Chern classes

Characteristic classes

- Measure how “twisted” a vector bundle is
- For real vector bundles we have **Stiefel-Whitney classes** and **Pontryagin classes**
- For complex vector bundles we have **Chern classes**
- Versions in topology, differential geometry, algebraic geometry, sheaf theory, number theory...
- Important to understand links between different versions

Vector bundles

A rank r complex vector bundle $E \to X$ over a topological space X is a family of vector spaces $\cong \mathbb{C}^r$ "continuously varying" over X.

- Topological space E with a map $\pi\colon E \to X$
- **(Locally trivial)** Every $x \in X$ has an open neighbourhood $U \subset X$ over which E is the "trivial" or product bundle: there exists an isomorphism $E|_U := \pi^{-1}(U) \xrightarrow[\sim]{g_U} U \times \mathbb{C}^r$
- **Linear structure on fibres preserved by changes in local trivialisations.** On overlaps the trivialisations g_U differ by *linear maps*; i.e.

$$g_{UV} \;:=\; g_U|_{U\cap V} \circ g_V^{-1}|_{U\cap V}$$

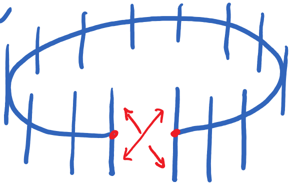
is multiplication by a map (**transition function**)

$$g_{UV}\colon U \cap V \to GL(r, \mathbb{C}).$$

Möbius band

Example of a **real** bundle over circle S^1.

$[0,1]\times\mathbb{R}/\sim$



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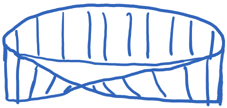
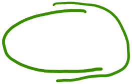


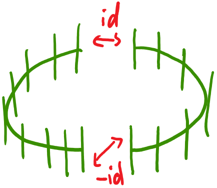
Glue by $(0,e)\sim(1,-e)$ $\forall e\in\mathbb{R}$

Möbius band

Ex check locally trivial over open cover



of S^1.



Ex: If put in two Möbius twists show resulting bundle is trivial. Further, show it can be untwisted if embedded in $\mathbb{R}^4$.

Exercises

So we may think of E as $\coprod_U (U \times \mathbb{C}^r)/\sim$, where we glue by the transition functions g_{UV}:

$$V \times \mathbb{C}^r \;\ni\; \; (x, e) \sim \big(x, g_{UV}(x)e\big) \; \;\in\; U \times \mathbb{C}^r.$$

Ex: Check defines an equivalence relation and quotient is E.

Ex: Show fibres are naturally vector spaces: if (x, e_1), (x, e_2) are points of the same fibre $E_x := \pi^{-1}(x)$ and $\alpha, \beta \in \mathbb{C}$ we can define $\alpha x_1 + \beta x_2 \in E_x$ and $0 \in E_x$ such that....

Ex: Define smooth vector bundle over a smooth manifold, algebraic bundle over an algebraic variety, real vector bundle, etc.

Sections

A section s of $\pi \colon E \to X$ is a continuous map $s \colon X \to E$ such that $\pi \circ s = \mathrm{id}_X$.

Can add, subtract, multiply by scalars. So get a vector space of sections $\Gamma(E)$.

Ex: A trivialisation of the bundle, i.e. an isomorphism

$$\begin{tikzcd} E \arrow[r, "\sim"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & X \times \mathbb{C}^r \arrow[d] \\ X \arrow[r, equal] & X, \end{tikzcd}$$

is the same thing as a choice of r sections $s_1, \ldots, s_r$ which form a **basis** at every point.
(I.e. $s_1(x), \ldots, s_r(x)$ is a basis of E_x at every $x \in X$.)

So a trivialisation of a line bundle $\iff$ a nowhere-zero section.

Homotopy invariance

Fact 1: Homotopic bundles are isomorphic.
Given $E \to X \times [0,1]$, let $E_t := E|_{X\times\{t\}}$. Then $E_0 \cong E_1$.

Fact 2: Bundles on contractible spaces X are trivial.
$X \simeq \{*\} \implies (E \to X) \cong (X \times \mathbb{C}^r)$.

Proofs using Tietze extension theorem; see e.g. Atiyah's *K-theory*.

