Tree-Based Methods

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Outline

- What is the tree-based methods?
- Why we need to use it?
- Examples
  - One-step Binomial Tree
  - Two-step Binomial Tree
- CRR Binomial Tree
  - Stock Price Movement in the Binomial Tree
  - Comparison CRR Binomial Price with BS Price
  - CRR Binomial Tree Price for an American or European Option

Tree-Based Methods

- Main idea
  - Trees make this possible by mapping out price movements of the underlying security.
  - These price movements are represented by a grid of equally spaced time steps, with a series of nodes at each step indicating the price of the security and of the option.
  - At each node, the security moves up or down by a certain amount, according to a pre-specified probability.
  - The price of the option is evaluated at each node, and then discounted back to obtain the price at the first node, representing time zero.

The reason why we use tree

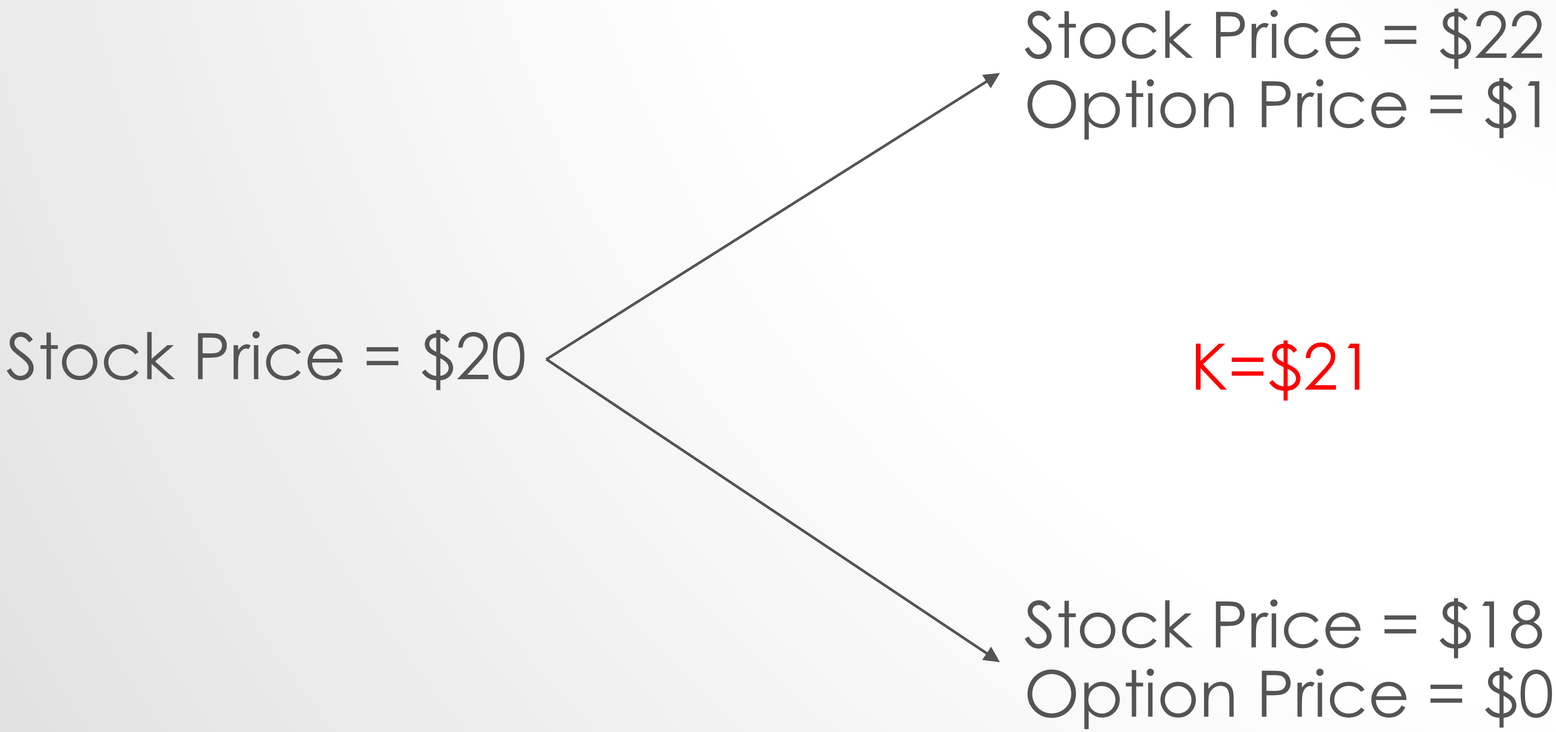
- Tree-based methods can be used for obtaining option prices, which are especially popular for pricing American options since many closed-form formulas currently available are for European options only.

