Mirror Symmetry Monodromy Actions on Derived Categories

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I, Michela Barbieri, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Given a linear toric Calabi-Yau geometric invariant theory (GIT) problem, Hori-Vafa toric homological mirror symmetry constructs a parameter space over which mirror Landau-Ginzburg models live, called the Fayet-Iliopoulos Parameter Space ($FIPS$). It is conjectured that there is a monodromy action of the fundamental group of $FIPS$ on the derived categories of the GIT quotients via spherical twists about spherical functors. We demonstrate the conjecture holds true for toric Calabi-Yau 3-folds of Picard rank 2.

Impact Statement

This thesis makes contributions to pure mathematics, in particular to algebraic geometry and homological mirror symmetry (HMS). We add to the body of evidence for the monodromy conjecture in HMS by demonstrating it holds in new examples. We also find unexpected symmetries of derived categories as a consequence.

Although highly theoretical in nature, advances in these areas have historically had a transformative influence far beyond mathematics itself. Ideas from geometry and category theory underpin modern developments in theoretical physics, including string theory, and provide the language through which deep structures in science can be described.

Pure mathematics also delivers long-term, often unexpected, benefits to technology and society. For example, abstract number theory has become the foundation of modern cryptography, and algebraic geometry underlies areas of coding theory and computer vision. By developing new conceptual tools within homological mirror symmetry, this research supports future interdisciplinary applications of mathematics, while contributing to the shared body of knowledge that underpins science as a whole.

Finally, this work has an immediate impact through the training and skills it represents. Undertaking this research has developed advanced problem-solving, abstraction, and reasoning skills that are valuable not only within academia but also in areas such as data science, finance, and artificial intelligence, where sophisticated mathematical thinking plays an increasingly important role.

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Contents

Chapter 1

Introduction

1.1 General Background

When mathematicians talk about mirror symmetry, they refer to a collection of mysterious relationships between different geometric objects spanning differential, symplectic, and algebraic geometry. The story begins in 1991 when string theorists Candelas, de la Ossa, Green, and Parkes published their famous paper [Can+91]. This paper caught the attention of mathematicians because the physicists appeared to have solved a mathematical problem that had occupied their minds for decades.

The question was, what is the number n_d of rational degree d curves on a quintic threefold? Mathematicians had only been able to work out n_d for $d < 4$ and were unsure whether the number was even finite for higher d. [Can+91] computed[1] n_d for $d < 11$, with the first three numbers matching those that the mathematicians had worked out. The physicists exploited what seemed like a magical duality between manifolds, now widely referred to as *mirror symmetry*.

The study of mirror symmetry since then has resulted in a plethora of interesting mathematics. Many unexpected relationships between different geometric objects have been observed as a result. One of the most profound developments to emerge from this work is homological mirror symmetry, an entirely new and rich area of mathematics that lies at the interface of algebraic geometry and symplectic geometry.

The core conjecture of homological mirror symmetry, first presented by Maxim Kontsevich at the 1994 International Congress of Mathematicians, roughly says the

[1] What was actually computed here were Gromov-Witten invariants, where the relationship with curve counts is a little subtle.

following. Given a *complex geometric object*[2] X, the homological mirror symmetry conjecture says that there is some *symplectic geometric object* X^v such that the following two puzzlingly different categories are equivalent

$$D^b(X) \cong \mathrm{Fuk}(X^v), \tag{1.1}$$

where the left hand side (the B-side) is the bounded derived category of coherent sheaves on X and depends only on the complex structure, while the right hand side (the A-side) is some version of the Fukaya category of X^v and depends only on the symplectic structure. Defining the Fukaya category in general is a really difficult problem, but fortunately in this thesis we only work on the B-side, where we refer the reader to [Huy06, Chapter 2] for a good introduction on derived categories.

We say that a pair (X, X^v) that satisfies (1.1) are *mirrors*. In fact, we have a whole family of mirrors to X on the A-side, each of which is symplectomorphic to X^v, but has a different complex structure. We denote the complex structure moduli space of the symplectic manifold X^v by $\mathscr{M}_{\mathrm{CS}}$.

Similarly, we can consider all the Kähler forms $\mathscr{M}_{\mathrm{KF}}$ (symplectic structures) on X and this should give us a moduli space of mirrors to X^v. Homological mirror symmetry implies that the two moduli spaces of mirrors are isomorphic, but we know that the symplectic story needs to be expanded, as $\mathscr{M}_{\mathrm{CS}}$ itself naturally carries a complex structure, while $\mathscr{M}_{\mathrm{KF}}$ is a real manifold locally identified with $H^2(X, \mathbb{R})$.

So we replace $\mathscr{M}_{\mathrm{KF}}$ with the conjectured *Stringy Kähler Moduli Space* $\mathscr{M}_{\mathrm{SKMS}}$ of X, which is larger[3] and complexified.

Recall that $\mathrm{Fuk}(X^v)$ is independent of complex structure, so over each point in $\mathscr{M}_{\mathrm{CS}}$ we have the same category. Nevertheless it turns out that this is only really true locally. This points to the existence of a local system of categories isomorphic to $\mathrm{Fuk}(X^v)$ over $\mathscr{M}_{\mathrm{CS}}$. By mirror symmetry, we should be able to carry this over to B-side local system over $\mathscr{M}_{\mathrm{SKMS}}$ with fibres isomorphic to $D^b(X)$. We call this the *mirror symmetry monodromy conjecture*.

There are various problems with the conjecture that make it very imprecise. The first is in particular related to B-side monodromy conjecture, as there is no rigorous or general definition of $\mathscr{M}_{\mathrm{SKMS}}$. $\mathscr{M}_{\mathrm{CS}}$ is easier to understand, so most B-side papers define $\mathscr{M}_{\mathrm{SKMS}}$ to be the complex structure moduli space of the mirror.

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[2] The use of the terms *complex geometric object* and *symplectic geometric object* here is deliberately vague as we will see that sometimes our B-side will be *stacks* and not varieties, and the A-side may be *Landau-Ginzburg models*.

[3] $\mathscr{M}_{\mathrm{SKMS}}$ includes other birational models of X.

The difficulties with the definition of $\mathscr{M}_{\text{SKMS}}$ are compounded by the technical limitations in the definition of the Fukaya category and the fact that defining what a local system of categories is a challenging task [PPS25]. The former is not an issue for us because we will be working exclusively on the B-side. We deal with the latter by analogy with classical local systems, and think about fundamental group actions of the moduli spaces via autoequivalences. This provides an algebraic description of a local system of categories.

The seminal example of this action is given in the profound paper [ST01] when X is the A_n-singularity. Seidel-Thomas produced novel autoequivalences on $D^b(X)$, called *spherical twists about spherical objects*, by exhibiting a configuration of Lagrangian spheres on the A-side mirror and studying the symplectic Dehn twists around these spheres. The braiding of symplectic Dehn twists manifests itself as a faithful[4] braid relation among the spherical twists.

Although the mirror symmetry monodromy conjecture is hard to state precisely in general, we can make sense of things in specific contexts where both the geometry is simpler and homological mirror symmetry is well-understood. The toric setting provides exactly such a context. Toric homological mirror symmetry has various formulations (see [HH22, Sec. 2] for a summary), but we work in the variation of Geometric Invariant Theory (GIT) and Hori-Vafa perspective.

In this thesis we will be looking at the algebraic side of toric homological mirror symmetry. The B-side objects arise as quotients in Geometric Invariant Theory (GIT). Specifically, we work with linear toric Calabi-Yau GIT problems (§2.1), which are given by an algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $V \cong \mathbb{C}^n$. The theory of GIT gives us various quotients X, and by so-called *wall-crossing* we get various equivalences and autoequivalences of the derived categories. Additionally, via *Hori-Vafa mirrors* [HV00] we have a heuristic recipe for constructing a substitute of $\mathscr{M}_{\text{CS}}$ called the Fayet-Iliopoulos Parameter Space ($FIPS$).

In our setting we therefore interpret the B-side mirror symmetry monodromy conjecture as saying that the fundamental group(oid) of $\pi_1(FIPS)$ acts on the derived categories of the GIT quotients via the wall-crossing equivalences. This is still an open problem which we seem to be far from solving. The main obstacle is that fundamental groups are really hard to understand in general. We will prove the conjecture for a special class of examples.

[4] Proving faithfulness was the hard part here.

1.2 Details and Structure

Having outlined the conceptual framework, we now turn to the technical details of our approach. We start with the B-side toric set-up, which we go through in Chapter 2. In §2.1 we exposit the theory of linear toric GIT, which explains how to take quotients when you have an algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $V \cong \mathbb{C}^n$. To obtain well-behaved quotients, you need to remove a certain zero locus from V before taking the naive quotient. This process is not unique and depends on a choice of *stability condition*, which in this setting is just a choice of character $\theta : T \to \mathbb{C}^*$ which lives in a rank r lattice $L^\vee$, called *the character lattice*. Different characters will give different birational but in general not isomorphic toric quotients.

The character lattice $L^\vee$ splits into positive cones where any two stability conditions that lie in the interior of the same cone have the same unstable locus. The top dimensional cones correspond to good quotients in the sense that they are *smooth Deligne-Mumford (DM) stacks*, which are stacks where the stabilisers are finite groups. The GIT quotients corresponding to the top dimensional cones are therefore the ones we refer to when we say GIT quotients. We call the decomposition of $L^\vee$ *a wall and chamber decomposition*, where two GIT quotients are said to be separated by a wall if their cones share a codimension 1 face.

We note that the wall and chamber decomposition of $L^\vee$ gives us the data of a toric fan, which we call *the secondary toric fan*. There is a geometric realisation of this fan $\mathfrak{F}$ which we call the *secondary toric stack*. By toric orbit-cone correspondence, for every GIT quotient there is a fixed point $c_X \in \mathfrak{F}$ under the dual torus $T^\vee$ action.

Let us consider a wall W in the secondary fan separating two GIT quotients X and X'. The *wall-crossing formula* tells us that

$$D^b(X) = \langle D^b(X'), D^b(Z), \ldots, D^b(Z) \rangle,$$

where Z is the T-quotient of the fixed locus of the action of the one-parameter family perpendicular to the wall, and the number of copies of Z can be easily computed combinatorially[5] (§2.3). We call X' *minimal* with respect to X, and X *maximal* with respect to X'.

The homological mirror symmetry story we will use only really works if our GIT problems satisfy a *Calabi-Yau condition* (§2.2). As one might guess from the name, the condition implies that all quotients are Calabi-Yau, i.e. have trivial canonical

[5] In fact, Z comes from a GIT problem called a *Higgs GIT problem*, whose data is a subset of the data of our original GIT problem. See Definition 2.9.

sheaf. It also implies that for any wall the number of copies of $D^b(Z)$ in the wall-crossing formula is zero, so that all your GIT quotients are derived equivalent.

In fact, if we fix a GIT quotient X, wall-crossing through a wall W actually gives us countably-infinite derived equivalences between X and the GIT quotient on the other side of the wall, which are called *window equivalences* [Seg11]. Going through one window and back through another produces a non-trivial autoequivalence of $D^b(X)$. For any wall we therefore have countably-infinitely-many non-trivial autoequivalences of $D^b(X)$.

It turns out that we can describe these autoequivalences as spherical twists about spherical functors [HS16], providing a concrete algebraic handle on the geometric wall-crossing phenomenon. Though the number of copies of $D^b(Z)$ in the SOD for X is 0, the toric variety Z is still important. For any wall W and corresponding Z we get a spherical functor (§2.4.2)

$$F : D^b(Z) \to D^b(X),$$

which is a generalisation of the spherical objects from [ST01]. The *spherical twist* $T_F : D^b(X) \to D^b(X)$ of a spherical functor F is an autoequivalence (§2.4).

It is worth noting that Z comes from a GIT problem itself which is not generally Calabi-Yau. Via repeated wall-crossing we find a semi-orthogonal decomposition

$$D^b(Z) = \langle D^b(Z_0), \dots, D^b(Z_0), \dots, D^b(Z_k), \dots D^b(Z_k) \rangle,$$

where Z_i are minimal GIT quotients, see equation (2.19). The details of this decomposition depends on some choices, but the multiplicity of each factor $D^b(Z_i)$ is well-defined [KS22]. Each restriction $F_i : D^b(Z_i) \to D^b(X)$ of F is a spherical functor and we can factor T_F in terms of these T_{F_i} by equation (2.27).

Having established a rich supply of autoequivalences on the B-side, we can now turn to the mirror side to complete our setup. Alongside our understanding of non-trivial autoequivalences of derived categories of Calabi-Yau toric GIT quotients, we also have another crucial tool: an explicit recipe for constructing the mirror side. We go through this part of the story in Chapter 3.

The *Hori-Vafa mirrors* are given by a *Landau-Ginzburg (LG) model* $((\mathbb{C}^*)^{n-r}, W)$ where $W : (\mathbb{C}^*)^{n-r} \to \mathbb{C}$ is a Laurent polynomial called the *superpotential* [HV00]. The mirror symplectic manifold is just the torus $(\mathbb{C}^*)^{n-r}$, specifically the dual torus mirror to a torus inside a GIT quotient X, but since X is a partial compactification of the torus we also have W as extra data on the A-side. A priori the coefficients of W live in $\mathbb{C}^n$ and can vary, giving us a family of mirror LG models. However, there are some coefficient values that are not allowed because they make $V(W)$ singular, and these forbidden values form what is called *the discriminant locus* $\nabla \subset \mathbb{C}^n$ (§3.1). The space of allowed coefficients over which the mirror LG-models live is called the *Fayet Iliopoulous Parameter Space* ($FIPS$) and we use this as a substitute for $\mathscr{M}_{\mathrm{CS}}$, or $\mathscr{M}_{\mathrm{SKMS}}$ on the original B-side.