So given a rank r bundle $E \to S^n$, we know that restricted to either hemisphere, it is trivial,

$$S^n = B_1^n \cup B_2^n, \qquad E|_{B_i^n} \cong B_i^n \times \mathbb{C}^r.$$

Glued over boundary $\partial B_1^n \cong S^{n-1}$ by a map $S^{n-1} \to \mathrm{GL}(r, \mathbb{C})$.

(Should really take B_i^n open, overlapping in an "annulus" $S^{n-1} \times (-\epsilon, \epsilon)$.)

Clutching construction

So rank r complex bundles on S^n are in 1-1 correspondence with homotopy classes of maps $S^{n-1} \to \mathrm{GL}(r, \mathbb{C})$, i.e. with

$$\pi_{n-1}(\mathrm{GL}(r, \mathbb{C})).$$

E.g. real version with $r = 1$ gives

$$\{\text{line bundles on } S^1\} \longleftrightarrow \pi_0(\mathrm{GL}(1, \mathbb{R})) = \pi_0(\mathbb{R}^\times) = \mathbb{Z}/2.$$

(The mod 2 integer is called the first Stiefel-Whitney class of the bundle.)

First Chern class

E.g. complex version with $r = 1$ gives

$$\{\text{line bundles on } S^2\} \longleftrightarrow \pi_1(\mathrm{GL}(1,\mathbb{C})) = \pi_1(\mathbb{C}^\times) = \mathbb{Z}.$$

This integer classifying the bundle is called its first Chern class, c_1.

Algebraic version: write

$$S^2 \cong \mathbb{P}^1 = \mathbb{C}_x \cup_{\mathbb{C}^\times} \mathbb{C}_y$$

glued over $\mathbb{C}^\times = \{x \neq 0\} = \{y \neq 0\}$ by $x = \frac{1}{y}$.

Then glue trivial bundles $\mathbb{C}_x \times \mathbb{C}$ to $\mathbb{C}_y \times \mathbb{C}$ by

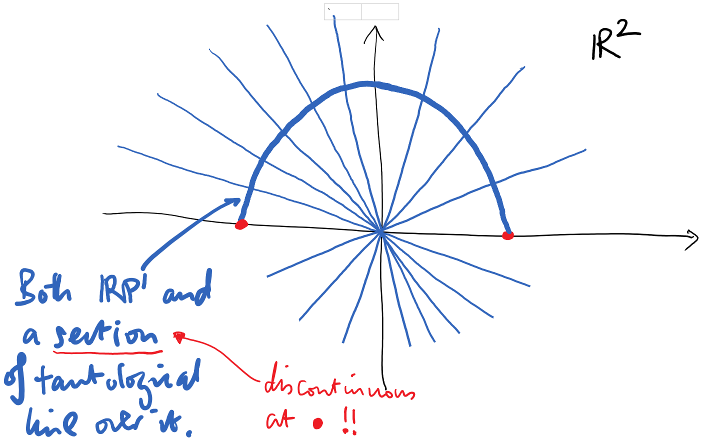
$$(x, t) \longmapsto \left(\frac{1}{x}, x^{-n} t\right) = (y, y^n t).$$

We call the resulting line bundle $\mathcal{O}(n)$ with $c_1 = n$.

Tautological bundle

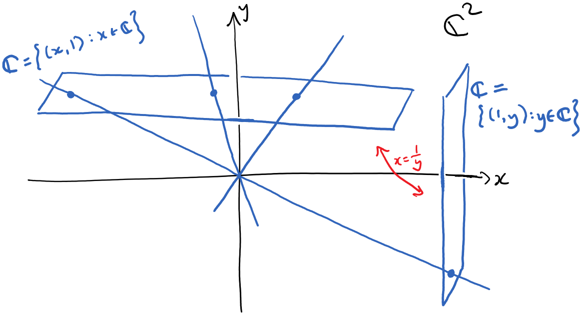
When $n = -1$ we get the **tautological bundle** $\mathcal{O}(-1) \to \mathbb{P}^1$.

