Optimal Stochastic Control for Pairs Trading

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Introduction

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**Pairs Trading** is an investment strategy used by many Hedge Funds. Consider two co-integrated and correlated stocks which trade at some spread. For a given period, we need to maximise the agent’s terminal utility of wealth subject to budget constraints.
Let $S_1$ and $S_2$ denote the co-integrated stock price which satisfies the stochastic differential equations

$$
\begin{align*}
    dS_1 &= (\mu_1 + \delta z(t)) S_1 dt + \sigma_1 S_1 dB_1, \\
    dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 \left( \rho dB_1 + \sqrt{1 - \rho^2} dB_2 \right),
\end{align*}
$$

where $B_1$ and $B_2$ are independent Brownian Motions.
The instantaneous co-integrating vector $z(t)$ is defined by

$$z(t) = a + \ln S_1(t) + \beta \ln S_2(t).$$  \hspace{1cm} (3)

The dynamics of $z(t)$ is a stationary Ornstein-Uhlenbeck process

$$dz = \alpha (\eta - z) dt + \sigma_{\beta} dB_t,$$

where $\alpha = -\delta$ is the speed of mean reversion,

$$\sigma_{\beta} = \sqrt{\sigma_1^2 + \beta^2 \sigma_2^2 + 2\beta \sigma_1 \sigma_2 \rho},$$

$$B_t = \frac{\sigma_1 + \beta \sigma_2 \rho}{\sigma_{\beta}} B_1 + \beta \frac{\sigma_2 \sqrt{1 - \rho^2}}{\sigma_{\beta}} B_2$$

is a Brownian motion adapted to $\mathcal{F}_t$ and

$$\eta = -\frac{1}{\delta} \left( \mu_1 - \frac{\sigma_1^2}{2} + \beta \left( \mu_2 - \frac{\sigma_2^2}{2} \right) \right)$$

is the equilibrium level.
The dynamic of wealth value is given by

\[ dW = \pi_1(t)dS_1 + \pi_2(t)dS_2. \] (4)

Substitute equation (1) and (2) into the value of wealth (4), then we obtain the SDE as below

\[ dW = \pi_1(t)(\mu_1 + \delta z(t))S_1 dt + \pi_2(t)\mu_2 S_2 dt \]
\[ + \pi_1(t)\sigma_1 S_1 dB_1 + \pi_2(t)\sigma_2 S_2 \left(\rho dB_1 + \sqrt{1 - \rho^2} dB_2\right). \] (5)
A pair of controls \((\pi_1, \pi_2)\) is said to be admissible if \(\pi_1\) and \(\pi_2\) are real-valued, progressively measurable, are such that, (1)(2)(5) define a unique solution \((W, S_1, S_2)\) for any \(t \in [0, T]\) and \((\pi_1, \pi_2, S_1, S_2)\) satisfy the integrability condition

\[
\mathbb{E} \int_t^T (\pi_1 S_1)^2 + (\pi_2 S_2)^2 \, ds < +\infty.
\]
Assume that the agent’s objective is

$$J(t, W, S_1, S_2) = \max_{(\pi_1, \pi_2) \in \mathcal{A}_t} \mathbb{E} \left[ U \left( \mathcal{W}_T^{t,W,S_1,S_2} \right) \right] ,$$

(6)

where $J(t, W, S_1, S_2)$ denote the value function, the agent seeks an admissible control pair $(\pi_1, \pi_2)$ that maximizes the utility of wealth at time $T$.