The non-toric part of the discriminant locus decomposes into irreducible components (3.6)

$$\nabla_{\text{non-toric}} = \bigcup_{i=0}^{k} \nabla_i.$$

An important part of our story is the observation, first noted in [APW17], that there is a correspondence between these components and the factors $D^b(Z_i)$ in the semi-orthogonal decomposition for $D^b(Z)$.

It's possible to see $FIPS$ as living inside the secondary toric stack $\mathfrak{F}$ (§3.1). Recall that for every GIT quotient X there is a fixed point $c_X \in \mathfrak{F}$ in the toric boundary. This makes the $\mathfrak{F}$ an interesting compactification of $FIPS$ because we can then see *large radius regions* of $FIPS$ that correspond to specific GIT quotients.

With both the B-side autoequivalences and the A-side parameter space in hand, we can now formulate more precisely how they should be related. The following conjecture captures the heart of our mirror symmetry monodromy story.

Main Conjecture (Conjecture 3.17)**.** Let X be a GIT quotient, and pick a basepoint in $FIPS$ corresponding to X. Then there is an action of $\pi_1(FIPS)$ on $D^b(X)$ in which a *meridian* (Definition 4.7) of ∇_i acts by a spherical twist about a spherical functor whose source category is $D^b(Z_i)$.

Remark 1.1. Conjecture 3.17 is a bit more detailed than the main conjecture above, and includes the action of the Picard groups of the GIT quotients.

This matching of meridians is non-canonical because meridians are only defined up to path conjugation, and the spherical twists T_{F_i} are only defined up to autoequivalences of $D^b(X)$. In rank 1 it is quite easy to see that the conjecture holds, see Example 3.19. The higher rank case is much harder.

Remark 1.2. One could also conjecture that the action is faithful, but proving faithfulness is hard and we won't be touching on it in this thesis.

Though the conjecture is still an open problem, it is widely believed to be true as it has been shown in various examples (see §1.4). In particular, the study of the fundamental group of $FIPS$ has allowed algebraic geometers to observe relations between equivalences of derived categories which were not at all obvious a priori, providing strong evidence that what has been shown so far is unlikely coincidental. This thesis adds to this body of work, proving the conjecture in new examples.

The major difficulty in proving the conjecture is that we don't know how to compute $\pi_1(FIPS)$ in general. It's not a surprise that in most of the cases where the conjecture has been shown $FIPS$ is particularly simple[6]. In our examples, we will see cases of $FIPS$ that are complements of higher degree singular curves, and we will have to carefully compute the fundamental group. In Chapter 4 we explain tools we will need to do this. Since our examples are rank 2 we will see that we can identify $FIPS$ with a curve complement modulo a finite group action. In §4.1 we go through the theory of how to compute fundamental groups of curve complements using the Zariski-Van Kampen method, and in §4.2 we explain how to deal with the finite group action to compute *the orbifold fundamental group*.

1.3 Main Results and Conjectures

To make progress on proving the conjecture, we focus on a particularly manageable class of examples that nonetheless captures the interesting features of the general phenomenon. These are toric Calabi-Yau 3-folds of Picard rank 2, which are given by convex 2D lattice polygons with 5 integral points. There are six such polygons up to equivalence [LZ11], depicted in Figure 1.1.

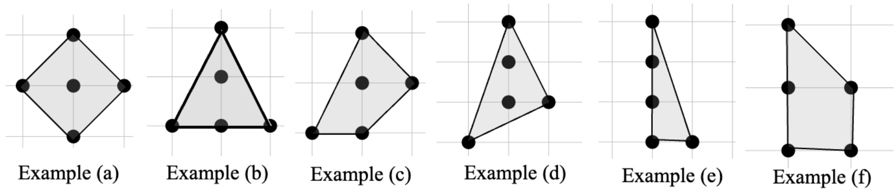


Figure 1.1: The primary polygons of toric Calabi-Yau threefolds of Picard rank 2.

Each polygon is a *primary polygon* (Definition 2.7) of a linear toric CY GIT problem $(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^5$, where we note that cones on triangulations of the primary polygons give toric fans for the GIT quotients.

Main Theorem (Thms. 5.2, 5.5, 5.6, 5.7)**.** The main conjecture holds for the linear toric CY GIT problems with primary polygons in Figure 1.1.

Remark 1.3. The cited Theorems prove the more specific Conjecture 3.17 in one GIT quotient for each case, but the more general conjecture for every GIT quotients

[6] E.g. the complement of a hyperplane arrangement.

follows. This is because the Calabi-Yau condition implies that derived categories of all the GIT quotients are equivalent (Corollary 2.13).

Examples (e) and (f) have been proved as part of a special class of examples coming for A_n surface singularities [DS15]. In Chapter 5, we will prove Examples (a)-(d), with one section dedicated per example.

To prove each example, we compute presentations for the fundamental groups of $FIPS$ and match generators to autoequivalences according to the conjecture. We then prove that the fundamental group relations also hold in the group of autoequivalences, thus proving that our action is well-defined. We remark again that these relations between autoequivalences, although not especially difficult to prove, are in general not at all obvious a priori.

Our study furthermore allows us to find and conjecture relations for cases beyond the examples we look at, such as the following example.

Example (Proposition 5.8)**.** Consider the smooth DM stack

$$X = [\mathbb{A}^3/_{n-1,-n,1}\mathbb{Z}_m],$$

where

$$n,m \in \mathbb{Z}_{\geq 0},\ 2 < n \leq \frac{m+1}{2},\ \gcd(n,m) = \gcd(n-1,m) = 1.$$

$\mathrm{Pic}(X) \cong \mathbb{Z}_m$, and each line bundle twist of the skyscraper sheaf of the origin $\mathscr{O}_0(l)$ is a spherical object. It is easy to see that the spherical twists about $\mathscr{O}_0(l)$ and $\mathscr{O}_0(l+1)$ satisfy the braid relation. Additionally, for $k = -n+1 \bmod m$ and $k = n \bmod m$ we show that the following relation holds

$$\left[T_{\mathscr{O}_0(k)}, T_{T_{\mathscr{O}_0(-1)}^{-1}\mathscr{O}_0}\right] = 1.$$

Furthermore, in §4.3, we conjecture some general relations between the autoequivalences associated to a specific choice of wall W. By the orbit-cone correspondence, a wall W in the secondary fan separating two quotients X and X' determines a rational toric boundary curve $C_W \subset \mathfrak{F}$ connecting fixed points c_X and $c_{X'}$ in the secondary toric stack $\mathfrak{F}$. We assume here that C_W is non-stacky and is contained in the discriminant locus[7]. By [KS22, Corollary 4.13], the discriminant locus ∇ intersects C_W in exactly one point, and each component ∇_Γ intersects with some multiplicity $m_\Gamma \in \mathbb{Z}_{\geq 0}$.

[7]It seems plausible that these constructions could be extended to the stacky setting, though the situation is less transparent.

Looking at a small analytic neighbourhood $U_W \subset FIPS$ of the intersection point, we find relations in fundamental group in a constructive way by performing blow-ups. The key idea is that the fundamental group of the blow-up without the exceptional divisor is the same as the original fundamental group, and that blowing up will separate components and resolve the singularities until they are ordinary. The meridians of the strict transforms and exceptional divisors should intuitively correspond to autoequivalences coming from W via Conjecture 3.17, which implies non-trivial relations between the autoequivalences. Unfortunately we did not find a nice closed presentation for the fundamental group, and in general these relations are quite unwieldy. Nevertheless, exploring possible implications of these blow-up relations could be an interesting avenue for future research.

For rank 2 GIT problems, some nice relations are implied. Note that in this case, the discriminant locus $\nabla \subset \mathfrak{F}$ is a curve. Excluding toric boundary components, it has at most two additional irreducible components, the principal component and another one, which we shall denote by ∇_0 and ∇_1 respectively. The curve C_W intersects ∇_0 at a smooth point[8] with some multiplicity $m \in \mathbb{Z}_{\geq 0}$, and ∇_1 at a smooth point with multiplicity either 0 or 1 [Kit19, Sec. 6.2].

Let $\mathscr{L} \in \operatorname{Pic} X$ be the canonical line-bundle corresponding to C_W by Lemma 3.15. Let T_{F_0} and T_{F_1} be two of the factors of the spherical twist T_F from crossing W, corresponding to ∇_0 and ∇_1 respectively. We make the following conjecture.

Conjecture (Conjecture 4.40)**.** Consider a rank 2 GIT problem with quotient X and adjacent wall W. If ∇_1 does not intersect C_W, then

$$((\otimes\mathscr{L}) \circ T_{F_0})^m = (T_{F_0} \circ (\otimes\mathscr{L}))^m.$$

If ∇_1 intersects C_W, then

$$\begin{aligned} T_{F_1} \circ T_{F_0} \circ F_1 &= (\otimes\mathscr{L}^{-1}) \circ F_1, \\ F_0 &= T_{(\otimes\mathscr{L})\circ F} \circ F_0. \end{aligned}$$

It's not hard to observe that the relations above are satisfied in our examples in Chapter 5 (the only examples with a boundary curve C_W that's non-stacky are Examples (a) and (c)). We highlight that in rank 2 the conjectured relations should be a direct corollary of the near-large-radius representation proved in [Kit19, Sec. 6]. Confirming this would require careful checking that our equivalence matching is the same, which we have not done.

[8] By Remark 6.6 in [Kit19] we know that the intersection occurs at smooth points of the discriminant components in rank 2 problems. We would guess this to be true in higher rank but do not know how to prove it.

1.4 Previous Results

To put our results in context, it is worth reviewing what has been accomplished previously and what obstacles remain. The homological mirror symmetry statement in our setting should say (for some definition of the Fukaya category of the LG-model) that $\mathrm{Fuk}((\mathbb{C}^*)^{n-r}, W)$ is equivalent to $D^b(X)$ for a GIT quotient X. If there was a mirror map that worked for the family of LG-models, and a monodromy action of $\pi_1(FIPS)$ on $\mathrm{Fuk}((\mathbb{C}^*)^{n-r}, W)$, we would be able to carry the action over to $D^b(X)$ and the B-side mirror symmetry monodromy conjecture would be proved.

A version of this programme was used successfully in [ŠB24], where the authors constructed an action of $\pi_1(FIPS)$ on the mirror wrapped Fukaya category of a generic fibre D of the superpotential W, and were able to use mirror symmetry to deduce an action on the derived category of the toric boundary of X.

However, if we want to upgrade this to an action on the full $D^b(X)$, the tools on the A-side are still insufficient. Currently the proven toric homological mirror statements (e.g. [Han19; HH22]) typically involve some extra choices (like a tropical limit), obstructing the construction of a monodromy action $\pi_1(FIPS)$ on $\mathrm{Fuk}((\mathbb{C}^*)^{n-r}, W)$.

For this reason, we instead use combinatorial observations to make a guess for how $\pi_1(FIPS)$ should act on the derived categories of toric GIT quotients, as is done in Conjecture 3.17. There are various papers that use this approach. For example, a representation of $\pi_1(FIPS)$ using variation of GIT was shown in [HS20] for *quasi-symmetric*[9] GIT problems. This was possible because for these problems, $FIPS$ is a hyperplane complement, whose fundamental group is much easier to understand than a general hypersurface complement. The authors were also able to use so-called *magic windows*, which relate the derived categories of all the GIT quotients simultaneously (generally windows are associated to a specific wall-crossing).

In [DS15], the authors show a faithful[10] action of mixed braid groups on a class of examples coming from A_n singularities by exploiting the existence of finite covers of the $FIPS$ which are themselves the $FIPS$ of a quasi-symmetric GIT problem. This covering strategy could be a promising approach to proving the conjecture in general, although given any GIT problem there is yet to be a general construction for a quasi-symmetric GIT problem that is related in this way. In this thesis, we deliberately deal with examples where we can't exploit the quasi-symmetric result.

[9] In the toric setting a quasi-symmetric GIT problem is a Calabi-Yau problem where the sums of weights on any line is zero.

[10] The faithfulness here followed by the faithfulness of the braid action in [ST01].

The thesis [Kit19] is also closely related to our work. For rank 2 examples, they construct a representation of the fundamental groupoid at the large radius region of a wall. They additionally build global representations of $FIPS$ in various examples, including one of rank 3. Finally, they propose a conjecture on the intersection multiplicities and semi-orthogonal decompositions (Theorem 3.22), which was proved in [HZ25].

For flopping curves in 3-folds, the stringy Kähler moduli space $\mathscr{M}_{\text{SKMS}}$ is defined in [DW25] as the quotient of normalised Bridgeland Stability conditions and is shown to be a 2-sphere with points missing. The authors also prove an action of the fundamental group of their $\mathscr{M}_{\text{SKMS}}$. Recently in [Var25], the author shows that flops of length 2 can also be constructed as quasi-symmetric GIT problems. The author moreover demonstrates that the recipe for building $\mathscr{M}_{\text{SKMS}}$ and a representation through variation of GIT coincides with that in [DW25].