Over $\mathbb{R}$ this is the Möbius bundle on $\mathbb{RP}^1 \cong S^1$:



Tautological bundle over $\mathbb{C}$

Over $\mathbb{C}$ we also see that $\mathcal{O}(-1)$ (defined as above with transition function $\frac{1}{x}$) is the tautological bundle $\mathcal{O}(-1) \hookrightarrow \underline{\mathbb{C}}^2$ over $\mathbb{P}^1$.



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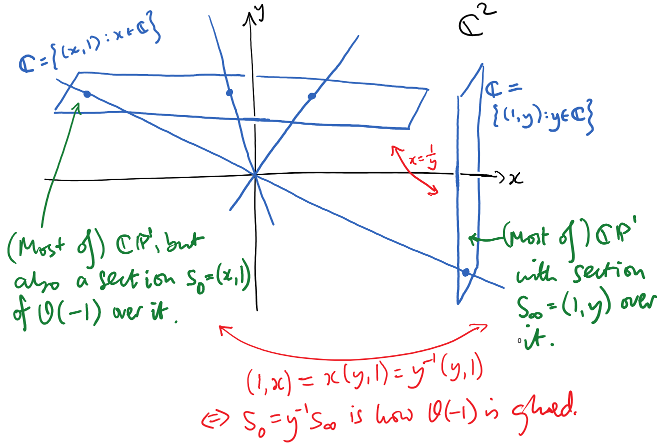


Zeros of sections

The $\mathcal{O}(n)$ line bundle over $\mathbb{P}^1$ was defined with transition function x^{-n}, gluing the section 1 over $\mathbb{C}_x$ to $x^{-n} = y^n$ over $\mathbb{C}_y$.

Therefore this defines a **global** holomorphic section of $\mathcal{O}(n)$, $n \geq 0$ with a **degree** n **zero** at $y = 0$.
(Meromorphic section with degree $|n|$ pole at $y = 0$ if $n < 0$.)

Similarly $p(x)$ over $\mathbb{C}_x$ is glued to $y^n p(y^{-1})$ over $\mathbb{C}_y$, so if $\deg p = n$ we get another algebraic/regular section over $\mathbb{P}^1$.
(Gives all sections from Spec & Proj lecture, $\Gamma(\mathcal{O}(n)) = \mathrm{Sym}^n(\mathbb{C}^2)^*$.)

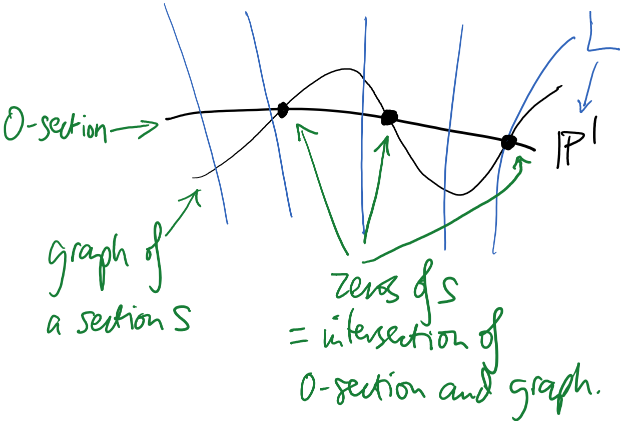
Again these all have n zeros.

Ex: When $n < 0$ we get a meromorphic section with n poles. Or instead glue 1 to an anti-holomorphic function across the circle $|x| = 1$ to give a (non-holomorphic) section with n zeros.

Intersecting with zero section

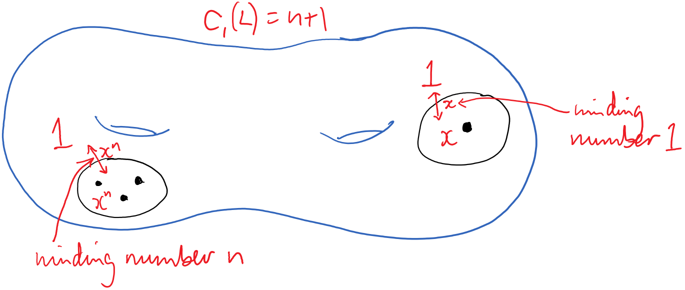
Indeed $c_1 = n$ is the number of zeros (counted with orientation and multiplicity) of **any** section of $\mathcal{O}(n)$.

In other words, $c_1(L)$ is the **intersection of the zero section of** L with itself (or equivalently the graph of any other section).