- Binomial and trinomial trees can be used to price many options, including plain vanilla options, but also exotic options such as barrier options, digital options, Asian options, and others.

- For trees, the price of a European option converges to the Black-Scholes price. Valuation of American options is done by assessing whether early exercise is profitable at each node in the tree.

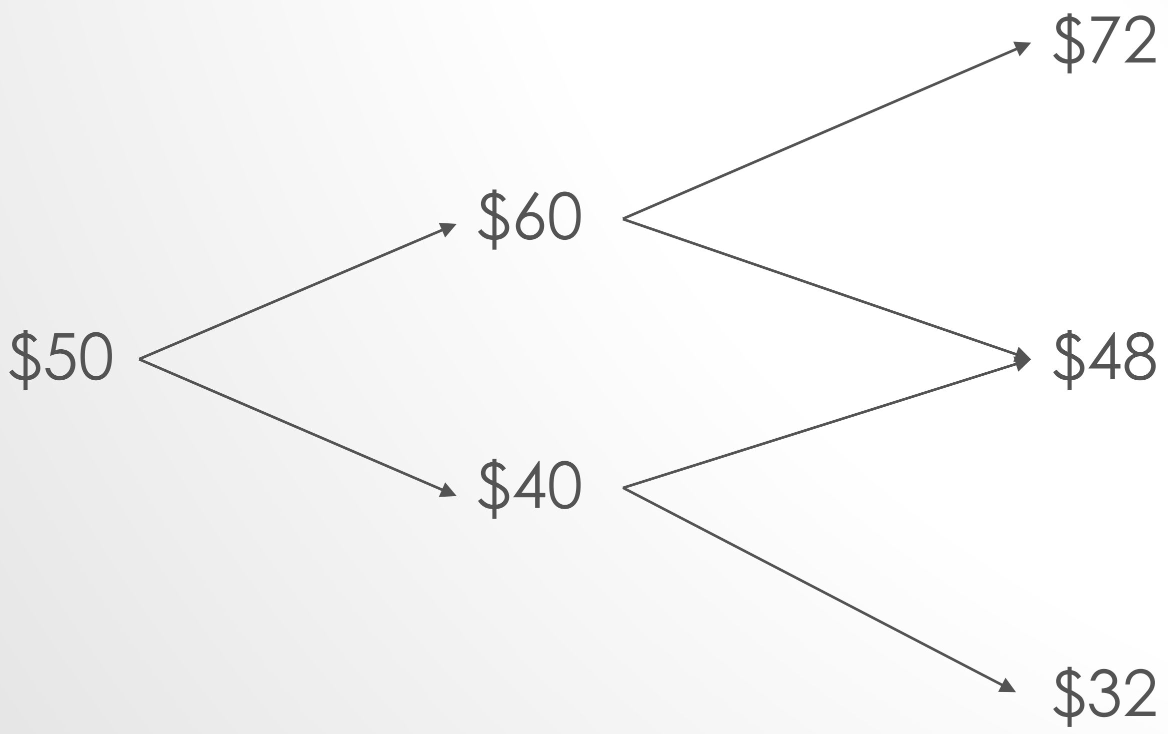
- The advantage of binomial and trinomial trees is that not only they can be used to value just about any type of option, but they are very easy to implement.