Less has been done on the A-side, however there is some related work. Here we consider the "purely symplectic" version of toric homological mirror symmetry, where X is a toric variety and we replace the Hori-Vafa potential in the mirror with a stop: a mostly Legendrian subset $\mathfrak{f}$ of the contact boundary $Y = \partial_\infty(\mathbb{C}^*)^{n-r}$. Associated with the pair $((\mathbb{C}^*)^{n-r}, \mathfrak{f})$ is a partially wrapped Fukaya category $\mathscr{W}((\mathbb{C}^*)^{n-r}, \mathfrak{f})$ which is derived equivalent to $D^b(X)$.

For example, each chart of $FIPS$ corresponding to quotient X contains a real torus, and in [Han19] they show that its monodromy induces an action on the A-side which is mirror to tensoring by a line bundle.

In [DK21; Zho25], the authors provide examples of derived autoequivalences arising from the microlocal sheaf theory of $((\mathbb{C}^*)^{n-r}, \mathfrak{f})$. In these papers, the window equivalences of [Seg11] are produced on the A-side by studying modifications of the FLTZ skeleta via stop removals. These modifications are inspired by (but not actually given by) deformations of the Legendrian stop. See, for instance, [Zho25, Sec. 0.3.1 or Example 1.1].

In joint work with Jeff Hicks and Andrew Hanlon (preprint in preparation) we construct a moduli space on the A-side for A_n examples and find an induced an action of the fundamental group on the partially wrapped Fukaya category. Moreover, we show that the braid group injects into this fundamental group, in line with [ST01].

Chapter 2

Toric Geometry and Derived Categories

2.1 Linear Toric Geometric Invariant Theory

We start with a choice of algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $V \cong \mathbb{C}^n$. We call this a *linear toric geometric invariant theory (GIT) problem* of rank r and dimension n. For a choice of coordinates on T and V the action is of the form

$$(\lambda_1, \dots, \lambda_r) \cdot (z_1, \dots, z_n) = (\lambda_1{}^{q_{11}} \lambda_2{}^{q_{21}} \dots \lambda_r{}^{q_{r1}} z_1, \dots, \lambda_1{}^{q_{1n}} \lambda_2{}^{q_{2n}} \dots \lambda_r{}^{q_{rn}} z_n), \quad (2.1)$$

where $q_{ij} \in \mathbb{Z}$ for $i = 1, \dots, r,\ j = 1, \dots, n$. For $j \in 1, \dots, n$, we define the r-dimensional vectors $q_j := (q_{ij})_{i=1}^r$ which we call the *weights* of the GIT problem.

We would like to construct a quotient space that parametrises the action's orbits, and GIT is the algebro-geometric theory that tells us how we should do this. More specifically, GIT produces *unstable loci*, which are orbits that we throw away before we quotient in order to get a 'good' or 'geometric' quotient [MFK94]. We call these quotients *GIT quotients*, which are often denoted $V /\!\!/ T$.

Example 2.1. Consider $\mathbb{C}^*$ acting on $\mathbb{C}^2$ linearly: $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. If we take the quotient space, we see that in the quotient topology (induced by the analytic topology) the orbit of the origin cannot be separated by disjoint open sets from any other orbit. Keeping the origin therefore yields a non-Hausdorff space, so we say that the origin is unstable and we have to remove it. As one would guess, we get GIT quotient

$$\mathbb{C}^2 /\!\!/ \mathbb{C}^* = \mathbb{C}^2 \backslash \{(0, 0)\} / \mathbb{C}^* = \mathbb{P}^1.$$

In general GIT quotients are not unique and depend on a choice of *stability condition*. In linear toric GIT problems the stability condition is just a choice of character

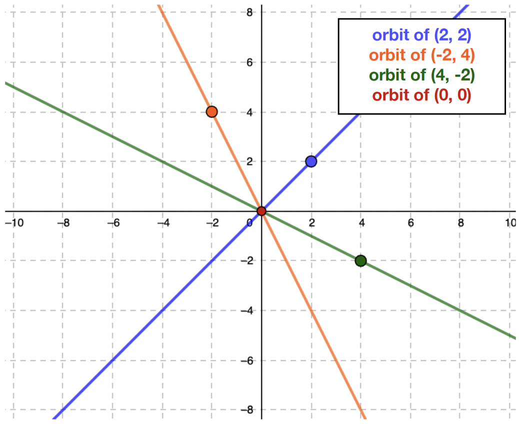


Figure 2.1: Orbits of the $\mathbb{C}^*$ action on $\mathbb{C}^2$ from Example 2.1.

$\theta : T \to \mathbb{C}^*$. The *character lattice* $L^\vee := \mathrm{Hom}(T, \mathbb{C}^*)$ is isomorphic to $\mathbb{Z}^r$, so θ lives in a rank r lattice. Different choices of θ lead us to remove different unstable loci thus give us non-isomorphic (but birational) quotients. We will now explain exactly how, given θ, we combinatorially determine the unstable locus (see equation (2.3)), where we use [CIJ18, Section 4] as our main reference.

We can think of the weights $q_1, \ldots, q_n$ as living in the character lattice $L^\vee$. The unstable locus is determined by the positive cones on subsets of the weights q_i that contain θ. For a subset $I \subset \{1, 2, \ldots, n\}$, and write I^c for the complement of I, and set

$$\angle_I = \left\{ \sum_{i \in I} c_i q_i \,\middle|\, c_i \in \mathbb{R}_{>0} \right\} \subseteq L^\vee_{\mathbb{R}}, \tag{2.2}$$

where we call $\angle_I$ *the strictly positive cone on weights* $\{q_i \mid i \in I\}$. We set $\angle_\emptyset = \{0\}$. Consider a *stability condition* $\theta \in L^\vee$ and set

$$\mathscr{A}_\theta = \left\{ I \subset \{1, 2, \ldots, n\} \mid \theta \in \angle_I \right\}.$$

We make the following assumptions on θ:

1. $\{1, 2, \ldots, n\} \in \mathscr{A}_\theta$,
2. for any $I \in \mathscr{A}_\theta$, $\{q_i \mid i \in I\}$ spans $L^\vee_{\mathbb{R}}$ over $\mathbb{R}$.

The first ensures that our quotient won't be empty and the second ensures that we are picking a generic stability condition, so that our quotient will be *smooth Deligne Mumford (DM) stack*, see Remark 2.3. Consider coordinates $z_1, \ldots, z_n$ on $\mathbb{C}^n$. We define the ideal

$$\mathscr{I}_\theta = \left\langle \prod_{i \in I} z_i \,\middle|\, I \in \mathscr{A}_\theta \right\rangle$$

and *the unstable locus of* θ to be

$$U_\theta = V(\mathscr{I}_\theta). \tag{2.3}$$

The *semi-stable locus* X_θ^{ss} is defined to be the complement $\mathbb{C}^n \setminus U_\theta$. The GIT quotient given by the stability condition $\theta \in L^\vee$ is the following *quotient stack* (see Remark 2.2):

$$X_\theta = [X_\theta^{ss}/T]. \tag{2.4}$$

Remark 2.2 (Quotient stacks)**.** The *quotient stack* $[X/G]$, where X is a variety and G is a possibly infinite group acting on X, is used to think about the G-equivariant geometry on X. When the action of G is free, then the quotient is simply the 'naive' Zariski topological quotient, also called the *coarse moduli space* in this context. When the action is not free, however, we have non-trivial stabiliser subgroups and the stack structure remembers this. A coherent sheaf on $[X/G]$ is a G-equivariant coherent sheaf on X, and if $X = V$ is a vector space then this is the data of a character $G \to \mathbb{C}^*$. The action of G on V induces a $\bar{G} = \mathrm{Hom}(G, \mathbb{C}^*)$-graded structure on the polynomial ring and we have

$$\mathrm{Coh}^G(V) \cong \bar{G}\text{-graded } \mathbb{C}[z_1, \ldots, z_n] \text{ modules.}$$

This implies that for $G = T$, coherent sheaves on $[V/T]$ are $\mathbb{Z}^r$-graded as $\mathbb{C}[z_1, \ldots, z_n]$-modules.

Remark 2.3 (Smooth DM stacks)**.** By smooth DM stack, we mean a geometric object $\mathscr{X}$ that is locally the quotient stack $[U/\Gamma]$ where $U \subset \mathbb{R}^n$ and Γ is a finite group. This in particular implies that stabiliser subgroups are finite[1].

The Picard group of a GIT quotient is given as a quotient of the character lattice $L^\vee \cong \mathbb{Z}^r$, more specifically

$$\mathrm{Pic}\, X_\theta = \frac{L^\vee}{\langle q_i \mid i \in K \rangle}, \tag{2.5}$$

where the subset $K \subseteq \{1, \ldots, n\}$ indexes the coordinate divisors that lie in the unstable locus, i.e. the k such that $\{z_k = 0\} \subseteq U_\theta$ [CLS24].

The space of stability conditions satisfying our assumptions has a *wall and chamber structure*. The *chamber* C_θ to which θ belongs is given by

$$C_\theta = \bigcap_{I \in \mathscr{A}_\theta} \angle_I,$$

[1]Every 'point' in a DM stack $x \in \mathscr{X}$ has a well-defined finite automorphism group Γ_x coming from any local description, which we refer to as the isotropy group of x. Hence when we refer to a point in $\mathscr{X}$, we mean a *gerbe* $B\Gamma_x = [\mathrm{Spec}\,\mathbb{C}/\Gamma_x]$.

and $X_\theta = X_{\theta'}$ if and only if $\theta' \in C_\theta$ or, equivalently, $\theta \in C_{\theta'}$. We call a codimension 1 sub-cone $W \subset C_\theta$ *a wall* to the chamber. This wall and chamber structure gives us a fan in $L^\vee \otimes \mathbb{R}$, which we call *the secondary fan*[2] Σ_{GKZ}. By construction, top-dimensional cones in Σ_{GKZ} are in one-to-one correspondence with GIT quotients. We note that the 1-dimensional cones in Σ_{GKZ} may be more than just the weights q_i when the rank of the linear toric GIT problem is greater than 2.

Example 2.4. Consider $\mathbb{C}^*$ acting on $\mathbb{C}^3$ via $\lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z)$. The character lattice is isomorphic to $\mathbb{Z}$. For $\theta > 0$, we have unstable locus $Z_+ = \{x = y = 0\}$. We have GIT quotient the total space

$$\mathbb{C}^3 /\!\!/_\theta \mathbb{C}^* = \mathbb{C}^3 \backslash \{x = y = 0\} / \mathbb{C}^* = \mathrm{Tot}\, \mathscr{O}(-1)_{\mathbb{P}^1_{x:y}}.$$

For $\theta < 0$, we have unstable locus $Z_- = \{z = 0\}$.

$$\mathbb{C}^3 /\!\!/_\theta \mathbb{C}^* = \mathbb{C}^3 \backslash \{z = 0\} / \mathbb{C}^* = \mathbb{A}^2_{x,y}.$$

Figure 2.2: Secondary fan of GIT problem $\mathbb{C}^3_{(1,1,-1)}$

There is a DM stack constructed using the secondary fan Σ_{GKZ}, called the *secondary stack* $\mathfrak{F}$. It is a toric DM stack given by a *stacky fan* as per [BCS04]. As a first approximation, a stacky fan is a simplicial fan with a distinguished lattice point on each ray in the fan. More precisely, we have the following definition.

Definition 2.5. [BCS04] A stacky fan $\mathbf{\Sigma}$ is a triple consisting of a finitely generated abelian group N, a simplicial fan Σ in $N_\mathbb{R}$, and a map $\beta : \mathbb{Z}^n \to N$ where n is the number of rays in Σ and the β-image of the standard basis in $\mathbb{Z}^n$ generates the rays in Σ.

A toric fan Σ produces a canonical stacky fan $\mathbf{\Sigma} := (N, \Sigma, \beta)$ where N is the distinguished cocharacter lattice and β is the map defined by the minimal lattice points on the rays. In [BCS04, Sec. 3], the authors show how a stacky fan encodes a group action on a quasi-affine variety, and the associated toric Deligne-Mumford

[2] Also known as the GKZ fan.

stack $\mathscr{X}(\mathbf{\Sigma})$ is defined as the quotient. Note that, if Σ corresponds to a smooth toric variety $X(\Sigma)$ and $\mathbf{\Sigma}$ is the canonical stacky fan associated to Σ, then we simply have $\mathscr{X}(\mathbf{\Sigma}) = X(\Sigma)$, as we would expect.

In this thesis we only deal with linear toric GIT problems of rank less than or equal to 2, for which the stacky fan of $\mathfrak{F}$ is

$$\mathbf{\Sigma_{GKZ}} = (L^\vee, \Sigma_{GKZ}, Q), \tag{2.6}$$

where $Q : \mathbb{Z}^n \to L^\vee$ is called the *weight map*, defined by the weights $q_1, \dots, q_n \in L^\vee$. For a general definition we refer the reader to [Kit19, Subsection 3.4]. A priori defining this stack from the secondary fan may seem pointless. We will see later that the secondary stack is a helpful compactification of one of our central objects of study, the *Fayet-Iliopoulos Parameter Space* (*FIPS*), see Definition 3.6 and equation (3.5).

We consider the cokernel $A^\vee : \mathbb{Z}^n \to N$ of the dual weight map $Q^\vee : L \to (\mathbb{Z}^n)^\vee$ (where we identify $(L^\vee)^\vee \cong L$). We note that N is a finitely generated group which may have torsion. We package this information into the short exact sequence

$$0 \to L \xrightarrow{Q^\vee} (\mathbb{Z}^n)^\vee \xrightarrow{A^\vee} N \to 0. \tag{2.7}$$

We can see $A := (A^\vee)^\vee : M \to \mathbb{Z}^n$ as a $n \times (n-r)$ dimensional matrix by picking a basis for the rank $n-r$ lattice $M = N^\vee$. We define $a_i = A^\vee((e_i)^\vee) \otimes \mathbb{R} \in N_\mathbb{R} \cong \mathbb{R}^{n-r}$ the *rays* of our GIT problem, which are integer vectors.