Clutching construction on arbitrary Riemann surfaces

Again line bundles = trivial bundles glued across circles/annuli.



$$\begin{aligned} c_1(L) &= \text{total winding number of transition functions} \\ &= \text{number of zeros of a section.} \end{aligned}$$

(So under line bundle $\leftrightarrow$ divisor correspondence, $c_1(\mathcal{O}(D)) = \deg D$.)

First Chern class on manifolds

More generally, for any complex line bundle L on a manifold X we define

$$c_1(L) \;:=\; [s^{-1}(0)] \;\in\; H_{\dim X-2}(X),$$

where s is any section transverse to the zero section.
(If s' is another choice let $s_t = s + ts'$. Then $[Z(s_t)]_{t\in[0,1]}$, is a chain interpolating between the two.)

In fact we define $c_1(L) \in H^2(X)$ to be the Poincaré dual of $[s^{-1}(0)]$ as (only) this cohomology classes will generalise to arbitrary topological spaces X.

For general X we can understand/define $c_1(L) \in H^2(X)$ by evaluating it on $[\Sigma] \in H_2(X)$, where $\Sigma \hookrightarrow X$ is a Riemann surface and

$$\big\langle c_1(L), [\Sigma]\big\rangle \;=\; c_1\big(L|_\Sigma\big).$$

Chern classes on manifolds

For any rank r complex vector bundle $E \to X$ pick a transverse C^∞-section s and define the **Euler class** or top Chern class

$$\begin{aligned} e(E) \;=\; c_r(E) \;:=\; [s^{-1}(0)] \quad &\in \quad H_{\dim X - 2r}(X) \\ &\cong \quad H^{2r}(X). \end{aligned}$$

Analogously define

$$c_k(E) \;\in\; H^{2k}(X)$$

to be Poincaré dual to the locus where $r-k+1$ generic sections fail to be linearly independent:

$$\big[Z(s_1 \wedge \ldots \wedge s_{r-k+1})\big] \;\in\; H_{\dim X - 2k}(X).$$

So $c_r(E) = e(E)$ while $c_1(E) = c_1(\Lambda^r E)$ and $c_i(E) = 0$ for $i > r$.
(When $k \neq 1, r$ then $c_k(E) \neq e(\Lambda^{r-k+1}(E))$ — this has the wrong degree, and $s_1 \wedge \ldots \wedge s_{r-k+1}$ is far from a generic section of $\Lambda^{r-k+1}(E)$.)

Whitney sum formula I

Given two generic sections $s_1 \in \Gamma(E_1)$, $s_2 \in \Gamma(E_2)$ we get a section $(s_1, s_2) \in \Gamma(E_1 \oplus E_2)$ and

$$\begin{aligned} e(E_1 \oplus E_2) &= [Z(s_1, s_2)] = [Z(s_1) \cap Z(s_2)] = [Z(s_1)] \cup [Z(s_2)] \\ &= e(E_1) \cup e(E_2) \in H^{2r_1+2r_2}(X). \end{aligned}$$

In particular for line bundles $c_2(L_1 \oplus L_2) = c_1(L_1) \cup c_1(L_2)$ and

$$\begin{aligned} c_1(L_1 \oplus L_2) &= c_1(\Lambda^2(L_1 \oplus L_2)) = c_1(L_1 \otimes L_2) = [Z(s_1 \otimes s_2)] \\ &= \big[Z(s_1) \cup Z(s_2)\big] = [Z(s_1)] + [Z(s_2)] \\ &= c_1(L_1) + c_1(L_2). \end{aligned}$$

We can write this as $c(L_1 \oplus L_2) = c(L_1) \cup c(L_2)$ where the total Chern class $c(E) := 1 + c_1(E) + c_2(E) + \ldots \in H^*(X)$.