Example I (One step Euro Call)



u = 1.1, d=0.9, r = 0.12, T = 0.25, fu = 1, and fd = 0.

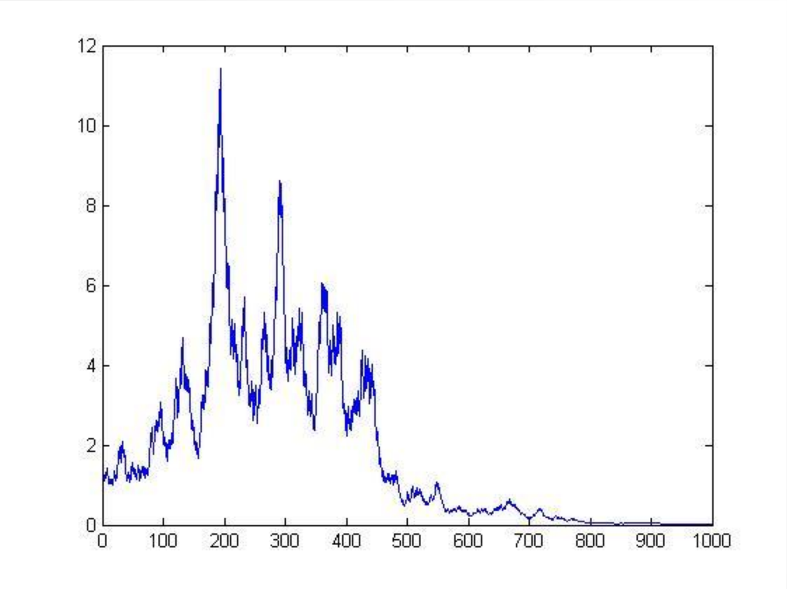
Example II (Two step Euro/Amer Put)

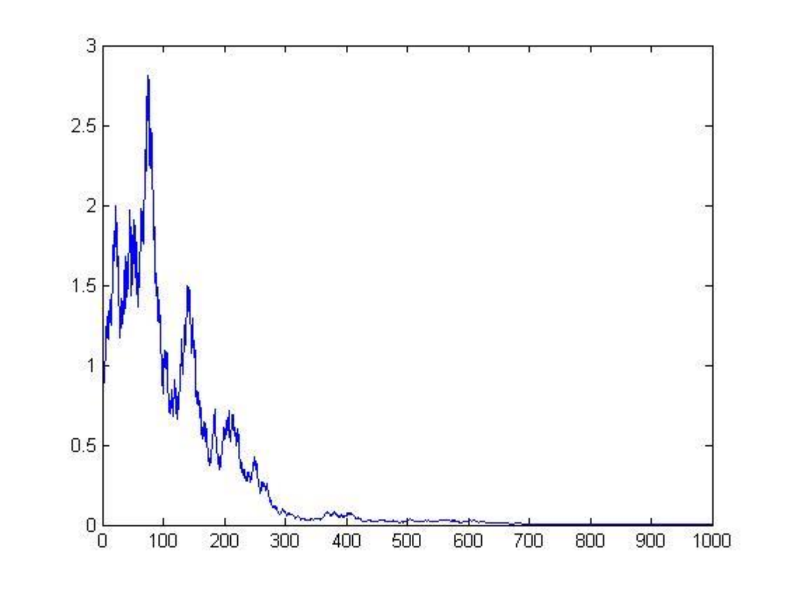


- What is the difference between Euro and Amer Put at node $40?

What is the problem when we use this style binomial tree?

- If it becomes N steps, what will happen when N is very large?





