned}
$$

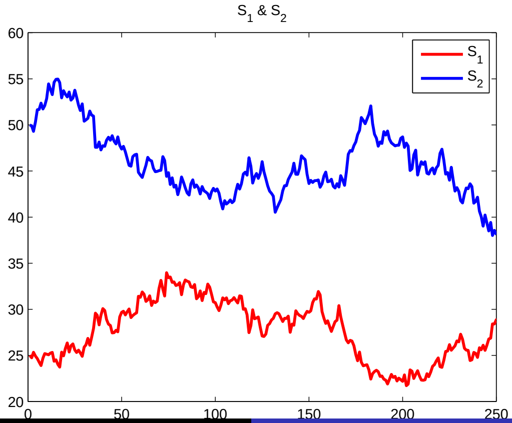
Optimal Control Pair (final)

$$\pi_1^* S_1 = \frac{(\mu_1 + \delta z)}{\gamma(1-\rho^2)\sigma_1^2} + \frac{\delta(-2a(t)(\mu_1 + \delta z) - b(t))}{\gamma} - \frac{\rho\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2}, \quad (16)$$

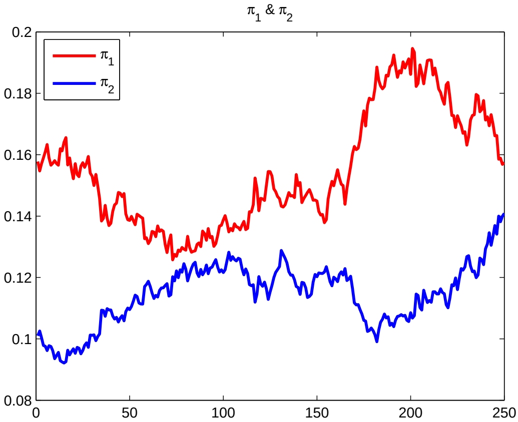
$$\pi_2^* S_2 = \frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2} + \frac{\delta\beta(-2a(t)(\mu_1 + \delta z) - b(t))}{\gamma} - \frac{\rho(\mu_1 + \delta z)}{\gamma(1-\rho^2)\sigma_1\sigma_2}. \quad (17)$$

The Dynamics of S_1 & S_2

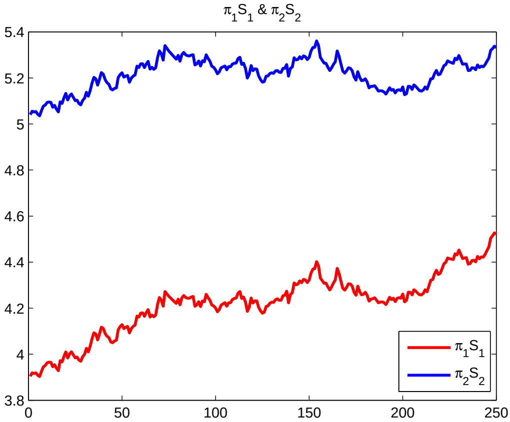
$$S_1^0 = 25, S_2^0 = 50, z(0) = 0.2, \mu_1 = 0.1, \mu_2 = 0.05$$
$$\sigma_1 = 0.4, \sigma_2 = 0.3, \delta = 0.2, \rho = -0.7, \beta = 0.5, \gamma = 0.5$$



The Dynamics of π_1 & π_2

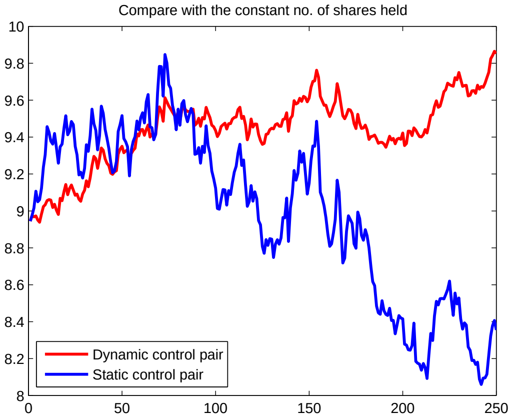


The Dynamics of $\pi_1 S_1$ & $\pi_2 S_2$



Comparison with the Static control pair

$\pi_1 = 0.1578$ & $\pi_2 = 0.1010$



Derive the explicit form for $a(t)$, $b(t)$ and $c(t)$ in Slides Page 14.