Whitney sum formula II

More generally for bundles E, F of ranks r, s use the decomposition

$$\Lambda^k(E \oplus F) \;\cong\; \bigoplus_{i=0}^{k} \Lambda^i(E) \otimes \Lambda^{k-i}(F)$$

and generic sections $e_1, \ldots e_k \in \Gamma(E)$ and $f_1, \ldots f_k \in \Gamma(F)$ to compute

$$Z\Big((e_1 \wedge \ldots \wedge e_k) \oplus (e_1 \wedge \ldots \wedge e_{k-1} \otimes f_k) \oplus \ldots$$
$$\ldots \oplus (e_1 \otimes f_2 \wedge \ldots \wedge f_k) \oplus (f_1 \wedge \ldots \wedge f_k)\Big).$$

Ex: Work it out and take Poincaré duals to give

$$c_{r+s-k+1}(E \oplus F) \;=\; c_r(E)c_{s-k+1}(F) + \ldots + c_{r-k+1}(E)c_s(F).$$

Deduce the **Whitney sum formula** $c(E \oplus F) = c(E)c(F)$.

Axiomatic approach

Knowing (or defining!) $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = [\mathbb{P}^{n-1}]$, the Whitney sum formula and functoriality is then enough to completely determine all Chern classes on all topological spaces.

Functoriality: $c(f^*E) = f^*c(E)$. **Ex:** Define f^*E and prove this using zero loci of sections when $f\colon X \to Y$ is a map of manifolds.

There are two steps to proving this:

- All rank r bundles on X are pull backs f^*Q of the **universal bundle on classifying space** $Q \to B\mathrm{GL}(r, \mathbb{C})$ by a map $f\colon X \to B\mathrm{GL}(r, \mathbb{C})$. (So only need to define c_i on one space.)
- **Splitting principle**: we may assume E is a direct sum of line bundles, without loss of generality.

Classifying space

Any bundle is a quotient of an infinite rank **trivial** bundle $\underline{\Gamma(E)}$

$$\underline{\Gamma(E)} \xrightarrow{\text{ev}} E \longrightarrow 0. \qquad (*)$$

(Or take a sufficiently large subbundle $\underline{\mathbb{C}}^N \subset \underline{\mathbb{C}}^\infty = \underline{\Gamma(E)}$, $N \gg 0$.)

Therefore it defines a map from X to the Grassmannian

$$\begin{aligned} f\colon X &\longrightarrow \mathrm{Gr}\,(\mathbb{C}^\infty, r), \\ x &\longmapsto (*)_x\,. \end{aligned}$$

There's a (tautological) universal quotient bundle $Q \to \mathrm{Gr}$,

$$\underline{\mathbb{C}}^\infty \longrightarrow Q \longrightarrow 0 \quad \text{on} \;\; \mathrm{Gr},$$

and it is tautological from $(*)$ that f pulls this back to give E,

$$f^*Q \;\cong E.$$

Classifying space II

Thus $\mathrm{Vect}_r(X) = [X, \mathrm{Gr}]$.

We call $\mathrm{Gr} = \mathrm{Gr}(\mathbb{C}^\infty, r)$ the **classifying space** $B\mathrm{GL}(r, \mathbb{C})$.

E.g. for $r = 1$ we have $B\mathbb{C}^\times = \mathbb{CP}^\infty$.

So any line bundle $\mathcal{L} \to X$ is $f^*\mathcal{O}(1)$ for some (homotopy class of) map $f\colon X \to \mathbb{CP}^\infty$
(or $f\colon X \to \mathbb{CP}^N$ for $N \gg 0$ if X is finite dimensional).

Then

$$c_1(\mathcal{L}) \;=\; f^*c_1(\mathcal{O}(1)) = f^*h,$$

where $h \in H^2(\mathbb{CP}^\infty)$ is the generator (the limit as $N \to \infty$ of the Poincaré duals of $\mathbb{CP}^{N-1} \subset \mathbb{CP}^N$, or the standard Kähler form).

Splitting principle

Given $E \to X$ (e.g. $Q \to \mathrm{Gr}$) there's a space dominating Y on which E splits as a sum of line bundles:

$$\pi\colon Y \to X \quad \text{such that} \quad \pi^*E \;\cong\; \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r\,,$$

with fibres $Y_x \;=\; \pi^{-1}(x)$ given by the flag manifolds

$$Y_x \;=\; \big\{\text{Linearly independent complex lines } L_1, \ldots, L_r \subset E_x\big\}.$$

There are universal/tautological bundles $\mathcal{L}_i$ on Y and it is then tautological that $\pi^*E \cong \bigoplus_{i=1}^r \mathcal{L}_i$.