Cox, Ross & Rubinstein (CRR)

- CRR Binomial Tree
  - Suppose an option with maturity *T* and strike *K* is to be priced, using a binomial tree with *n* time increments on a stock with spot price *S* with volatility *σ* when the risk free rate is *r*.
  - The stock moves up in increments of $u = exp(\sigma\sqrt{dt})$
  - The stock moves down in increments of $d = \frac{1}{u}$
  - Each time step of length is $dt = \frac{T}{n}$
  - The probability of an up move is $p = \frac{exp(r \times dt) - d}{u - d}$
  - The probability of a down move is $1 - p$

The stock Price movements in the Binomial Tree

	A	B	C	D	E
1	0	1×dt	2×dt	3×dt	T = 4×dt
2					
3					Su^4
4				Su^3	
5			Su^2		Su^3d
6		Su		Su^2d	
7	S		S		S
8		Sd		Sud^2	
9			Sd^2		Sud^3
10				Sd^3	
11					Sd^4
12					
13					
14	S(1,1) = S	S(1,2) = Su	S(1,3) = Su^2	S(1,4) = Su^3	S(1,5) = Su^4
15		S(2,2) = Sd	S(2,3) = S	S(2,4) = Su^2d	S(2,5) = Su^3d
16			S(3,3) = Sd^2	S(3,4) = Sud^2	S(3,5) = S
17				S(4,4) = Sd^3	S(4,5) = Sud^3
18					S(5,5) = Sd^4

European Option using CRR

- Euro Call

$$Call = exp(-rT)\sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}max\left(Su^id^{n-i}-K,0\right)$$

where

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

This formula represents the expected value of the option at the final time step, discounted by the risk-free rate.

- Euro Put

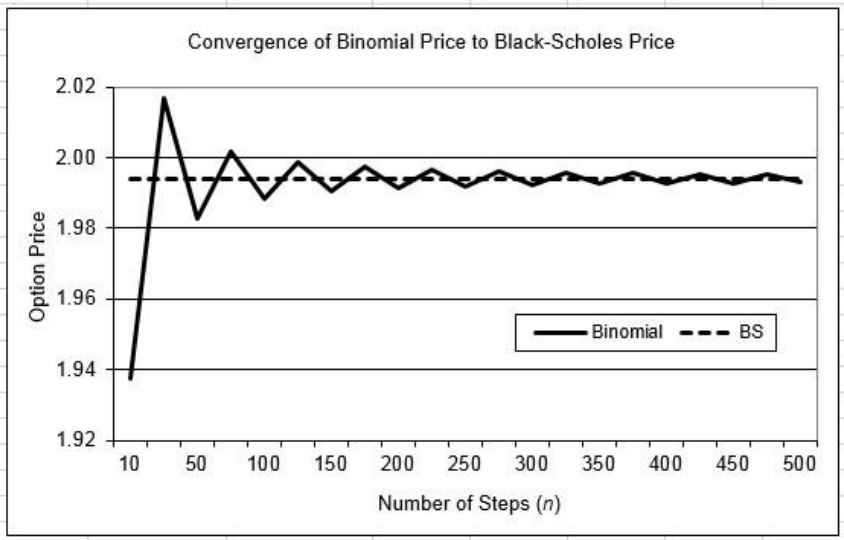
To price a Euro Put, replace $Su^id^{n-i} - K$ by $K - Su^id^{n-i}$

European Option using CRR (cont)

Stock Price (S)	30
Strike Price (K)	30
Years to Maturity (T)	0.4167
Interest Rate (r)	0.05
Volatility (v)	0.3
Steps (n)	20
Type (*Call* or *Put*)	Put

Binomial	1.96551
Black-Scholes	1.99411

# Steps	Binomial	BS
10	1.9373	1.9941
25	2.0166	1.9941
50	1.9826	1.9941
75	2.0016	1.9941
100	1.9884	1.9941
125	1.9986	1.9941
150	1.9903	1.9941
175	1.9973	1.9941
200	1.9912	1.9941
225	1.9966	1.9941
250	1.9918	1.9941
275	1.9961	1.9941
300	1.9922	1.9941
325	1.9958	1.9941
350	1.9925	1.9941
375	1.9956	1.9941
400	1.9927	1.9941
425	1.9954	1.9941
450	1.9928	1.9941
475	1.9953	1.9941
500	1.9930	1.9941



