The Heston Model

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Introduction

1. **Why the Black-Scholes model is not popular in the industry?**
2. **What is the stochastic volatility models?**
   Stochastic volatility models are those in which the variance of a stochastic process is itself randomly distributed.

A general expression for non-dividend stock with stochastic volatility is as below:

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dW_t^1 \ , \quad (1)$$

$$dv_t = \alpha(S_t, v_t, t)dt + \beta(S_t, v_t, t)dW_t^2 \ , \quad (2)$$

with

$$dW_t^1 dW_t^2 = \rho dt \ ,$$

where S_t denotes the stock price and v_t denotes its variance.

Examples:

- Heston model
- SABR volatility model
- GARCH model
- 3/2 model
- Chen model

The Heston model is a typical Stochastic Volatility model which takes $\alpha(S_t, v_t, t) = \kappa(\theta - v_t)$ and $\beta(S_t, v_t, t) = \sigma\sqrt{v_t}$, i.e.

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \ , \tag{3}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t} \ , \tag{4}$$

with

$$dW_{1,t} dW_{2,t} = \rho dt \ , \tag{5}$$

where θ is the long term mean of v_t, κ denotes the speed of reversion and σ is the volatility of volatility.
The instantaneous variance v_t here is a CIR process (square root process).

Let $x_t = \ln S_t$, the risk-neutral dynamics of Heston model is

$$dx_t = \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW^*_{1,t} \ , \quad (6)$$

$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}dW^*_{2,t} \ , \quad (7)$$

with

$$dW^*_{1,t}dW^*_{2,t} = \rho dt \ . \quad (8)$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \frac{\kappa\theta}{\kappa+\lambda}$.

Using these dynamics, the probability of the call option expires in-the-money, conditional on the log of the stock price, can be interpreted as risk-adjusted or risk-neutral probabilities.

Hence,

$$F_j(x, v, T; \ln K) = Pr(x(T) \geq \ln K | x_t = x, v_t = v) \ .$$

The price of vanilla call option is:

$$C(S,v,t) = SF_1 - e^{-r(T-t)}KF_2 \ , \tag{9}$$

where F_1 and F_2 should satisfy the PDE (for $j = 1,2$)

$$\begin{aligned} &\frac{1}{2}v\frac{\partial^2 F_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 F_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F_j}{\partial v^2} \\ &+(r+u_j v)\frac{\partial F_j}{\partial x} + (a_j - b_j v)\frac{\partial F_j}{\partial v} + \frac{\partial F_j}{\partial t} = 0 \ . \end{aligned} \tag{10}$$

The parameter in Equation (10) is as follows

$$u_1 = \frac{1}{2}, \qquad u_2 = -\frac{1}{2}, \qquad a = \kappa\theta, \qquad b_1 = \kappa+\lambda-\rho\sigma, \qquad b_2 = \kappa+\lambda.$$

The simulated variance can be inspected to check whether it is negative ($v < 0$). In this case, the variance can be set to zero ($v = 0$), or its sign can be inverted so that v becomes $-v$. Alternatively, the variance process can be modified in the same way as the stock process, by defining a process for natural log variances by using Itô's lemma

$$d \ln v_t = \frac{1}{v_t}\left(\kappa^*(\theta^* - v_t) - \frac{1}{2}\sigma^2\right) dt + \sigma \frac{1}{\sqrt{v_t}} dW^*_{2,t} \ . \qquad (11)$$

The Heston model can be discretized as following

$$\begin{aligned}
\ln S_{t+\Delta t} &= \ln S_t + \left(r - \frac{1}{2} v_t \right) \Delta t + \sqrt{v_t}\sqrt{\Delta t}\epsilon_{S,t+1} \ , \\
\ln v_{t+\Delta t} &= \ln v_t + \frac{1}{v_t} \left(\kappa^*(\theta^* - v_t) - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma \frac{1}{\sqrt{v_t}} \sqrt{\Delta t}\epsilon_{v,t+1} \ .
\end{aligned}$$

Shocks to the volatility, $\epsilon_{v,t+1}$, are correlated with the shocks to the stock price process, $\epsilon_{S,t+1}$. This correlation is denoted ρ, so that $\rho = Corr(\epsilon_{S,t+1}, \epsilon_{v,t+1})$ and the relationship between the shocks can be written as

$$\epsilon_{v,t+1} = \rho\epsilon_{S,t+1} + \sqrt{1-\rho^2}\epsilon_{t+1}$$

where ϵ_{t+1} are independently with $\epsilon_{S,t+1}$.

Heston (1993) Call Price by Monte Carlo	
Spot Price (S)	100
Strike Price (K)	100
Risk Free Rate (r)	0.05
Time to Maturity ($T-t$)	30
Rho (ρ)	-0.7
Kappa (κ)	2
Theta (θ)	0.01
Lambda (λ)	0.05
Volatility of Variance (σ)	0.1
Current variance (v)	0.01
Number of Simulations	5,000
Heston Call Price	1.3444

Figure: Heston (1993) Call Price by Monte Carlo

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Option Base 1

' Heston Call Price by Monte Carlo Simulation

Function HestonMC(kappa, theta, lambda, rho, sigmav, daynum, startS, r, startv, K, ITER)
Dim allS() As Double, Stock() As Double

simPath = 0
ReDim allS(daynum) As Double, Stock(ITER) As Double
deltat = (1 / 365)

For itcount = 1 To ITER
    lnSt = Log(startS)
    lnvt = Log(startv)
    curv = startv
    curS = startS
        For daycnt = 1 To daynum
            e = Application.NormSInv(Rnd)
            eS = Application.NormSInv(Rnd)
            ev = rho * eS + Sqr(1 - rho ^ 2) * e
            'update the stock price
            lnSt = lnSt + (r - 0.5 * curv) * deltat + Sqr(curv) * Sqr(deltat) * eS
            curS = Exp(lnSt)
            lnvt = lnvt + (1 / curv) * ((kappa + lambda) * (kappa * theta / (kappa + lambda) - curv) _
                   - 0.5 * sigmav ^ 2) * deltat + sigmav * (1 / Sqr(curv)) * Sqr(deltat) * ev
            curv = Exp(lnvt)
            allS(daycnt) = curS
        Next daycnt
    simPath = simPath + Exp((-daynum / 365) * r) * Application.Max(allS(daynum) - K, 0)
Next itcount
  HestonMC = simPath / ITER
End Function
```

Figure: VBA code for Heston (1993) Call Price by Monte Carlo

Additional Exercise

Use the Closed-Form Approach to implement Heston Call & Put.