Fact $\pi^*\colon H^*(X) \to H^*(Y)$ is an **injection**. So pulling back $c(E)$ loses no information, and

$$\pi^*c(E) \;=\; c(\pi^*E) \;=\; c(\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r) \;=\; c(\mathcal{L}_1) \cdot \ldots \cdot c(\mathcal{L}_r).$$

Upshot

Given $E \to X$ there's a diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"'] & B(\mathbb{C}^\times)^r \;=\; (\mathbb{CP}^\infty)^r \\ X \end{tikzcd}$$

such that $\pi^*\colon H^*(X) \to H^*(Y)$ is an injection, and $c(E) \in H^*(X)$ is the unique class such that

$$\pi^* c(E) \;=\; f^*\big[(1+h_1)\cdot\ldots\cdot(1+h_r)\big].$$

So the splitting principle, the Whitney sum formula, and $c_1(\mathcal{O}(1)) = h$ determine all Chern classes uniquely.
(Existence takes a bit – not much – more work, e.g. computing $H^*(\mathrm{Gr})$.)

Ex: Corollary: if E has rank r then $c_i(E) = 0\ \forall i > r$.

Grothendieck's definition

On the projective bundle $\pi\colon \mathbb{P}(E) \to X$ we have the tautological inclusion

$$\mathcal{O}_{\mathbb{P}(E)}(-1) \ \hookrightarrow \ \pi^* E.$$

Since the quotient is a bundle of rank $r-1$,

$$c_r\big(\pi^* E/\mathcal{O}_{\mathbb{P}(E)}(-1)\big) \ = \ 0.$$

By the Whitney sum formula, this is the degree r part of

$$\pi^* c(E)/c\big(\mathcal{O}_{\mathbb{P}(E)}(-1)\big) \ = \ \pi^* c(E)/(1-h),$$

where $h = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. Thus

$$h^r + \pi^* c_1(E) h^{r-1} + \ldots + \pi^* c_{r-1}(E) h + \pi^* c_r(E) \ = \ 0. \qquad (*)$$

Fact: $H^*(\mathbb{P}(E)) = H^*(X) \oplus H^*(X)h \oplus \ldots \oplus H^*(X)h^{r-1}$ as a vector space. So h^r can be written uniquely in this basis to give $(*)$ and thus define $c_i(E)$.

Chern-Weil approach for line bundles

If X is a manifold we can pick a connection A on $\mathcal{L} \to X$.

Its curvature F_A is a closed 2-form $dF_A = 0$.

Changing $a \mapsto A + a \implies F_A \mapsto F_A + da$ so $[F_A] \in H^2(X, \mathbb{R})$ independent of A. In fact it is

$$\frac{[F_A]}{2\pi i} = [c_1(\mathcal{L})] \in H^2(X, \mathbb{Z})/\mathrm{torsion}.$$

Let's prove this for X a Riemann surface and $\mathcal{L}$ described by the clutching construction.

Connections and clutching construction

Write $X = U \cup_{S^1} D^2$ where D is a disc and S^1 is an annulus thickening its boundary.

Write $\mathcal{L}$ as $\underline{\mathbb{C}}_U \cup_\phi \underline{\mathbb{C}}_{D^2}$ for a transition function $\phi\colon S^1 \to \mathbb{C}^\times$ of winding number $n = c_1(\mathcal{L})$.

Put the trivial connection d on $\underline{\mathbb{C}}_U$. In the trivialisation $\underline{\mathbb{C}}_D$ restricted to the annulus this is the connection

$$d + \phi^{-1} d\phi$$

since this annihilates ϕ^{-1} (which is glued to 1 on U).

Extend this to any connection $d + a$ over D^2 and compute

$$\begin{aligned} \int_X F_A &= \int_{D^2} F_A = \int_{D^2} da = \int_{S^1} a \\ &= \int_{S^1} \frac{d\phi}{\phi} = \int_{S^1} d \log \phi = 2\pi i n. \end{aligned}$$

Chern-Weil theory

Manifold X, rank r bundle $E \to X$, connection A. Form

$$p\Big(\frac{F_A}{2\pi i}\Big) \;\in\; H^{2k}(X,\mathbb{R})$$

for any ad-invariant $\big(p(N^{-1}MN) = p(M)\big)$ polynomial of $\operatorname{End}\mathbb{C}^r$.

