The Heston Model

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Introduction

Stochastic Volatility
Generalized SV models
The Heston Model
Vanilla Call Option via Heston

Monte Carlo simulation of Heston
Itô’s lemma for variance process
Euler-Maruyama scheme
Implement in Excel&VBA

Additional Exercise
Introduction

1. Why the Black-Scholes model is not popular in the industry?

2. What is the stochastic volatility models?
   Stochastic volatility models are those in which the variance of a stochastic process is itself randomly distributed.
A general expression for non-dividend stock with stochastic volatility is as below:

\[
\begin{align*}
    dS_t &= \mu_t S_t dt + \sqrt{v_t} S_t dW^1_t, \\
    dv_t &= \alpha(S_t, v_t, t) dt + \beta(S_t, v_t, t) dW^2_t,
\end{align*}
\]

(1)\hspace{1cm}(2)

with

\[dW^1_t dW^2_t = \rho dt,\]

where \(S_t\) denotes the stock price and \(v_t\) denotes its variance.

**Examples:**

- Heston model
- SABR volatility model
- GARCH model
- 3/2 model
- Chen model
The Heston model is a typical Stochastic Volatility model which takes $\alpha(S_t, v_t, t) = \kappa(\theta - v_t)$ and $\beta(S_t, v_t, t) = \sigma \sqrt{v_t}$, i.e.

\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t}, \\
    dv_t &= \kappa(\theta - v_t) dt + \sigma \sqrt{v_t} dW_{2,t},
\end{align*}

with

\begin{equation}
    dW_{1,t} dW_{2,t} = \rho dt,
\end{equation}

where $\theta$ is the long term mean of $v_t$, $\kappa$ denotes the speed of reversion and $\sigma$ is the volatility of volatility. The instantaneous variance $v_t$ here is a CIR process (square root process).
Let $x_t = \ln S_t$, the risk-neutral dynamics of Heston model is

$$
\begin{align*}
    dx_t &= \left(r - \frac{1}{2} v_t\right) dt + \sqrt{v_t} dW_{1,t}^*, \\
    dv_t &= \kappa^* (\theta^* - v_t) dt + \sigma \sqrt{v_t} dW_{2,t}^*,
\end{align*}
$$

(6) (7)

with

$$
dW_{1,t}^* dW_{2,t}^* = \rho dt .
$$

(8)

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \frac{\kappa \theta}{\kappa + \lambda}$.

Using these dynamics, the probability of the call option expires in-the-money, conditional on the log of the stock price, can be interpreted as risk-adjusted or risk-neutral probabilities. Hence,

$$
F_j(x, v, T; \ln K) = Pr(x(T) \geq \ln K | x_t = x, v_t = v).
$$
The price of vanilla call option is:

$$C(S, \nu, t) = SF_1 - e^{-r(T-t)}KF_2,$$

where $F_1$ and $F_2$ should satisfy the PDE (for $j = 1, 2$)

$$\frac{1}{2}\nu \frac{\partial^2 F_j}{\partial x^2} + \rho\sigma \nu \frac{\partial^2 F_j}{\partial x \partial \nu} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 F_j}{\partial \nu^2} + (r + u_j \nu) \frac{\partial F_j}{\partial x} + (a_j - b_j \nu) \frac{\partial F_j}{\partial \nu} + \frac{\partial F_j}{\partial t} = 0. \quad (10)$$

The parameter in Equation (10) is as follows

$$u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = \kappa \theta, \quad b_1 = \kappa + \lambda - \rho \sigma, \quad b_2 = \kappa + \lambda.$$
The simulated variance can be inspected to check whether it is negative \((v < 0)\). In this case, the variance can be set to zero \((v = 0)\), or its sign can be inverted so that \(v\) becomes \(-v\). Alternatively, the variance process can be modified in the same way as the stock process, by defining a process for natural log variances by using Itô’s lemma

\[
d\ln v_t = \frac{1}{v_t} \left( \kappa^*(\theta^* - v_t) - \frac{1}{2} \sigma^2 \right) dt + \sigma \frac{1}{\sqrt{v_t}} dW_{2,t}^* .
\]  