Given a GIT problem (2.7) and a choice of stability condition $\theta \in L^\vee$, we can also describe our GIT quotients X_θ as toric DM stacks given by stacky fans as per [BCS04]. The the toric fan is given by

$$\Sigma_\theta = \{\sigma_I \subset N_\mathbb{R} \cong \mathbb{R}^{n-r} \mid I^c \in \mathscr{A}_\theta\}. \tag{2.8}$$

We need the data of a choice of generators for the one-dimensional cones of Σ_θ. So let $\beta := A^\vee : \mathbb{Z}^n \to N$ and consider the subset $S \subset \{1, \dots, n\}$ by:

$$S = \{i \in \{1, \dots, n\} \mid \{i\}^c \notin \mathscr{A}_\theta\}.$$

In words, S contains the rays a_i whose cone does not give a one-dimensional cone in Σ_θ. Then the stacky fan of X_θ is given by the triple data

$$\mathbf{\Sigma}_\theta = (N, \Sigma_\theta, \beta'), \tag{2.9}$$

where $\beta' : \mathbb{Z}^{n-|S|} \to N$ is obtained from β by deleting the columns corresponding to the subset S. The toric DM stack obtained from Section 3 of [BCS04] coincides precisely with our GIT quotient as given by the quotient stack (2.4) [CIJ18, Sec. 4].

We finish this subsection with another perspective to equation (2.5) on how to compute the Picard group of X_θ

$$\operatorname{Pic} X_\theta \cong \frac{L^\vee}{\langle q_i \mid i \in S \rangle}. \tag{2.10}$$

2.2 Calabi-Yau Condition

In variation of GIT, toric homological mirror symmetry only really makes sense if a linear toric GIT problem satisfies a *Calabi-Yau* condition.

Definition 2.6 (Calabi-Yau)**.** We say a linear toric GIT problem with T acting on V is *Calabi-Yau* (CY) if T acts through $SL(V)$.

The Calabi-Yau condition will imply that our GIT quotients are all Calabi-Yau, in that they have trivial canonical sheaf. If T acts through $SL(V)$, then for all $(\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$, we require the matrix by which the torus acts

$$\begin{pmatrix} {\lambda_1}^{q_{11}} {\lambda_2}^{q_{21}} \dots {\lambda_r}^{q_{r1}} & 0 & \cdots & 0 \\ 0 & {\lambda_1}^{q_{12}} {\lambda_2}^{q_{22}} \dots {\lambda_r}^{q_{r2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {\lambda_1}^{q_{1n}} {\lambda_2}^{q_{2n}} \dots {\lambda_r}^{q_{rn}} \end{pmatrix}$$

to have determinant 1. This means that

$${\lambda_1}^{q_{11}+\dots+q_{1n}} {\lambda_2}^{q_{21}+\dots+q_{2n}} \dots {\lambda_r}^{q_{r1}+\dots+q_{rn}} = 1,$$

for all $\lambda_i \in \mathbb{C}^*$. So we need the the following sums to be 0:

$$q_{m1} + \dots + q_{mn} = 0,$$

for all $m \in \{1, \dots, r\}$. In other words, we need the sum of the weights to be the zero vector.

Definition 2.7 (Primary Polytope)**.** The *primary polytope* $P \subset N_\mathbb{R} \cong \mathbb{R}^{n-r}$ of a Calabi-Yau linear toric GIT problem is the convex hull of the rays $a_i = A^\vee((e_i)^\vee) \otimes \mathbb{R} \in N_\mathbb{R} \cong \mathbb{R}^{n-r}$.

Because of the Calabi-Yau condition, we can choose a basis for M so that the first column of A as a matrix is a list of 1s. This implies that P lives at height 1 and is $n-r-1$ dimensional.

It is worth noting here that the GIT fan Σ_θ in the stacky fan $\mathbf{\Sigma}_\theta$ (equation (2.9)) is given by taking the positive cone over a (*coherent*)[3] P_θ of P [Gel+94, Ch. 7].

[3] See [Gel+94, p.218] for the definition of coherent triangulation.

Example 2.8. Consider $\mathbb{C}^*$ acting on $\mathbb{C}^3$ via $\lambda \cdot (x,y,z) = (\lambda x, \lambda y, \lambda^{-2} z)$, i.e. that our GIT problem is given by weight matrix $Q = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$. Note that this problem is Calabi-Yau because the weights sum to 0. By a very similar stability analysis to Example 2.4, we find that for stability conditions $\theta \in L^\vee \cong \mathbb{Z}$ our GIT quotients are

$$X_\theta = \begin{cases} \left[\mathbb{C}^3 \setminus \{x=y=0\}/\mathbb{C}^*\right] = \operatorname{Tot} \mathscr{O}(-2)_{\mathbb{P}^1_{x,y}}, & \text{for } \theta > 0, \\ \left[\mathbb{C}^3 \setminus \{z=0\}/\mathbb{C}^*\right] = [\mathbb{A}^2/\mathbb{Z}_2], & \text{for } \theta < 0. \end{cases}$$

We have short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{Q^T} \mathbb{Z}^3 \xrightarrow{A^T} \mathbb{Z}^2 \to 0,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

The primary polytope $P \subset \mathbb{Z}^2 \otimes \mathbb{R} \cong \mathbb{R}^2$ is one-dimensional of length 2, and we have two (coherent) triangulations, P itself and P 'broken in half'. These correspond to our two GIT quotients.

In the next section we see how we can relate the derived categories of different GIT quotients. In particular, for Calabi-Yau problems, the quotients are derived equivalent (Corollary 2.13).

2.3 Wall-Crossing Formula

Let us consider an algebraic torus $T \cong (\mathbb{C}^*)^r$ acting on a vector space $V \cong \mathbb{C}^n$, with GIT problem packaged by the short exact sequence (see (2.7))

$$0 \to L \xrightarrow{Q^\vee} (\mathbb{Z}^n)^\vee \xrightarrow{A^\vee} N \to 0,$$

where Q is the weight map. The seminal paper [Hal15] establishes the theory through which the derived categories of GIT quotients can be studied. The key idea is to work on the derived category of T-equivariant coherent sheaves on V, denoted $D^b_T(V)$. In particular, given any choice of stability condition θ, the author proves the existence of a *semi-orthogonal decomposition*[4] (SOD) of $D^b_T(V)$ of which one component is equivalent to $D^b(X_\theta)$ under restriction.

[4] A semi-orthogonal decomposition of a triangulated category $\mathscr{T}$ is a decomposition of $\mathscr{T}$ into triangulated subcategories

$$\mathscr{T} = \langle \mathscr{T}_1, \ldots, \mathscr{T}_n \rangle$$

such that the categories together generate $\mathscr{T}$ under shifting (think of shifting chain complexes) and taking mapping cones [Huy06, Chapter 1].

Building on this, the general theory of 'windows' allows us to explicitly relate the derived categories of GIT quotients separated by a wall[5] [BFK19; Seg11].

Let C_+ and C_- be two adjacent chambers of the secondary fan separated by a wall W, and labelled such that C_+ lies on the same side of W as *the character*

$$\det V := q_1 + \cdots + q_n \in L^\vee.$$

Let $\lambda_W \in L$ be the primitive 1-parameter subgroup normal to this wall, oriented so that

$$\kappa := (\det V)(\lambda_W) \geq 0. \tag{2.11}$$

Write $X_\pm$ for the GIT quotients corresponding to these two chambers.

Associated to any wall W, we have a 'sub-GIT problem' built from a subset of the data of our original GIT problem. More specifically, we can in general define the notion of *Higgs GIT problems*.

Definition 2.9 (Higgs GIT Problem)**.** Consider a subset $J \subseteq \{1, ..., n\}$ and consider the corresponding set of weights $Q(J)$. We define $L_J^\vee$ as the primitive sublattice generated by these weights

$$L_J^\vee = L^\vee \cap \langle Q(J) \rangle_\mathbb{R} \subset L_\mathbb{R}^\vee.$$

We define a linear toric GIT problem via the weight map $Q_J : \mathbb{Z}^J \to L_J^\vee$, which we call *Higgs GIT problem* associated to J.

Now that we have the general definition of a Higgs GIT problem, we can explain the Higgs GIT problem coming from W.

Definition 2.10 (Higgs GIT for W, Z_W)**.** Consider a wall W and an indexing subset $\mathscr{T} \subset \{1, \ldots, n\}$ for the weights lying on the subspace $\langle W \rangle_\mathbb{R} \subseteq L_\mathbb{R}^\vee$ spanned by the wall over $\mathbb{R}$. The *Higgs GIT problem for W* is the Higgs GIT problem for $\mathscr{T}$ as per Definition 2.9. This gives a rank $(r-1)$, dimension $|\mathscr{T}|$, linear toric GIT problem.

We also define Z_W to be a GIT quotient of the Higgs GIT problem for W, where we choose a stability condition in $L_\mathscr{T}^\vee$ that is inside the cone over the weights lying on W.

Remark 2.11. The vector space corresponding to $\mathbb{Z}^\mathscr{T}$ is the fixed subspace $V^{\lambda_W} \subseteq V$. Also $Q(W)$ span $\langle W \rangle_\mathbb{R} \subseteq L_\mathbb{R}^\vee$ by definition, so $L_\mathscr{T}^\vee$ is exactly the orthogonal to λ_W, i.e. it's the character lattice of T/λ_W. Hence the Higgs GIT problem

[5]The theory of 'windows' applies to a general GIT quotient of a variety by a reductive group. However, currently you cannot use this theory to compare two different GIT quotients unless you assume that the wall-crossing is of a particularly simple form, which is the case in linear toric GIT.

for W is just describing the action of T/λ_W on V^{λ_W}, and Z_W is the GIT quotient $V^{\lambda_W} /\!\!/ (T/\lambda_W)$ with stability condition lying inside the cone on the weights in W in T/λ_W after quotienting [KS22, Sec. 3]. Let $X_0^{ss} \subset V$ be the semi-stable locus of a stability condition lying on the wall (in the original GIT problem given by T acting on V). We can write:

$$Z_W = [V^{\lambda_W} \cap X_0^{ss}/(T/\lambda_W)].$$

Theorem 2.12. [BFK19, Thm. 5.2.1] There exists a semi-orthogonal decomposition (ignoring ordering)

$$D^b(X_+) = \langle D^b(X_-), D^b(Z_W), \ldots, D^b(Z_W) \rangle, \tag{2.12}$$

where Z_W is as in Definition 2.10, and the number of copies of $D^b(Z_W)$ is κ (equation (2.11)).

We say that X_- is *minimal* with respect to X_+ and we will call equation (2.12) the *wall-crossing formula*.

Corollary 2.13. The GIT quotients of a linear toric Calabi-Yau GIT problem are derived equivalent.

Proof. This follows because the Calabi-Yau condition implies $\det V = 0$. □

In fact, between two adjacent linear toric CY GIT quotients separated by a wall, there are countably infinitely many equivalences between them that come from window equivalences [Seg11]. This is a special case of the Bondal-Orlov conjecture, which says that birational Calabi-Yaus are always derived equivalent. This has been proved for three-dimensions by Bridgeland [Bri02] and using different tools by van den Bergh [Ber04]. The conjecture remains open in general, largely because we don't know enough about non-toric higher dimensional birational geometry.

2.4 Spherical Functors

The goal of this section will be to construct *spherical functors* from wall-crossing in a linear toric Calabi-Yau GIT problem. These will give us autoequivalences of the derived categories of our quotients that are closely related to window equivalences. We start first by explaining some general theory about spherical functors.

2.4.1 General Theory

It is a difficult task in general to construct interesting autoequivalences of a derived category. We know by classical results of Bondal and Orlov that in the Fano or anti-Fano case all equivalences arise from automorphisms of varieties. This is far from true in the Calabi-Yau case, where we have already seen that window equivalences imply that some symmetries of the derived category hide birational information too.

The simplest examples of non-obvious autoequivalences of the derived categories are *spherical twists* about *spherical objects*, defined by Seidel-Thomas in their seminal paper [ST01]. Let X be a smooth toric DM stack over a field k.

Definition 2.14. [Huy06, Chapter 8] An object $\mathscr{E} \in D^b(X)$ is called *spherical* if

1. $\mathscr{E} \otimes \omega_X \cong \mathscr{E}$, and
2. $\mathrm{Hom}^i(\mathscr{E},\mathscr{E}) \cong \begin{cases} k & \text{if } i = 0, \dim(X), \\ 0 & \text{otherwise.} \end{cases}$

The name comes from the fact that condition (2) can be rephrased

$$\mathrm{Hom}^i(\mathscr{E},\mathscr{E}) \cong H^i(S^{\dim(X)},k),$$

where $S^{\dim(X)}$ denotes the $\dim(X)$-dimensional sphere. In the context of homological mirror symmetry, spherical objects are mirrors to Lagrangian spheres on the A-side.