European Option using CRR (cont)

```
Function BlackScholes(S, K, r, T, v, PutCall As String)
    d1 = (Log(S / K) + r * T) / (v * Sqr(T)) + 0.5 * v * Sqr(T)
    CallPrice = S * Application.NormSDist(d1) - K * Exp(-r * T) * Application.NormSDist(d1 - v * Sqr(T))
    Select Case PutCall
        Case "Call": BlackScholes = CallPrice
        Case "Put": BlackScholes = CallPrice + K * Exp(-r * T) - S
    End Select
End Function
```

```
Function EuroBin(S, K, T, rf, sigma, n, PutCall As String)
    dt = T / n
    u = Exp(sigma * (dt ^ 0.5))
    d = 1 / u
    p = (Exp(rf * dt) - d) / (u - d)
    EuroBin = 0
    For i = 0 To n
      Select Case PutCall
        Case "Call": EuroBin = EuroBin + Application.Combin(n, i) * p ^ i * (1 - p) ^ (n - i) * Application.Max(S * u ^ i * d ^ (n - i) - K, 0)
        Case "Put": EuroBin = EuroBin + Application.Combin(n, i) * p ^ i * (1 - p) ^ (n - i) * Application.Max(K - S * u ^ i * d ^ (n - i), 0)
      End Select
    Next i
    EuroBin = Exp(-rf * T) * EuroBin
End Function
```

American Option using CRR

- Amer Call at node (i, j) is given by

$$\max\{S_{i,j} - K, e^{-rdt}[pS_{i,j+1} + (1-p)S_{i+1,j+1}]\}$$

  It's **NEVER** logical to exercise an American call option early (non-dividend stock). Why?

- Amer Put
  To price a Amer Put, replace $S_{i,j} - K$ by $K - S_{i,j}$

American Option using CRR (cont)

	A	B	C	D
1	Asset price		S	50
2	Exercise price		K	40
3	Interest rate		r	0.05
4	Volatility		sigma	0.3
5	Time to maturity		T	2
6	Number of time periods		N	100
7				
8	**American Put**			2.470282188

American Option using CRR (cont)

- Part of VBA code for Amer Put.

```
For j = 1 To N
    S = S * u2
    PutV(j) = Application.Max(K - S, 0)
Next j
For i = N - 1 To 0 Step -1                      ' back up in time to date 0
    S = S0 * d ^ i                              ' stock price at bottom node at date i
    PutV(0) = Application.Max(K - S, dpd * PutV(0) + dpu * PutV(1))
    For j = 1 To i                              ' step up over nodes at date i
        S = S * u2
        PutV(j) = Application.Max(K - S, dpd * PutV(j) + dpu * PutV(j + 1))
    Next j
Next i
American_Put_Binomial = PutV(0)                 ' put value at bottom node at date 0
End Function
```

Extras

- Develop a Pricing and risk management sheet for Euro/Amer Option by using CRR Tree.

	A	B	C	D	H	I	J
1							
2		**CRR Binomial Tree Price for an American or European Option**					
3							
4							
5		Spot Price (*S*)	30				
6		Strike Price (*K*)	30				
7		Years to Maturity (*T*)	0.4167				
8		Interest Rate (*rf*)	0.05				
9		Volatility (*v*)	0.3				
10		Steps (*n*)	100				
11		Type (*Call* or *Put*)	Put				
12		Option (*Amer* or *Euro*)	Amer				
13							
14							
15		**Option Price**	2.0462				
16		**Delta (Δ)**	-0.4358				
17		**Gamma (Γ)**	0.0723				
18		**Theta (Θ)**	-2.1702				
19		**Vega (V)**	7.5356				
20		**Rho (ρ)**	-4.6991				
21							