**Specifying $U(W)$.** Now let us assume the utility function like

$$U(W) = -\exp(-\gamma W) ,$$

(7)

which is the CARA (Constant Absolute Risk Aversion) utility, where $\gamma > 0$ is constant and equal to the absolute risk aversion.
We expect the value function $J(t, W, S_1, S_2)$ satisfy the following HJB partial differential equation:

$$
J_t + \max_{\pi_1, \pi_2} \left[ (\pi_1 (\mu_1 + \delta z) S_1 + \pi_2 \mu_2 S_2) J_W + (\mu_1 + \delta z) S_1 J_{S_1} + \mu_2 S_2 J_{S_2} \\
+ \pi_1 \sigma_1^2 S_1^2 J_{W S_1} + \pi_2 \rho \sigma_1 \sigma_2 S_1 S_2 J_{W S_1} \\
+ \pi_2 \sigma_2^2 S_2^2 J_{W S_2} + \pi_1 \rho \sigma_1 \sigma_2 S_1 S_2 J_{W S_2} \\
+ \frac{1}{2} \left( \pi_1^2 \sigma_1^2 S_1^2 + \rho \pi_1 \pi_2 \sigma_1 \sigma_2 S_1 S_2 + \pi_2^2 \sigma_2^2 S_2^2 \right) J_{WW} \\
+ \frac{1}{2} \sigma_1^2 S_1^2 J_{S_1 S_1} + \rho \sigma_1 \sigma_2 S_1 S_2 J_{S_1 S_2} + \frac{1}{2} \sigma_2^2 S_2^2 J_{S_2 S_2} \right] = 0 , \quad (8)
$$

with the final condition

$$
J(T, W, S_1, S_2) = U(W_T) = - \exp(-\gamma W_T) . \quad (9)
$$
Ansatz

\[ S_1 = e^x, \quad S_2 = e^y, \quad J(t, W, S_1, S_2) = -e^{-\gamma W}g(t, x, y). \]

Then the transformed HJB equation will be

\[
g_t = \max \left[ (\pi_1(\mu_1 + \delta z)S_1 + \pi_2\mu_2S_2)\gamma g - (\mu_1 + \delta z)g_x - \mu_2g_y \\
+ \pi_1\sigma_1^2S_1\gamma g_x + \pi_2\rho\sigma_1\sigma_2S_2\gamma g_x + \pi_2\sigma_2^2S_2\gamma g_y + \pi_1\rho\sigma_1\sigma_2S_1\gamma g_y \\
- \frac{1}{2} (\pi_1^2\sigma_1^2S_1^2 + 2\pi_1\pi_2\rho\sigma_1\sigma_2S_1S_2 + \pi_2^2\sigma_2^2S_2^2) \gamma^2 g \\
- \frac{1}{2} \sigma_1^2(g_{xx} - g_x) - \frac{1}{2} \sigma_2^2(g_{yy} - g_y) - \rho\sigma_1\sigma_2g_{xy} \right],
\]

subject to

\[
g(T, x, y) = 1.
\]
Optimal Control Weights (initial)

\[ \pi_1^* = \frac{(\mu_1 + \delta z)}{\gamma(1 - \rho^2)\sigma_1^2 S_1} + \frac{g_x}{\gamma gS_1} - \rho \frac{\mu_2}{\gamma(1 - \rho^2)\sigma_1\sigma_2 S_1}, \quad (12) \]

\[ \pi_2^* = \frac{\mu_2}{\gamma(1 - \rho^2)\sigma_2^2 S_2} + \frac{g_y}{\gamma gS_2} - \rho \frac{(\mu_1 + \delta z)}{\gamma(1 - \rho^2)\sigma_1\sigma_2 S_2}. \quad (13) \]
Particular Case $\delta = 0$

If we set $\delta = 0$, the value function is independent of the stocks and the amount invested in each stock are constant. Then the closed form for the value function and the amount corresponding to $\delta = 0$ will be

$$g(t, x, y) = \exp \left[ -\frac{1}{2(1 - \rho^2)} \left( \frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - 2\frac{\rho\mu_1\mu_2}{\sigma_1\sigma_2} \right) (T - t) \right],$$

$$\pi_1^* S_1 = \frac{\mu_1}{\gamma(1 - \rho^2)\sigma_1^2} - \frac{\rho\mu_2}{\gamma(1 - \rho^2)\sigma_1\sigma_2},$$

$$\pi_2^* S_2 = \frac{\mu_2}{\gamma(1 - \rho^2)\sigma_2^2} - \frac{\rho\mu_1}{\gamma(1 - \rho^2)\sigma_1\sigma_2}. $$
Reduce the HJB equation into a one-dimensional equation by letting