Ex: $p(F_A)$ closed and changes by an exact form under $A \mapsto A + a$.

E.g. p could be tr $\big($giving $c_1(E)\big)$ or det $\big($giving $c_r(E)\big)$ or any other symmetric polynomial in the eigenvalues like $\det \Lambda^r$.

If p is integral the result is an integral characteristic class.

Theorem $c(E) = \det\Big(\mathrm{id} + \frac{F_A}{2\pi i}\Big)$ in $H^*(X)/$torsion, i.e.

$$1 + c_1(E) + c_2(E) + \dots \;=\; 1 + \frac{\operatorname{tr} F_A}{2\pi i} - \frac{\operatorname{tr}(F_A \wedge F_A)}{4\pi^2} + \dots$$

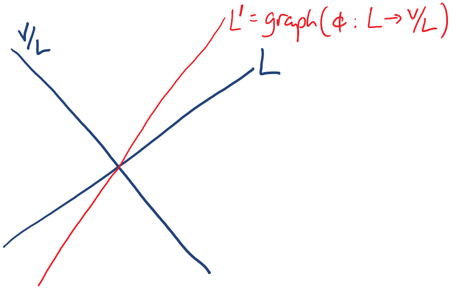
$\big(H^*(B\mathrm{GL}(r,\mathbb{C}) = \big\{$Ad-invariant polynomials$\big\}$.$\big)$

Tangent bundle to projective space

Let $V := \mathbb{C}^{n+1}$ so that $\mathbb{P}(V) = \mathbb{P}^n$. Then

$$T_{\mathbb{P}^n} \;=\; \mathcal{O}(-1)^* \otimes \frac{V}{\mathcal{O}(-1)}\,.$$

Sketch: a point of $\mathbb{P}(V)$ is a complex line $L \leq V$. Pick any complement to write $V = L \oplus V/L$. Then nearby lines in $V \;\longleftrightarrow$ graphs of linear maps $L \to V/L$. So tangent space $= L^* \otimes (V/L)$.



Chern classes of projective space

Applying Whitney sum formula to $T_{\mathbb{P}^n} = \underline{V}(1)/\mathcal{O}$ gives

$$c(T_{\mathbb{P}^n}) \;=\; c\big(\underline{V}(1)\big)/c(\mathcal{O}) = c\big(\mathcal{O}(1)^{\oplus(n+1)}\big) = (1+h)^{n+1},$$

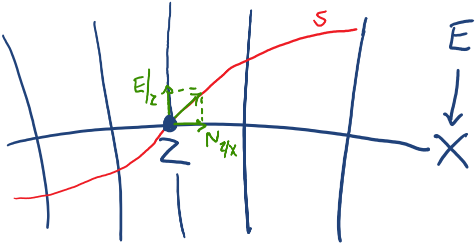
where $h = c_1(\mathcal{O}(1))$ is the hyperplane class Poincaré dual to $\mathbb{P}^{n-1} \subset \mathbb{P}^n$.

E.g. $c_n(T_{\mathbb{P}^n}) = (n+1)h^n$ so integrating gives $e(\mathbb{P}^n) = n+1$.

Hypersurfaces in projective space

Ex: "**Adjunction**". If $s \in \Gamma(E)$ is transverse to the zero section, show its zero locus $Z = s^{-1}(0)$ has normal bundle

$$N_{Z/X} \;=\; E|_Z.$$



Ex: Hence work out the total Chern class $c(T_{X_d})$ of a degree d hypersurface $X_d \subset \mathbb{P}^n$ (the zero locus of a section of $\mathcal{O}(d)$). Apply to $n = 3$, $d = 4$ to find $c_1(T_S)$ and $e(S)$ for S a "K3 surface".

Exercises

Ex: Compute $c_i(\mathrm{End}(E))$ in terms of $c_i(E)$ for E a rank 2 bundle. (Hint: splitting principle.)
Why did you find $c_1 = 0 = c_3 = c_4$?

Ex: (1) Compute $c_4(\mathrm{Sym}^3(E))$ in terms of $c_i(E)$ for E a rank 2 bundle. (Hint: splitting principle.)