Using the cone construction we define a Fourier-Mukai (FM) kernel $\mathscr{P}_{\mathscr{E}} \in D^b(X \times X)$ [Huy06, p.167]:

$$\mathscr{P}_{\mathscr{E}} := C\left(q^*\mathscr{E}^\vee \otimes p^*\mathscr{E} \to \mathscr{O}_\Delta\right),$$

where $q, p : X \times X \to X$ are the projections onto the first and second factors of $X \times X$ respectively, and $\mathscr{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset X \times X$.

More precisely, $\mathscr{O}_\Delta = i_*\mathscr{O}_X$, where i is the diagonal embedding $i : X \hookrightarrow X \times X$. The morphism $q^*\mathscr{E}^\vee \otimes p^*\mathscr{E} \to \mathscr{O}_\Delta$ defining $\mathscr{P}_{\mathscr{E}}$ is the composition of

$$q^*\mathscr{E}^\vee \otimes p^*\mathscr{E} \to i_*i^*(q^*\mathscr{E}^\vee \otimes p^*\mathscr{E}) \cong i_*(\mathscr{E}^\vee \otimes \mathscr{E}),$$

with the direct image of the trace map $tr : \mathscr{E}^\vee \otimes \mathscr{E} \to \mathscr{O}_X$ [Huy06, p.77]. We note here that the FM kernel is not well-defined, as cones are only defined up to non-unique isomorphisms. This isn't an issue though, as isomorphic FM kernels induce naturally isomorphic FM transforms.

Definition 2.15 (Spherical Twist about Spherical Object)**.** The *spherical twist* $T_{\mathscr{E}} : D^b(X) \to D^b(X)$ associated to a spherical object $\mathscr{E} \in D^b(X)$ is by definition the Fourier Mukai transform with kernel $\mathscr{P}_{\mathscr{E}} \in D^b(X \times X)$.

Note that the spherical twist $T_{\mathscr{E}}$ is mirror to Dehn twisting over the mirror Lagrangian sphere to $\mathscr{E}$ on the A-side.

The following formula is very helpful

$$T_{\mathscr{E}}(\mathscr{F}) \cong C\left(\operatorname{Hom}(\mathscr{E},\mathscr{F}) \otimes \mathscr{E} \to \mathscr{F}\right), \tag{2.13}$$

where the map is given by evaluation and all the functors are derived.

The formula for the spherical twist may look familiar, and this is because at the level of K-theory it acts as reflection through the hyperplane orthogonal to the Mukai vector $v(\mathscr{E})$. So in a sense the spherical twist is a categorification of reflection.

Lemma 2.16. For a spherical twist $T_{\mathscr{E}}$ about a spherical object $\mathscr{E} \in D^b(X)$:

1. $T_{\mathscr{E}} \cong T_{\mathscr{E}[k]}$ for any $k \in \mathbb{Z}$,
2. $T_{\mathscr{E}}(\mathscr{E}) \cong \mathscr{E}[1-\dim(X)]$, and
3. for any $\mathscr{F} \in D^b(X)$ with $\operatorname{Hom}(\mathscr{E},\mathscr{F}) = 0$, $T_{\mathscr{E}}\mathscr{F} \cong \mathscr{F}$.

Proof. We start with (1.). We recall that for any $\mathscr{F} \in D^b(X)$, $\operatorname{Hom}(\mathscr{E},\mathscr{F})$ is defined to be the complex with

$$\operatorname{Hom}^i(\mathscr{E},\mathscr{F}) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(\mathscr{E}^j,\mathscr{F}^{j+i}).$$

So $\operatorname{Hom}(\mathscr{E}[k],\mathscr{F}) = \operatorname{Hom}(\mathscr{E},\mathscr{F})[-k]$. Additionally, for any $\mathscr{A},\mathscr{B} \in D^b(X)$, $\mathscr{A} \otimes \mathscr{B}$ is given by the complex

$$(\mathscr{A} \otimes \mathscr{B})^i = \bigoplus_{j+k=i} \mathscr{A}^j \otimes \mathscr{B}^k,$$

so $\mathscr{A}[n] \otimes \mathscr{B}[m] = (\mathscr{A} \otimes \mathscr{B})[n+m]$ for $n,m \in \mathbb{Z}$. Hence we find $\operatorname{Hom}(\mathscr{E}[k],\mathscr{F}) \otimes \mathscr{E}[k] = \operatorname{Hom}(\mathscr{E},\mathscr{F})[-k] \otimes \mathscr{E}[k] = \operatorname{Hom}(\mathscr{E},\mathscr{F}) \otimes \mathscr{E}$. The result follows.

(2.) follows from the Definition 2.14 of spherical object, because

$$\operatorname{Hom}(\mathscr{E},\mathscr{E}) \otimes \mathscr{E} = \mathscr{E} \oplus \mathscr{E}[-\dim(X)],$$

and taking to cone over $\mathscr{E} \oplus \mathscr{E}[-\dim(X)] \to \mathscr{E}$ yields $\mathscr{E}[1-\dim(X)]$. Lastly, (3.) follows directly by definition, as $T_{\mathscr{E}}\mathscr{F} = C(0 \to \mathscr{F}) = \mathscr{F}$.

$\square$

Proposition 2.17. [ST01, Theorem 1.2] The spherical twist $T_{\mathscr{E}} : D^b(X) \to D^b(X)$ about any spherical object $\mathscr{E} \in D^b(X)$ is an autoequivalence.

By Proposition 2.17, we have an inverse $T_{\mathscr{E}}^{-1}$. The FM kernel of $T_{\mathscr{E}}^{-1}$ is not as easily described as for $T_{\mathscr{E}}$, though we know that the inverse is a FM transform by, for example, [Huy06, Corollary 5.17]. We can give a similar formula to (2.13) for it though

$$T_{\mathscr{E}}^{-1}(\mathscr{F}) = C(\mathscr{F} \to \mathrm{Hom}(\mathscr{F},\mathscr{E})^{\vee} \otimes \mathscr{E}). \tag{2.14}$$

Theorem 1.2 in [ST01] also includes the following result.

Proposition 2.18. [ST01, Theorem 1.2] Let $\mathscr{E},\mathscr{F} \in D^b(X)$ be spherical objects such that $\dim \mathrm{Hom}(\mathscr{E},\mathscr{F}) = 1$. Then the spherical twists satisfy the braid relation

$$T_{\mathscr{E}}T_{\mathscr{F}}T_{\mathscr{E}} = T_{\mathscr{F}}T_{\mathscr{E}}T_{\mathscr{F}}.$$

Spherical twists about *spherical functors* are generalisations of spherical twists about spherical objects. Our main reference for spherical twists and functors is [Add16, Section 1].

Definition 2.19 (Twists)**.** Consider an exact functor $F : \mathscr{A} \to \mathscr{B}$ between triangulated categories $\mathscr{A}$ and $\mathscr{B}$, with left and right adjoints $L,R : \mathscr{B} \to \mathscr{A}$. We define the *twist* $T_F : \mathscr{B} \to \mathscr{B}$ to be the cone on the counit of the adjunction $FR \xrightarrow{\varepsilon} \mathrm{id}_{\mathscr{B}}$, so that there is an exact triangle

$$FR \xrightarrow{\varepsilon} \mathrm{id}_{\mathscr{B}} \to T_F, \tag{2.15}$$

and the *cotwist* $C_F : \mathscr{A} \to \mathscr{A}$ to be the cone on the unit of the adjunction $\mathrm{id}_{\mathscr{A}} \xrightarrow{\eta} RF$, so that there is an exact triangle

$$\mathrm{id}_{\mathscr{A}} \xrightarrow{\eta} RF \to C_F. \tag{2.16}$$

Strictly speaking, at this stage the twist and cotwist are not well-defined. This is again because cones are non-functorial and are only defined up to non-unique isomorphism. In our examples though (see Subsection 2.4.2), our functor F is a FM transform and it follows that T_F and C_F can be given as FM transforms too (because R, FR, LF are FM transforms). This bypasses the problem with cones not being functorial[6], because although the FM kernel will be only defined up to isomorphism, FM transforms of isomorphic FM kernels yield FM transforms that are naturally isomorphic. So the functors are defined up to natural isomorphism.

Definition 2.20 (Spherical Functors)**.** We define F to be a *spherical functor* if C_F is an equivalence and $R \cong C_F L$. We call the twist (Definition 2.19) T_F a *spherical twist* if F is a spherical functor.

[6] Twists and cotwists can also be well-defined in the general setting via *dg-enhancements* [AL17].

Theorem 2.21. [Rou06; AL17] The spherical twist $T_F : \mathscr{B} \to \mathscr{B}$ about a spherical functor $F : \mathscr{A} \to \mathscr{B}$ is an autoequivalence.

The inverse spherical twist $T_F^{-1} : \mathscr{B} \to \mathscr{B}$ is given by:

$$T_F^{-1} \to \mathrm{id}_{\mathscr{B}} \xrightarrow{\eta} FL. \tag{2.17}$$

Example 2.22. How is a spherical twist about a spherical functor a generalisation of the spherical twist about a spherical object? Suppose we have a spherical object $\mathscr{E} \in D^b(X)$. We define:

$$F = \mathscr{E} \otimes - : D^b(pt) \to D^b(X).$$

We can prove that F is a spherical functor, and since $R = \mathrm{RHom}(\mathscr{E}, -)$, the twist $T_F : D^b(X) \to D^b(X)$ is exactly like the twist $T_{\mathscr{E}}$ we defined earlier. Note that $R \cong C_F L$ just amounts to condition (1) in Definition 2.14.

Example 2.23. Spherical twists about spherical functors are also generalisations of line-bundle twists $\otimes \mathscr{L} : D^b(X) \to D^b(X)$. Let $i : X \hookrightarrow \mathrm{Tot}\, \mathscr{L}^{-1}$ be the inclusion of the zero-section in the total space of the dual line-bundle $\mathscr{L}^{-1}$. It's easy to show that the spherical twist of the derived pullback $i^* : D^b(\mathrm{Tot}\, \mathscr{L}^{-1}) \to D^b(X)$ realises the line-bundle twist up to a shift. Via line-bundle divisor correspondence $\mathscr{L} = \mathscr{O}(D)$, we can alternatively do this by taking the spherical twist about the derived pushforward I_* of the inclusion $I : D \hookrightarrow X$ when D is effective.

Lemma 2.24. Let $\Phi : D^b(X) \to D^b(X)$ be an autoequivalence. Then

$$\Phi \circ T_F \circ \Phi^{-1} \cong T_{\Phi \circ F}.$$

Proof. Taking cones commutes with applying exact functors, so the conjugation of T_F by Φ passes inside the cone. This means that

$$\Phi \circ T_F \circ \Phi^{-1} = C(\Phi F R \Phi^{-1} \to \mathrm{id}_{D^b(X)}),$$

and note that the right adjoint to ΦF is $R\Phi^{-1}$. □

2.4.2 Spherical Twists in linear toric Calabi-Yau GIT

Let us consider a linear toric Calabi-Yau GIT problem with $T \cong (\mathbb{C}^*)^r$ acting on $V \cong \mathbb{C}^n$. Take a GIT quotient X and an adjacent wall W in the secondary fan. In this subsection we will explain how to obtain a spherical functor $F : D^b(Z_W) \to D^b(X)$ where Z_W is the variety in the wall-crossing formula (2.12). Note that in [HS16], Halpern-Leistner and Shipman proved that the autoequivalences coming from window equivalences through wall-crossing are the same as those from these spherical functors.

In §2.1 we defined the unstable locus for a choice of stability condition $\theta \in L^\vee$ in a linear toric GIT problem. The unstable locus can also be determined via the *Hilbert-Mumford Criterion*, an approach that works in more general non-abelian GIT problems too. The key idea is that to analyse the stability of points in V, it's enough to study their stability under one-parameter subgroups [MFK94]. A point will then be (semi-)stable if it is (semi-)stable for all one-parameter subgroups.

Via the Hilbert-Mumford numerical criterion one can produce a special stratification of the unstable locus $U_\theta \subset V$ by disjoint locally closed T-equivariant subvarieties, called Kempf-Ness (KN) strata:

$$U_\theta = \bigsqcup_{i=1}^{k} S_{\lambda_i}$$

where $\lambda_1, \dots, \lambda_k \in L$ is an ordered list of 'destablising' one-parameter subgroups determined by the choice of stability $\theta \in L^\vee$ and S_i are *blades* as defined as [HS16, Sec. 2]:

$$S_i = \{x \in V \setminus \sqcup_{j>i} S_j \mid \lim_{t\to\infty} \lambda_i(t) \cdot x \in Z^i\},$$

where $Z^i \subseteq V$ is the fixed locus under the $\mathbb{C}^*$ action of λ_i. Note that we have a projection $\Pi_i : S_i \to Z^i$ defined by $\Pi_i(x) = \lim_{t\to\infty} \lambda_i(t) \cdot x$, making S_i a vector bundle over Z^i [Bia73, Thm. 4.1].

Now consider two chambers C_+ and C_- that share a wall. Let λ_+ be a one-parameter subgroup normal to the wall that is pointing into the chamber for C_+. The key idea is that we can find compatible lists of destabilising one-parameter subgroups for the chambers C_+ and C_-.

More explicity, for C^+ we have ordered list of destabilising one-parameter subgroups $\lambda_+, \lambda_2, \dots, \lambda_k$, and for C^- we have ordered list of destabilising one-parameter subgroups $\lambda_-, \lambda_2, \dots, \lambda_k$, where the point is that apart from the first one parameter subgroups differing by multiplication by -1, the rest of the one parameter subgroups are the same. This means that the stratifications of the unstable loci of C_- and C_+ only differ by $S_+ := S_{\lambda_+}$ and $S_- := S_{\lambda_-}$, the strata associated to λ_+ and λ_- in the aforementioned ordered lists for C_+ and C_-.