\[ X = \mu_1 + \delta z = \mu_1 + \delta(a + x + \beta y), \]

and by using the exponential change of variable

\[ \Phi(t, X) = -\ln(g(t, x, y)) \]

i.e. \( g = \exp(-\Phi). \) Then we will obtain the linear parabolic PDE

\[
\Phi_t = -\frac{1}{1 - \rho^2} \left( \frac{1}{2} \left( \frac{X^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} \right) - \rho \frac{\mu_2 X}{\sigma_1 \sigma_2} \right) + \frac{1}{2} (\sigma_1^2 + \beta \sigma_2^2)(\delta \Phi_X) \\
- \frac{1}{2} (\sigma_1^2 + \beta^2 \sigma_2^2 + 2 \sigma_1 \sigma_2 \beta \rho)(\delta^2 \Phi_{XX}). \tag{14}
\]

for any real number \( X \) and time \( 0 \leq t < T \) and is subject to the terminal condition

\[ \Phi(T, X) = 0. \tag{15} \]
Assume $\Phi(t, X) = a(t)X^2 + b(t)X + c(t)$ is an explicit solution of the PDE in (14), where the coefficient $a, b, c$ are given by

$$a(t) = \frac{1}{2} \frac{(T - t)}{(1 - \rho^2)\sigma_1^2},$$

$$b(t) = -\frac{1}{4} \frac{(\sigma_1^2 + \beta \sigma_2^2)\delta}{(1 - \rho^2)\sigma_1^2} (T - t)^2 - \frac{\rho \mu_2}{(1 - \rho^2)\sigma_1\sigma_2} (T - t),$$

$$c(t) = \frac{1}{2} \frac{\mu_2^2}{(1 - \rho^2)\sigma_2^2} (T - t) + \frac{1}{4} \frac{(\sigma_1^2 + \beta \sigma_2^2 + 2 \sigma_1 \sigma_2 \beta \rho)\delta^2}{(1 - \rho^2)\sigma_1^2} (T - t)^2$$

$$+ \frac{1}{4} \frac{\mu_2 (\sigma_1^2 + \beta \sigma_2^2)\delta \rho}{(1 - \rho^2)\sigma_1\sigma_2} (T - t)^2 + \frac{1}{24} \frac{(\sigma_1^2 + \beta \sigma_2^2)^2 \delta^2}{(1 - \rho^2)\sigma_1^2} (T - t)^3.$$
Optimal Control Pair (final)

\[
\pi_1^* S_1 = \frac{(\mu_1 + \delta z)}{\gamma(1 - \rho^2)\sigma_1^2} + \frac{\delta(-2a(t)(\mu_1 + \delta z) - b(t))}{\gamma} - \frac{\rho \mu_2}{\gamma(1 - \rho^2)\sigma_1 \sigma_2},
\]

\[
\pi_2^* S_2 = \frac{\mu_2}{\gamma(1 - \rho^2)\sigma_2^2} + \frac{\delta \beta(-2a(t)(\mu_1 + \delta z) - b(t))}{\gamma} - \frac{\rho (\mu_1 + \delta z)}{\gamma(1 - \rho^2)\sigma_1 \sigma_2}.
\]
The Dynamics of $S_1$ & $S_2$

$S_1^0 = 25, \ S_2^0 = 50, z(0) = 0.2, \mu_1 = 0.1, \mu_2 = 0.05$
$\sigma_1 = 0.4, \sigma_2 = 0.3, \delta = 0.2, \rho = -0.7, \beta = 0.5, \gamma = 0.5$
The Dynamics of $\pi_1$ & $\pi_2$
The Dynamics of $\pi_1 S_1$ & $\pi_2 S_2$
Comparison with the Static control pair

\[ \pi_1 = 0.1578 \quad \text{and} \quad \pi_2 = 0.1010 \]
Derive the explicit form for $a(t)$, $b(t)$ and $c(t)$ in Slides Page 14.