## HOMOGENEOUS DYNAMICS

Exercises marked in one $*$ or $**$ are more difficult.

### 1. Doubling map

Recall that the doubling map is the map defined by

$$\begin{array}{rcccc} m & := & S^1 & \longrightarrow & S^1 \\ & & x & \longmapsto & 2x \end{array}$$

where $S^1 = \mathbb{R}/\mathbb{Z}$.

**Exercise 1.** *Show that there exists a point $x \in S^1$ such that $\{m^n(x) \mid n \in \mathbb{Z}\}$ is dense.*

**Exercise 2.** *Show that there exists a point $x \in S^1$ such that $\mathcal{O}(x) := \{m^n(x) \mid n \in \mathbb{Z}\}$ is a Cantor set.*

**Exercise* 3.** *Show that there exists a point $x \in S^1$ such that $\{m^n(x) \mid n \in \mathbb{Z}\}$ is a Cantor set of arbitrary Hausdorff dimension.*

**Exercise 4.** *Show that the Lebesgue measure on $S^1$ is invariant under $m$ (recall that a measure is invariant under $m$ if and only if for any measurable set $A$, $\mu(m^{-1}(A)) = \mu(A)$.*

**Exercise* 5.** *Assuming that the Lebesgue measure is ergodic for $m$, prove that almost every $x$ is such that $\mathcal{O}(x)$ is dense.*

**Exercise* 6.** *Show that the Lebesgue measure is ergodic for $m$.*

### 2. Rotations

Recall that the rotation of angle $\alpha$ is the map defined by

$$\begin{array}{rcccc} R_\alpha & := & S^1 & \longrightarrow & S^1 \\ & & x & \longmapsto & x+\alpha \end{array}$$

where $S^1 = \mathbb{R}/\mathbb{Z}$.

**Exercise 7.** *Show that if $\alpha \notin \mathbb{Q}$, then every orbit under $R_\alpha$ is dense.*

**Exercise 8.** *Show that the Lebesgue measure is invariant under $R_\alpha$. Show that if $\alpha \notin \mathbb{Q}$, it is ergodic.*

**Exercise* 9.** *Show that if $\alpha \notin \mathbb{Q}$, the Lebesgue measure is the only invariant measure.*

### 3. Skew-products

For these exercises we consider the map defined by

$$\begin{array}{rcccc} T & := & S^1 \times S^1 & \longrightarrow & S^1 \times S^1 \\ & & (x,y) & \longmapsto & (x+\alpha, y+x) \end{array}.$$

**Exercise 10.** *Show that the Lebesgue measure is invariant under $T$.*

**Exercise 11.** *Show that the Lebesgue measure is ergodic for $T$ if $\alpha \notin \mathbb{Q}$ (hint: use Fourier series and Birkhoff's ergodic theorem).*

**Exercise* 12.** *Assuming the only invariant measure of $T$ is the Lebesgue measure when $\alpha \notin \mathbb{Q}$, show that*

$$\frac{1}{n}\sum_{k=1} \cos(2\pi k^2 \alpha) = 0.$$

**Exercise** 13.** *Show that the only invariant measure of $T$ is the Lebesgue measure.*

## 4. Horocycle flow

In this series of exercises, $G = \mathrm{SL}(2,\mathbb{R})$ and $h_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

**Exercise 14.** *Show that the Haar measure of $G$ is invariant under the action on the left of $(h_s)$ on $G/\Gamma$.*

**Exercise 15.** *Show that $(h_s)$ has at least one periodic orbit.*

We recall the statement of Ratner's theorem.

**Theorem 1.** *Let $G$ be a finite dimension Lie group, $\Gamma$ a lattice and $H$ a 1-parameter subgroup of $\Gamma$. Then for every $x \in G/\Gamma$, the closure of $H \cdot x$ in $G/\Gamma$ is homogeneous; meaning that there exists $L$ a sub-Lie group of $G$ such that*

$$\overline{H \cdot x} = S \cdot x.$$

*Moreover, $\Gamma \cap S$ is a lattice in $S$.*

**Exercise* 16.** *Using Ratner's theorem, show that any orbit of $(h_s)$ on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ is either closed or dense.*