Ex: (2) The Grassmannian $\mathrm{Gr}(2,4)$ of 2-planes in $\mathbb{C}^4$ has a universal subbundle $\mathcal{U} \hookrightarrow \underline{\mathbb{C}}^4$. Describe a cycle Poincaré dual to $c_2(\mathcal{U}^*)$. (Hint: use $(\underline{\mathbb{C}}^4)^* \twoheadrightarrow \mathcal{U}^*$ to pick a section of $\mathcal{U}^*$.)

Ex: (3) Describe a cycle Poincaré dual to $c_1(\mathcal{U}^*)$.
(Hint: pick two sections of $\mathcal{U}^*$ and see where they're linearly dependent.)

Ex: From (1,2,3) show $\int_{\mathrm{Gr}(2,4)} c_4(\mathrm{Sym}^3 \mathcal{U}^*) = 27$.

Ex: Identify $\mathrm{Gr}(2,4)$ with $\{\text{lines } \mathbb{P}^1 \subset \mathbb{P}^3\}$. Let $s \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(3))$ cut out a cubic surface $S \subset \mathbb{P}^3$. Show s defines a section of $\mathrm{Sym}^3 \mathcal{U}^* \to \mathrm{Gr}(2,4)$ cutting out the (27) lines in $\mathbb{P}^3$ which lie in S.

Exercise: Segre classes

We defined $c_i(E)$ as (Poincaré dual to) the locus where $r-i+1$ generic sections fail to be linearly independent, i.e. the $x \in X$ s.t.

$$\mathbb{C}^{r-i+1} \xrightarrow{s_1(x),\dots,s_{r-i+1}(x)} E_x$$

fails to be injective.

Similarly we can define the ith **Segre class** $s_i(E) \in H^{2i}(X)$ to be ($(-1)^i$ times by the Poincaré dual to) the locus where $r+i-1$ generic sections fail to generate E, i.e. the $x \in X$ where

$$\mathbb{C}^{r+i+1} \xrightarrow{s_1(x),\dots,s_{r+i+1}(x)} E_x$$

fails to be surjective.

Ex: Show that for line bundles, $s_i(\mathcal{L}) = (-1)^i c_1(\mathcal{L})^i$.

In fact $s(E) = c(E)^{-1}$, where $s(E) := 1 + s_1(E) + s_2(E) + \dots$.

Čech cohomology formulation

Let $\mathcal{O}$ denote the sheaf of (holomorphic, or algebraic, or C^∞, or ...) functions, and $\mathcal{O}^\times$ the (multiplicative) sheaf of invertible functions.

Then the exact sequence (in Euclidean topology)

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \longrightarrow 0$$

induces the long exact sequence of Čech cohomology groups

$$H^1(X, \mathcal{O}^\times) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}).$$

Consider an element $e \in H^1(X, \mathcal{O}^\times)$ (invertible e_{UV} on each overlap $U \cap V$ satisfying $e_{UV} e_{VW} e_{WU} = 1$ on $U \cap V \cap W$) to be the transition functions for a line bundle $\mathcal{L}$.

Ex: Identify $\delta(e) \in H^2(X, \mathbb{Z})$ with $c_1(\mathcal{L})$ for X a Riemann surface. (Hint: use clutching construction. Lift all $e_{UV} \in \mathcal{O}^\times_{U \cap V}$ to $\log(e_{UV}) \in \mathcal{O}_{U \cap V}$ compatibly *except* for the winding number of e_{UV}, which gives $\mathbb{Z}$ ambiguity.)

Exercise: homotopy/homology groups

Ex: $L \to \Sigma$ a line bundle over a Riemann surface, with sphere (S^1) bundle) $S(L) \to \Sigma$ and LES of homotopy groups

$$\dots \longrightarrow \pi_2(S(L)) \longrightarrow \pi_2(\Sigma) \xrightarrow{\partial} \pi_1(S^1) \longrightarrow \pi_1(S(L)) \longrightarrow \dots$$

When $\Sigma = \mathbb{P}^1$ show $\partial \colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by $c_1(L) \in \mathbb{Z}$.

For more general Σ replace π_* by H_* and ∂ by a differential in the Leray spectral sequence.

What about arbitrary X?