Using KN stratifications and the Hilbert-Mumford numerical criterion, [BFK19, Thm. 5.1.4] prove the following disjoint unions

$$X_0^{ss} = X_+^{ss} \sqcup S_+ = X_-^{ss} \sqcup S_-.$$

where $X_0^{ss} \subset V$ is the semi-stable locus for a stability condition lying in the wall, and $X_+^{ss} \subset V$ and $X_-^{ss} \subset V$ denote the semi-stable loci associated to C_+ and C_- respectively.

We have GIT quotients $X_+ = [X_+^{ss}/T]$ and $X_- = [X_-^{ss}/T]$. Note that the torus $T/\lambda_\pm$ acts naturally on Z, the fixed locus under $\lambda_\pm$. We consider the diagram from [HS16, Sec. 3.1]

$$\begin{tikzcd} {[E/T]} \arrow[r, "i"] \arrow[d, "\pi"] & X_+ \\ Z_W & \end{tikzcd}$$

where we note that $Z_W = [(Z \cap X_0^{ss})/(T/\lambda_\pm)]$ by definition (Remark 2.11), and $E := S_- \cap X_+^{ss}$. The projection π is induced by the projection $\Pi_- : S_- \to Z$, and is a weighted projective (stacky) bundle. The map i is given by inclusion.

Theorem 2.25 (GIT Spherical Functors)**.** The functor $F := i_* \pi^*$ (where all of the functors are derived) is a spherical functor.

Proof. This follows via the EZ-diagram construction in [Hor04] and the fact that $\mathscr{O}_{[E/T]}$ is a EZ-spherical object [HS16, Remark 3.7]. □

Remark 2.26. A helpful way to think about the content of the theorem above is that over each point $z \in Z_W$, the fibre of E is a weighted projective space P, and the fibre-wise normal bundle N satisfies $\det(N) = K_P$, because of the Calabi-Yau condition. So E is just a family (over Z_W) of standard orbifold flops.

Note that the spherical twist T_F is well-defined because F is given by a FM kernel[7]. For any morphism of smooth DM stacks $f : X \to Y$, the right adjoint $f^! : D^b(X) \to D^b(Y)$ to (derived) f_* is given by [Huy06, Corollary 3.35]:

$$f^! : \mathscr{E} \mapsto Lf^*(\mathscr{E}) \otimes \omega_f[\dim(f)],$$
$$\omega_f := \omega_X \otimes f^* \omega_Y^*, \ \dim(f) := \dim(X) - \dim(Y).$$

Therefore the right adjoint to F is $R = \pi_* i^!$, with the pushforward derived.

Though the number of copies of $D^b(Z_W)$ in $D^b(X)$ is 0 (Corollary 2.13), we use the wall-crossing formula on $D^b(Z_W)$ itself in the Higgs GIT problem for W (Definition

[7]The FM kernel of pushforward and pullback are given by the structure sheaf of the graph in the product.

2.10), which may not be Calabi-Yau. Upon applying the wall-crossing formula once (Theorem 2.12), we find

$$D^b(Z_W) = \langle D^b(Z'), D^b(Y), ..., D^b(Y)\rangle, \tag{2.18}$$

where Z' is a GIT quotient in the Higgs GIT problem for W that is separated by a wall W' from Z_W, and Y is a subvariety coming from a the Higgs GIT problem associated to W'. Notice that this is also a Higgs GIT problem coming from the original GIT problem Q. We can iteratively apply the wall-crossing formula on $D^b(Z')$ and $D^b(Y)$ until we can no longer cross to a more minimal quotient. This always terminates after a finite number of steps, as each step strictly decreases the rank of the GIT problems. We reach *a minimal quotient*, and we get an SOD (ignoring ordering):

$$D^b(Z_W) = \langle D^b(Z_0), \dots, D^b(Z_0), \dots, D^b(Z_k), \dots, D^b(Z_k), \rangle \tag{2.19}$$

where each Z_i is a minimal GIT quotient for a Higgs problem of our original GIT problem. In [KS22] the subcategories occurring in the final decomposition and their multiplicities are independent of all wall-crossing choices.

Theorem 2.27. Suppose you have a linear toric GIT problem, a wall W in the secondary fan, and a SOD for $D^b(Z_W)$:

$$D^b(Z_W) = \langle \mathscr{A}_1, \dots, \mathscr{A}_m\rangle.$$

Then the restriction of the spherical functor $F : D^b(Z_W) \to D^b(X)$ to components in the SOD (2.19) is spherical functor $F_i : \mathscr{A}_i \to D^b(X)$, where we mean restriction in the sense of [Huy06, Chapter 1]. Additionally, the spherical twist T_F factors by the spherical twists about the F_i:

$$T_F \cong T_{F_1} \circ \cdots \circ T_{F_m}. \tag{2.20}$$

Proof. The formal result is [HS16, Theorem 4.14] and it applies in this situation since the cotwist around F is (up to a shift) the Serre functor on $D^b(Z_W)$. □

In [KS22], they show that the irreducible factors Z_i come from Higgs problems associated to *relevant subspaces*, as defined below.

Definition 2.28. Let $H \subseteq L^\vee_{\mathbb{R}}$ be a proper subspace (i.e. not all of $L^\vee_{\mathbb{R}}$), and let $\sigma_H \subseteq H$ be the cone on the weights lying on H. We call $H \subset L^\vee_{\mathbb{R}}$ a *relevant subspace* if $\sigma_H = H$.

Remark 2.29. In [KS22], they do not require a relevant subspace to be proper, while in [Kit19, Lemma 3.47] they do. If we do allow $L_{\mathbb{R}}^{\vee}$ to be considered relevant, than it being relevant equates to the original problem having non-empty minimal GIT quotient. We chose to only consider proper subspaces in this thesis as it yields a cleaner correspondence between certain relevant subspaces and components of the *discriminant locus*, as we will see in §3.1, particular Proposition 3.16.

A relevant subspace H defines a Higgs GIT problem via the weight matrix $Q(H)$ of weights lying on H. These Higgs GIT problems will have a non-empty minimal GIT quotient Z_H. The important point is that Z_H are precisely the factors $Z_0,\dots,Z_k$ in the SOD (2.19) for $D^b(Z_W)$. In other words, the relevant subspaces H are in one-to-one correspondence with the irreducible factors.

By Theorem 2.27, for each wall and associated spherical functor $F : D^b(Z_W) \to D^b(X)$, the twist factors according to the SOD (2.19) for $D^b(Z_W)$. So for each relevant subspace H, we get a spherical functor $F_H : D^b(Z_H) \to D^b(X)$ via restriction from F.

Chapter 3

Toric Homological Mirror Symmetry

In Chapter 2 we explained what we mean by a linear toric CY GIT problem. In doing so, we developed the B-side, or 'algebraic' side, of the homological mirror symmetry (HMS) context that this thesis is set in. There are various formulations of toric HMS, but we will be particularly interested in the *Hori-Vafa* perspective.

Let us consider a torus $T \cong (\mathbb{C}^*)^r$ acting on $V \cong \mathbb{C}^n$, with data packaged by the short exact sequence

$$0 \to L \xrightarrow{Q^\vee} (\mathbb{Z}^n)^\vee \xrightarrow{A^\vee} N \to 0. \tag{3.1}$$

where Q is the weight map. Assume that our problem also satisfies the CY condition (2.6) and recall that our GIT quotients are therefore derived equivalent by Corollary 2.13. If we choose a basis for $M = N^\vee$, A as a map is represented by $n \times (n-r)$-dimensional integer matrix. In this chapter, we will think of A as a set of the matrix rows $a_i = A^\vee((e_i)^\vee) \otimes \mathbb{R} \in N_\mathbb{R} \cong \mathbb{R}^{n-r}$, otherwise called rays of our GIT problems. The rays are integer vectors, so belong to $\mathbb{Z}^{n-r} \subseteq \mathbb{R}^{n-r}$. So, from now on A is a finite set of n, $(n-r)$-dimensional integer vectors.

The *Hori-Vafa superpotential* W associated to our GIT problem is the Laurent polynomial with Newton polytope given by the rays of our GIT problem [HV00]. The allowed coefficients of W lie outside a *discriminant locus* $\nabla \subseteq \mathbb{C}^n$ which is the zero-locus of what is called the *principal A-determinant* E_A (Definition 3.3). E_A is studied in detail by Gelfand, Kapranov and Zelevinski in [Gel+94] and we will review some of their most important results in §3.1.

This gives us a parameter space over which our mirror LG-models live, which we will call the *Fayet Iliopoulos Parameter Space* $FIPS$ (Definition 3.6).

Homological mirror symmetry predicts that $\pi_1(FIPS)$ acts on the derived categories of our GIT quotients. The upshot of this chapter will be that the discriminant locus decomposes into irreducible components

$$\nabla = \bigcup_{i=0}^{k} \nabla_i,$$

and each such component will correspond to a variety Z_i from the SOD in equation (2.19). This was first observed in [APW17], and hence leading them to conjecture that the action of $\pi_1(FIPS)$ is such that a *meridian* (Definition 4.7) of ∇_i will act on the derived category of a GIT quotient X by twists by spherical functors $F_i : D^b(Z_i) \to D^b(X)$ as described in §2.4.2. There is some deliberate ambiguity here; there is no canonical meridian around ∇_i (even up to homotopy), so the functors $F_i : D^b(Z_i) \to D^b(X)$ are at best defined up to composition by autoequivalences.

3.1 $FIPS$ and GKZ Discriminants

In this section we discuss the *principal A-determinant* E_A as defined in [Gel+94]. A friendly and brief introduction to the topic can be found in [Seg24].

Let $A \subseteq \mathbb{Z}^{n-r}$ be the set of rays of a n-dimensional, rank-r, linear toric CY GIT problem, so that $|A| = n$.

Remark 3.1. The theory in [Gel+94] is built more generally for any finite subset A of integer vectors.

We take $n-r$ variables $x_1,...,x_{n-r}$. There is a generic Laurent polynomial W corresponding to our linear toric CY GIT problem that is a sum of the monomials for every element of A, depending on n coefficients $c_1,\ldots,c_n \in \mathbb{C}$. Explicitly, to any ray $a_i \in \mathbb{Z}^{n-r} \subseteq N_{\mathbb{R}}$, we associate the Laurent monomial $x^{a_i} = x_1^{(a_i)_1} \ldots x_{n-r}^{(a_i)_{n-r}}$. The Hori-Vafa superpotential is given by

$$W(x_1,\ldots,x_{n-r}) = \sum_{a_i \text{ ray}} c_i x^{a_i}. \tag{3.2}$$

We call the pair $((\mathbb{C}^*)^{n-r}, W)$ a *Landau-Ginzburg model*, which we refer to as *Hori-Vafa mirrors* to the GIT problem.

The hypersurface given by the vanishing

$$V(W) \subseteq \mathbb{C}^{n-r},$$

is singular if and only if there is an $(x_i) \in \mathbb{C}^{n-r}$ such that these $n-r+1$ polynomials vanish simultaneously:

$$W,\ \partial W/\partial x_1,\ldots,\partial W/\partial x_{n-r}.$$

So $V(W)$ is smooth for a generic choice of parameters, and singular for some locus in $\mathbb{C}^n$. Note that the locus where W is singular is always irreducible [Gel+94, Ch. 9], and if it is codimension 1 we define the *A-discriminant* to be the polynomial[1]

$$\Delta_A \in \mathbb{C}[c_1, \dots, c_n], \tag{3.3}$$

cutting it out. Otherwise we formally declare $\Delta_A := 1$. We write $\nabla_A := V(\Delta_A) \subseteq \mathbb{C}^n$. Note that for subsets $\Gamma \subseteq A$, we can define $\Delta_\Gamma \in \mathbb{C}[c_1, \dots, c_n]$ and $\nabla_\Gamma \subseteq \mathbb{C}^n$ in the same way, by considering the Laurent polynomial $W_\Gamma(x_1, \dots, x_{n-r})$ of monomials coming from Γ.

In very nice examples (like Example 3.13 or Examples (a) and (b) in Chapter 5) we can very easily compute Δ_A by solving the system of equations by hand. In general though computing Δ_A is difficult, where the discriminant of the cubic already requires significant work.

There is a general method which in principle allows one to compute any resultant called *the Cayley Method* [Gel+94, Ch. 2], although it quickly becomes computationally involved. Fortunately for us, Macaulay2's "HolonomicSystems" package implements the algorithm, which we will be using for Examples (c) and (d) in Chapter 5. See Appendix A for a sample of code.

Remark 3.2. There is an insightful definition of the A-discriminant in terms of projective duality, see [Seg24, Subsec. 2.1].

Recall Definition 2.7 of the primary polytope $P \subseteq N_\mathbb{R} \cong \mathbb{R}^{n-r}$ as the real convex hull of the rays $a_i = A^\vee((e_i)^\vee) \otimes \mathbb{R}$. Consider a face $\Gamma \subseteq P$, where we consider faces of any dimension, including P itself. We consider all the rays that lie in this face, and (abusing notation) write $\Gamma \subseteq A$ to denote the subset of these rays too.

Definition 3.3 (Principal A-Determinant)**.** We define the *principal A-determinant* as the product

$$E_A = \prod_{\text{faces } \Gamma \subseteq A} \Delta_\Gamma^{u_\Gamma}, \tag{3.4}$$

where u_Γ are certain multiplicities that are addressed in [Gel+94, p.299][2] but that we won't explain or need in this thesis. Note that the product runs over all faces of A of all dimensions, including A itself.