(11)
The Heston model can be discretized as following

\[
\ln S_{t+\Delta t} \quad = \quad \ln S_t + \left( r - \frac{1}{2} \nu_t \right) \Delta t + \sqrt{\nu_t} \sqrt{\Delta t} \epsilon_{S,t+1},
\]

\[
\ln \nu_{t+\Delta t} \quad = \quad \ln \nu_t + \frac{1}{\nu_t} \left( \kappa^* (\theta^* - \nu_t) - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \frac{1}{\sqrt{\nu_t}} \sqrt{\Delta t} \epsilon_{\nu,t+1}.
\]

Shocks to the volatility, \( \epsilon_{\nu,t+1} \), are correlated with the shocks to the stock price process, \( \epsilon_{S,t+1} \). This correlation is denoted \( \rho \), so that \( \rho = Corr(\epsilon_{S,t+1}, \epsilon_{\nu,t+1}) \) and the relationship between the shocks can be written as

\[
\epsilon_{\nu,t+1} = \rho \epsilon_{S,t+1} + \sqrt{1 - \rho^2} \epsilon_{t+1}
\]

where \( \epsilon_{t+1} \) are independently with \( \epsilon_{S,t+1} \).
The Heston Model

<table>
<thead>
<tr>
<th>Heston (1993) Call Price by Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot Price ($S$)</td>
</tr>
<tr>
<td>Strike Price ($K$)</td>
</tr>
<tr>
<td>Risk Free Rate ($r$)</td>
</tr>
<tr>
<td>Time to Maturity ($T - t$)</td>
</tr>
<tr>
<td>Rho ($\rho$)</td>
</tr>
<tr>
<td>Kappa ($\kappa$)</td>
</tr>
<tr>
<td>Theta ($\theta$)</td>
</tr>
<tr>
<td>Lambda ($\lambda$)</td>
</tr>
<tr>
<td>Volatility of Variance ($\sigma$)</td>
</tr>
<tr>
<td>Current variance ($v$)</td>
</tr>
<tr>
<td>Number of Simulations</td>
</tr>
</tbody>
</table>

| Heston Call Price                      | 1.3444 |

**Figure:** Heston (1993) Call Price by Monte Carlo
Option Base 1

' Heston Call Price by Monte Carlo Simulation

Function HestonMC(kappa, theta, lambda, rho, sigmav, daynum, startS, r, startv, K, ITER)
Dim allS() As Double, Stock() As Double

simPath = 0
ReDim allS(daynum) As Double, Stock(ITER) As Double
deltat = (1 / 365)

For itcount = 1 To ITER
    lnSt = Log(startS)
    lnvt = Log(startv)
    curv = startv
    curS = startS
    For daycnt = 1 To daynum
        e = Application.NormSInv(Rnd)
        eS = Application.NormSInv(Rnd)
        ev = rho * eS + Sqr(1 - rho ^ 2) * e
        'update the stock price
        lnSt = lnSt + (r - 0.5 * curv) * deltat + Sqr(curv) * Sqr(deltat) * eS
        curS = Exp(lnSt)
        lnvt = lnvt + (1 / curv) * ((kappa + lambda) * (kappa * theta / (kappa + lambda) - curv) -
            0.5 * sigmav ^ 2) * deltat + sigmav * (1 / Sqr(curv)) * Sqr(deltat) * ev
        curv = Exp(lnvt)
        allS(daycnt) = curS
    Next daycnt
    simPath = simPath + Exp((-daynum / 365) * r) * Application.Max(allS(daynum) - K, 0)
Next itcount
HestonMC = simPath / ITER
End Function

Figure: VBA code for Heston (1993) Call Price by Monte Carlo

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Additional Exercise

Use the Closed-Form Approach to implement Heston Call & Put.