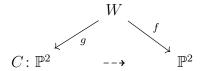
LSGNT Blow ups exercises

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- 1. Let $I \subset \mathbb{C}[x_1,\ldots,x_n]$ be an ideal and let $I=(f_1,\ldots,f_r)$ and $I=(f'_1,\ldots,f'_s)$ be two sets of generators for I. Using the explicit description of the blow up in terms of generators, let $X \subset \mathbb{A}^n \times \mathbb{P}^{r-1}$ be the blow up using f_1,\ldots,f_r and let $X' \subset \mathbb{A}^n \times \mathbb{P}^{s-1}$ be the blow up using f'_1,\ldots,f'_s . Show that $X \cong X'$.
- 2. Let $L \subset \mathbb{A}^n$ be a linear subspace. What is the exceptional divisor of blow up at L?
- 3. Compute the resolution of $\{x^p + y^q = 0\} \subset \mathbb{A}^2_{\mathbb{C}}$ using blow ups. How many blow ups does it take?
- 4. Compute the blow up of the Whitney umbrella $\{x^2 + y^2z = 0\} \subset \mathbb{A}^3_{\mathbb{C}}$ at the origin in each of the three natural coordinate charts.
- 5. Blow up $\operatorname{Spec}(\mathbb{Z}[x])$ at the point (p, x). What is the exceptional divisor?
- 6. Let a_1, \ldots, a_n be positive integers. We define the weighted blow up of \mathbb{A}^n at $I = (x_1, \ldots, x_n)$ to be $\mathbf{Proj}_{\mathbb{A}_n} R(I)$ where we stipulate that x_i has graded weight $= a_i$. Show that $\{x^p + y^q = 0\}$ can be resolved by a single weighted blow up.
- 7. Show that the blow up of \mathbb{P}^2 in two points is isomorphic to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. Try to generalise this statement by replacing \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ by arbitrary Hirzebruch surfaces.
- 8. We define the *Cremona involution* to be the rational map $C: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by C([x:y:z]) = [1/x:1/y:1/z]. Show that C is a birational map. Where is C not well defined? (Hint: its only at finitely many points).

9. Show that there exists a diagram



where f and g are each a blow of \mathbb{P}^2 at 3 points.

- 10. Let $S^3 \subset \mathbb{C}^2$ be the unit sphere about the origin. Let $X \to \mathbb{C}^2$ be the blow up at the origin with exceptional divisor E. Show that there is a morphism $X \to E$. Show that the restricted map (of topological spaces) $S^3 \to E$ is the Hopf fibration.
- 11. The surface singularity $\{xy+z^n=0\}\subset \mathbb{A}^3$ is a type of Du Val singularity called an A_n -singularity. Resolve this singularity by blow ups. What is the exceptional locus of this resolution? What is intersection pairing of the exceptional locus?
- 12. The other types of Du Val singularities are D_n -singularities given by $\{x^2 + zy^2 + y^{n-1} = 0\}$ where $n \geq 4$, E_6 -singularities given by $\{x^2 + y^3 + z^4 = 0\}$, E_7 -singularities given by $\{x^2 + y^3 + yz^3 = 0\}$ and E_8 -singularities given by $\{x^2 + y^3 + z^5 = 0\}$. Compute the resolutions of these singularities.
- 13. Let X be a smooth surface and let $\sum E_i \subset X$ be a simple normal crossings divisor. We define the dual graph $D(\sum E_i)$ as follows. The vertices of $D(\sum E_i)$ correspond to the irreducible components of $\sum E_i$ and there is an edge between v_{E_i} and v_{E_j} for each point of $E_i \cap E_j$. Compute the dual graphs of the exceptional divisors on the resolutions of each Du Val singularity, and compare them with the corresponding Dynkin diagrams.
- 14. Show that the blow up of a scheme X along an effective Cartier divisor $D \subset X$ is an isomorphism. Show by example that if D is not Cartier then the blow up need not be an isomorphism.