Remark 3.4. If A' is a vertex of A then $W_{A'}$ is a single monomial, and $\nabla_{A'}$ is just the coefficient of this monomial. So each variable lying at a vertex of A appears as a factor of E_A.

[1] The locus where W is singular is not always codimension 1, see Example 3.13.

[2] We're citing results in [Gel+94] by page number because their numbering conventions can be confusing.

Definition 3.5. We define the *discriminant locus* of A to be $\nabla := V(E_A) \subseteq \mathbb{C}^n$.

By construction, we note that the Laurent polynomial W is quasi-homogeneous with respect to the action of the weights in the matrix representation for $A^\vee$. Hence, $\nabla \subseteq \mathbb{C}^n$ is invariant under the torus action of $(\mathbb{C}^*)^{n-r}$ defined by A^T as a weight matrix. We can now finally define $FIPS$.

Definition 3.6. The *Fayet-Iliopoulous parameter space* ($FIPS$) is defined as the quotient stack (see Remark 2.2)

$$FIPS = \left[(\mathbb{C}^n \backslash V(E_A)) \big/_A (\mathbb{C}^*)^{n-r} \right],$$

where $(\mathbb{C}^*)^{n-r}$ acts with weight matrix given by A^T and E_A is defined in Definition 3.3.

We call the irreducible factor Δ_A the *principal discriminant* and ∇_A the *principal component*. We will see that although we take the product over all faces, only certain faces will actually matter, see Definition 3.12.

Remark 3.7 (Resultants)**.** The A-discriminant and the principal A-determinant are both defined in [Gel+94] in terms of a more general problem called 'resultants'. The definition we give for E_A is actually Theorem 1.2 on page 299, with the original definition on page 297.

One of the most beautiful results in [Gel+94] is Theorem 1.4 in Chapter 10, which states that the extremal monomial terms of E_A are in 1-1 correspondence with the (coherent) triangulations of the primary polytope P. More specifically they can be computed as the product of the following formula for each simplex in a triangulation

$$(\text{volume of simplex} \times \text{vertices})^{\text{volume}}.$$

This is completely non-obvious and shows the depth of the combinatorial results in the book. The argument relates to extremal toric degenerations of the embedded projective variety coming from A that we aluded to in Remark 3.2.

The same theorem also gives us another very interesting result.

Theorem 3.8. [Gel+94, Ch.10, Thm 1.4] The Newton polytope of E_A is dual to the secondary fan.

Theorem 3.8 implies that the secondary toric stack $\mathfrak{F}$, defined in equation (2.6), is a natural compactification of $FIPS$:

$$FIPS \subseteq \mathfrak{F}. \tag{3.5}$$

To see this, we note that by Theorem 3.8, E_A extends to a global section of a line bundle on $\mathfrak{F}$ and hence we can consider the vanishing locus ∇ of E_A as inside $\mathfrak{F}$. This gives us *FIPS* as a complement inside $\mathfrak{F}$, which will be a useful perspective, for example because toric orbit-cone correspondence [CLS24, Chapter 3] implies that the fixed torus points in $\mathfrak{F}$ are in 1-1 correspondence with GIT quotients.

Corollary 3.9. [KS22, Corollary 4.11] The non-toric components of $\nabla \subseteq \mathfrak{F}$ avoid all toric fixed points in $\mathfrak{F}$.

Definition 3.10 (Large Radius Limit)**.** Given a linear toric CY GIT problem with GIT quotient X, we define the *large radius limit* of X to be the corresponding fixed point $c_X \in \mathfrak{F}$.

From the Corollary above, for each GIT quotient X, there is an (analytic) open neighbourhood $V_X \subseteq \mathfrak{F}$ of $c_X \in \mathfrak{F}$ which is invariant under the compact torus $(S^1)^r \subseteq T$ and avoids the non-toric parts of the discriminant. Additionally, V_X is disjoint from $V_{X'}$ for all GIT quotients $X' \neq X$.

Definition 3.11 (Large Radius Region)**.** We define the *large radius region* $L_X \subseteq FIPS$ to be the intersection of $FIPS$ with $V_X \subseteq \mathfrak{F}$ of c_X.

Not all subsets $\Gamma \subseteq A$ give us non-trivial discriminant components. A certain type will be particularly important for us, those associated to *minimal faces* of our primary polytope P.

Definition 3.12. [KS22, Sec. 4.2.2] A face $\Gamma \subseteq P$ of the primary polytope is called *minimal* if every ray in Γ is linearly dependent on the other rays in Γ. Any face that is not minimal is called *redundant.*

Redundant faces that are not vertices $\Gamma \subseteq A$ do not contribute to the product in Definition 3.3 of E_A, because the locus where W_Γ is singular will be codimension greater than 1. Therefore, only vertices and minimal faces $\Gamma \subseteq A$ have non-trivial Δ_Γ [Kit19][Rem. 3.7], where the vertices only contribute toric boundary divisors. We have the decomposition of the non-toric part of the discriminant locus into irreducible components

$$\nabla_{\text{non-toric}} = \bigcup_{\text{min faces } \Gamma \subseteq A} \nabla_\Gamma. \tag{3.6}$$

Since the primary polytope P is $(n-r-1)$-dimensional, that means that its number of vertices is bounded below by $n-r$. An important consequence of this is that we can use the $(\mathbb{C}^*)^{n-r}$ action in the definition of $FIPS$ to scale $(n-r)$-coordinates to

1, with possibly some residual finite group G acting (a semi-direct product of cyclic groups) as a result of multiple scaling choices. So in fact this means we can write

$$FIPS = [(\mathbb{C}^r \backslash \bar{\nabla})\,/\,G], \tag{3.7}$$

where $\bar{\nabla}$ is the image of ∇ under the quotient map. This gives us $FIPS$ clearly as a smooth DM stack (see 2.3), specifically a hypersurface complement modulo a finite group.

Example 3.13. Consider the GIT problem with weight matrix $Q = \begin{pmatrix} 2 & -1 & -1 & 0 \end{pmatrix}$. We have the following matrix representation for A (the dual of the cokernel of $Q^\vee$, $A^\vee$)

$$A = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}.$$

The primary polygon is as in Figure 3.1 below. We have only have one minimal face, that is the convex hull of Γ. Additionally, we know that the coefficients corresponding to the vertices are factors of E_A (see Remark 3.4). Therefore $\nabla = V(bcd\Delta_\Gamma) \subseteq \mathbb{C}^5$ by Equation (3.6).

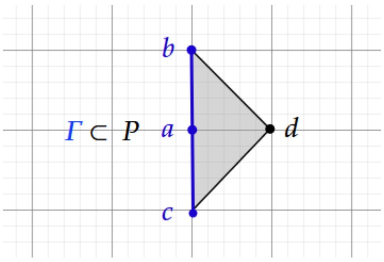


Figure 3.1: The primary polygon P and minimal face $\Gamma \subseteq P$.

We now compute the discriminant components. We assign independent variables x, y, z to the columns of A. By definition (3.2), we have superpotential

$$\begin{aligned} W(x,y,z) &= ax + bxz + cxz^{-1} + dxy, \\ &= x(a + bz + cz^{-1} + dy). \end{aligned}$$

Consider $\Gamma = \{1,2,3\}$. We have associated Laurent poynomial and derivatives

$$\begin{aligned} W_\Gamma(x,y,z) &= x(a+bz+cz^{-1}), \\ \frac{\partial W_\Gamma}{\partial x} &= a+bz+cz^{-1}, && (3.8) \\ \frac{\partial W_\Gamma}{\partial y} &= 0, \\ \frac{\partial W_\Gamma}{\partial z} &= x(b-cz^{-2}). && (3.9) \end{aligned}$$

(3.9) $= 0$ implies $z = \pm\sqrt{\frac{c}{b}}$. Plugging this into (3.8) $= 0$ we find $\Delta_\Gamma = a^2 - 4cb$. So

$$FIPS = [\mathbb{C}^4 \backslash V(bcd(a^2-4cb))/_A(\mathbb{C}^*)^3].$$

We can use the rank 3 torus action to scale out non-zero coordinates b, c, d from $FIPS$ and we get the smooth DM stack

$$FIPS \cong [\mathbb{C}_a \backslash \{\pm 2\} \,/\, \mathbb{Z}_2].$$

Note that in our definition of the Hori-Vafa superpotential W (equation (3.2)), we first fixed bases for N and $M = N^\vee$. We can, however, write W in a more coordinate free way.

Tensoring the short exact equation (3.1) and its dual yields two exact sequences of tori:

$$L_{\mathbb{C}^*} \xrightarrow{Q^\vee_\times} (\mathbb{C}^*)^n \xrightarrow{A^\vee_\times} N_{\mathbb{C}^*},$$
$$M_{\mathbb{C}^*} \xrightarrow{A_\times} (\mathbb{C}^*)^n \xrightarrow{Q_\times} L^\vee_{\mathbb{C}^*}.$$

The Hori-Vafa superpotential is also given by $W : M_{\mathbb{C}^*} \to \mathbb{C}$:

$$W(x) = \langle c, A^\vee_\times(x)\rangle,$$

where $c \in \mathbb{C}^n$ are the coefficients.

Consider the discriminant component $\nabla_A \subseteq \nabla \subseteq \mathbb{C}^n$. Recall that ∇_A is invariant under rescaling of $x \in M_{\mathbb{C}^*}$ as W is quasi-homogeneous with respect to the action of $A^\vee$. We can replace ∇_A with the open subset $\nabla_A \cap (\mathbb{C}^*)^n$ and quotient by $M_{\mathbb{C}^*}$ to obtain a subvariety

$$D_A \subseteq L^\vee_{\mathbb{C}^*},$$

called the *reduced A-discriminantal variety*.

Definition/Theorem 3.14. [Gel+94, Ch. 9.1.C] There is a useful dominant rational map

$$\mathbb{P}L_{\mathbb{C}} \to D_A \subseteq L^{\vee}_{\mathbb{C}^*},$$

called *Horn uniformisation* given by

$$[\lambda] \mapsto Q_{\times} \circ Q^{\vee}(\lambda).$$

In explicit coordinates, this says:

$$[\lambda_1,\dots,\lambda_r] \mapsto \left(\prod_{i=1}^{n} \left(\sum_{k=1}^{r} Q_{ik}\lambda_k \right)^{Q_{i1}}, \dots, \prod_{i=1}^{n} \left(\sum_{k=1}^{r} Q_{ik}\lambda_k \right)^{Q_{ir}} \right).$$

3.2 Monodromy Action Conjecture

Now that we understand the geometry of $FIPS$, we will be able to state our main conjecture for how the fundamental group(oid) $\pi_1(FIPS)$ acts on the derived categories of our GIT quotients.

Consider a GIT problem as given by (3.1). Recall Definition 3.6 of $FIPS$ for a linear toric CY GIT problem:

$$FIPS = \left[(\mathbb{C}^n \backslash \nabla) \big/_{A} (\mathbb{C}^*)^{n-r} \right].$$

where $\nabla = V(E_A)$.

By (3.5) we see $FIPS$ as a complement in $\mathfrak{F}$, which gives us a large radius limit $c_X \in \mathfrak{F}$ (Definition 3.10) and large radius region $L_X \subseteq FIPS$ (Definition 3.11) for every GIT quotient X.

The following result shows that locally for each L_X there is a canonical action of the fundamental group via line bundle twists of X. We will use these to build our conjecture for the global action.

Lemma 3.15. [Kit19, Lemma 5.4] Let X be a GIT quotient of a linear toric CY GIT problem. There is a canonical isomorphism between $\pi_1(L_X)$ and $\operatorname{Pic} X$.

In the case when all the toric divisors near c_X are in the discriminant locus ∇, L_X is a punctured polydisk [Kit19, Sec. 5.1], that is, is homeomorphic to $(D^*)^r$, where $D^* \subseteq \mathbb{C}$ is a open disk in the analytic topology with a point missing. This implies that $\pi_1(L_X) \cong \mathbb{Z}^r$. In other cases, some of the toric stacky divisors containing c_X may not be in ∇ and may have an orbifold locus. Then the orbifold fundamental group (see §4.2) is $\pi_1(L_X) \cong \mathbb{Z}^k \times G$ where $k \leq r$ and G is a finite abelian group.

Recall the wall-crossing SOD given in equation (2.19) for Z_W associated to a wall W in the secondary fan. The next proposition is what allows us to relate factors of this SOD to discriminant components. In §3.1 we discussed how minimal faces correspond to discriminant components (equation (3.6)), and we recall that the SOD factors correspond to relevant subspaces, see Definition 2.28.

Proposition 3.16. [KS22, Prop. 4.15], [Kit19, Lemma 3.47] Let $\Gamma \subseteq P$ be a minimal face of the primary polytope $P \subseteq \mathbb{R}^{n-r}$ (Definition 3.12). By abuse of notation denote by $\Gamma \subseteq A$ also the indexing subset $\{1,\dots,n\}$ of rays that lie inside the minimal face. Consider the subspace $H_{\Gamma^c} \subseteq L_{\mathbb{R}}^{\vee} \cong \mathbb{R}^r$ that is linearly spanned by the weights $Q(\Gamma^c)$. H_{Γ^c} is a relevant subspace (Definition 2.28), and moreover the map $\Gamma \to H_{\Gamma^c}$ is a bijection between the minimal faces of P and the relevant subspaces of $L_{\mathbb{R}}^{\vee}$.

So given a minimal face $\Gamma \subseteq P$, we can consider the Higgs GIT problem associated to the relevant subspace H_{Γ^c}, whose minimal GIT quotient we will denote Z_Γ.

The correspondence in Proposition 3.16 was first noted in [APW17, Conjecture 5] and prompted a non-canonical guess for how the fundamental group of $FIPS$ acts on the derived categories of our GIT quotients. The idea is that a *meridian* (see Definition 4.7) in $\pi_1(FIPS)$ of a discriminant component ∇_Γ should correspond to a spherical twist about a spherical functor $F_\Gamma : D^b(Z_\Gamma) \to D^b(X)$. Combining this with Lemma 3.15 we get our monodromy action conjecture.

Conjecture 3.17 (Monodromy Action Conjecture)**.** Given a linear toric CY GIT quotient X and a basepoint $b_X \in L_X$, there is an action of $\pi_1(FIPS, b_X)$ on $D^b(X)$ via spherical twists about spherical functors which satisfies:

1. The restricted representation to $\pi_1(L_X, b_X)$ is via the canonical isomorphism in Lemma 3.15.

2. A meridian $\pi_1(FIPS, b_X)$ around a discriminant component $\nabla_\Gamma \subseteq FIPS$ corresponds to a spherical twist about a spherical functor $F_\Gamma : D^b(Z_\Gamma) \to D^b(X)$, up to composition by autoequivalences.

Remark 3.18. In Chapter 5, we will be proving Conjecture 3.17 for a class of GIT problems, but for only one GIT quotient in each case. Note that Conjecture 3.17 implies the less general Main Conjecture in §1.2, because linear toric CY GIT quotients are derived equivalent (Corollary 2.13).

Example 3.19 (Rank 1)**.** Let us now consider a linear toric CY GIT problem of rank equal to 1, e.g. a one-dimensional algebraic torus $T \cong \mathbb{C}^*$ acting on a vector space

V. $L^\vee \cong \mathbb{Z}$ and there are two possible GIT quotients, which we denote by X_+ and X_- for positive and negative stability conditions. If we split V by positive, zero, and negative weights as $V_+ \oplus V_0 \oplus V_-$ then $X_\pm$ is a vector bundle over $\mathbb{P}V_\pm \times V_0$, a stacky weighted projective space. Since $\dim L^\vee_\mathbb{R} = 1$, we only have one relevant subspace (Definition 2.28), that is 0. By Proposition 3.16, we therefore know that we only have one minimal face of the primary polytope. This implies that we only have one non-trivial, non-toric component of the discriminant locus (equation (3.6)) $\nabla \subseteq \mathbb{C}^n$.

By Horn uniformisation (Definition/Theorem 3.14),

$$(\nabla \cap (\mathbb{C}^*)^n)/M_{\mathbb{C}^*} \subseteq L^\vee_{\mathbb{C}^*} \subseteq [\,\mathbb{C}^n \,/\, M_{\mathbb{C}^*}\,],$$

is just a single point, which we call the conifold point. Recall the definition of the $FIPS$:

$$FIPS = [\,(\mathbb{C}^n \backslash \nabla)\,/\,M_{\mathbb{C}^*}\,],$$

where $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^{n-1}$. For a choice of coordinates on $M_{\mathbb{C}^*}$ and by equation (3.7), $FIPS$ is isomorphic either to $\mathbb{C}^* \setminus pt$ or to the DM stack $[(\mathbb{C}\backslash\bar{\nabla})\,/\,G]$, where G is a finite group, pt is the conifold point, and $[\bar{\nabla}\,/\,G] = pt \subseteq \mathbb{C}^*$.

Pick a basepoint close to the origin in either case, which should correspond to the large radius region of one of the GIT quotients, say X_+ without loss of generality. In the former case, we have a freely generated group

$$\pi_1(FIPS) \cong \langle l, x \rangle,$$

where l is a loop around the conifold point p and x is a loop around the origin 0. In the latter case, the origin is a gerbe BG, and we have the semi-direct product[3]

$$\pi_1(FIPS) \cong \langle l \rangle \rtimes G$$

where l is the same loop as before.

The loop x, or equivalently a generator of G which viewed in $\pi_1(FIPS)$, will be represented by a generator of the Picard group of X_+ via Lemma 3.15.

More interesting is the representation of the loop l which will be a spherical twist about a spherical functor as given by Theorem 2.25. If there are no zero weights then the resulting autoequivalence is the spherical twist $T_S : D^b(X_+) \to D^b(X_+)$ around the skyscraper sheaf along the zero section in X_+:

$$S = \mathscr{O}_{\mathbb{P}V_+},$$

[3]This is a simple case of Lemma 4.34, which we forward reference.

or any line bundle twist of it. If there are zero weights we upgrade this to a twist around the spherical functor $F : D^b(V_0) \to D^b(X_+)$.

We can build an analogous representation with basepoint choice in the large radius region of X_-. Showing that two such representations are compatible in the fundamental groupoid amounts to showing that $T_S \circ (\otimes \mathscr{O}_{X_+}(1))$ corresponds to tensoring by $\mathscr{O}_{X_-}(1)$ under wall-crossing (§2.3). An explanation for why this works can be found in [Kit19, Sec. 2.1.].

Example 3.20 (Example 3.13 continued)**.** The GIT problem in Example 3.13 is a $\mathbb{C}^*$ action on $\mathbb{C}^4$ given by

$$\lambda \cdot (\alpha, \beta, \gamma, \delta) = (\lambda^2 \alpha, \lambda^{-1} \beta, \lambda^{-1} \gamma, \delta).$$

Recall that the secondary toric stack $\mathfrak{F}$ is defined to be the quotient stack as-

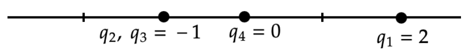


Figure 3.2: The secondary fan of Example 3.13.

sociated the GKZ fan $\boldsymbol{\Sigma_{GKZ}}$ defined in equation (2.6). Following the conventions of [BCS04], $\mathfrak{F}$ is isomorphic to the weighted projective stack $\mathbb{P}^1(2,1) = [(\mathbb{C}^2 \setminus (0,0))/_{2,1}\mathbb{C}^*]$ with coordinates q_1, q_2 or q_1, q_3. We arbitraily choose the former. A positive stability condition will have us remove $\{\alpha = 0\}$. We scale out the α-coordinate but end up with a residual $\mathbb{Z}_2$ action with weight 1 on the β and γ coordinates, so we have quotient $X_+ = [\mathbb{C}^2_{\beta,\gamma,\delta} /_{1,1,0} \mathbb{Z}_2]$. A negative stability condition will have us remove $\{\beta = \gamma = 0\}$, and so we have quotient $X_- = \mathscr{O}(-2)_{\mathbb{P}^1_{\beta:\gamma}} \times \mathbb{C}_\delta$. So $\operatorname{Pic} X_+ = \mathbb{Z}_2$ and $\operatorname{Pic} X_- = \mathbb{Z}$.

To $\Gamma = \{1,2,3\}$ we have corresponding relevant subspace $H_{\{4\}} = 0$ via Proposition 3.16 and it follows that $Z_\Gamma = \mathbb{C}$. From previous iterations of this example we know that

$$FIPS \cong [\mathbb{C} \backslash \{\pm 2\} \,/\, \mathbb{Z}_2].$$

Note that we can also see $FIPS$ as a complement of $V(E_A)$ in $\mathfrak{F}$ (equation (3.5)), where $V(E_A)$ is cut out by the global section $q_2(q_1 - 4q_2^2)$. $[0:1] \in \mathfrak{F}$ is the large radius limit of X_+, and picking a base-point in $FIPS$ close to $[0:1]$ we get orbifold fundamental group (see §4.2 for the relevant background)

$$\pi_1(FIPS) = \langle l, g \,|\, g^2, [\sigma, g] \rangle,$$

where g corresponds to a loop around the origin and l is a loop around the conifold point. $\pi_1(FIPS)$ acts on $D^b(X_+)$, where g is a line-bundle twist by the generator of Pic $X_+ = \mathbb{Z}_2$ and l is the spherical twist about spherical functor $F = i_*\pi^* : D^b(\mathbb{C}_\delta) \to D^b(X_+)$ (Theorem 2.25), where the map i is an inclusion map and π is a $B\mathbb{Z}_2$-bundle:

$$\begin{array}{ccc} [\mathbb{C}_\delta/_0\mathbb{Z}_2] & \xrightarrow{i} & X_+ \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathbb{C}_\delta & & \end{array}$$

Note that if we didn't have relation σ^2 in $\pi_1(FIPS)$, the Picard group of X_+ wouldn't be represented which is why the stack structure of $FIPS$ and the fact that the orbifold fundamental groups remembers it is important.

By the orbit-cone correspondence, given a wall W in the secondary fan separating two quotients X and X' we have a rational stacky curve $C_W \subseteq \mathfrak{F}$ in the toric boundary of $\mathfrak{F}$ as a toric stack connecting c_X and $c_{X'}$.

Lemma 3.21. [KS22, Corollary 4.13] The discriminant locus[4] $\nabla \subseteq \mathfrak{F}$ intersects C_W in exactly one point.

Each irreducible component of the discriminant locus $\nabla_\Gamma \subseteq \mathfrak{F}$ will intersect C_W with a multiplicity $m^A_\Gamma \in \mathbb{Z}_{\geq 0}$. In fact, by [Kit19, Remark 6.6], if a component ∇_Γ intersects C_W then it will be smooth at that point (using Horn uniformisation). Note also that for every relevant subspace $H = \Gamma^c$, there is a well-defined multiplicity $m^B_\Gamma \in \mathbb{Z}_{\geq 0}$ with which $D^b(Z_\Gamma)$ appears in the semi-orthogonal decomposition (2.18) for $D^b(Z_W)$. We have the following beautiful result, which adds to the supporting evidence for Conjecture 3.17.

Theorem 3.22. [Kit19; KS22; HZ25; HS20] $m^A_\Gamma = m^B_\Gamma$.

Inspired again by [APW17], Theorem 3.22 was first conjectured in [Kit19], shown for rank 2 in [KS22], and finally proved in general in [HZ25].

Consider a small tubular neighbourhood $L_W \subseteq FIPS$ of C_W and define it as the *large radius region of W*. For the rank 2 case, where $FIPS$ is two-dimensional, a near large radius limit version of the monodromy conjecture is proved in [Kit19, Sec. 6]. More specifically, they show that the fundamental groupoid $\pi_1(L_W, \{b_X, b_{X'}\})$

[4] Excluding C_W if it is itself a component of ∇.

on two large radius region base-points $b_X \in L_X$ and $b_{X'} \in L_{X'}$ acts via (*fractional*) window equivalences that come from crossing W.

In §4.3, we analyse the fundamental group near the intersection of C_W and ∇, and conjecture relations between autoequivalences associated to a wall W. More specifically, these autoequivalences are the factors of the spherical twist associated to W and the line bundle twist of the canonical line bundle $\mathscr{L} \in \operatorname{Pic} X$ corresponding to C_W.

Chapter 4

The Fundamental Group of $FIPS$

In this chapter we will explain the background we need to be able to compute $\pi_1(FIPS)$. Recall from equation (3.7) that we can write $FIPS$ in the following form

$$FIPS = [\,(\mathbb{C}^r \setminus \bar{\nabla})\,/\,G\,].$$

This shows clearly that $FIPS$ is a smooth DM stack (see Remark 2.3), specifically a hypersurface complement modulo a finite group.

The examples we consider in Chapter 5 are rank 2 so we are really looking at plane curve complements quotiented by finite groups. First, in §4.1, we review the classical Zariski-van Kampen method for computing the fundamental groups of curve complements. Then, in §4.2, we explain how to deal with the finite group action by defining the *orbifold fundamental group* of a smooth DM chart in terms of loops of the overlying space.

4.1 Fundamental Groups of Curve Complements

Ignoring the action of G in (3.7) and possible stackiness, we see $FIPS$ as a hypersurface complement. In our examples $r = 2$, so we are considering plane curve complements $\mathbb{C}^2 \setminus \mathscr{C}$. The theory of fundamental groups of curve complements is very classical, and in this section we present the key results that are relevant to us, taking much inspiration from [Cog11].

The central idea is the Zariski-Van Kampen method, which gives us a finite presentation of $\pi_1(\mathbb{C}^2 \setminus \mathscr{C})$. Originally sketched by Zariski [Zar29], the proof was completed in 1933 by Van-Kampen [Van33]. It starts by choosing a linear projection $\mathbb{C}^2 \setminus \mathscr{C} \to \mathbb{C}$ which outside of a finite set of critical points $S \subset \mathbb{C}$ is a locally trivial fibration.

For a locally trivial fibration $\rho : X \to M$, we can consider a *monodromy* action of paths in the base to loops in fibres. Let us denote our fibres by F and consider a choice of section $s : M \to X$. The monodromy action is a right action of the fundamental groupoid $\Pi_1(M)$ of paths on the base of the fibration on the fibre fundamental groups $\pi_1(F)$.

More explicitly, for any two basepoints $p_1, p_2 \in M$, a path $\gamma \in \pi_1(M, p_1, p_2)$ from p_1 to p_2, and $g \in \pi_1(F, q_1 = s(p_1))$, one obtains a new loop $g^\gamma \in \pi_1(F, q_2 = s(p_2))$. The new loop g^γ is constructed via the Homotopy Lifting Property, where we define a homotopy in the base $h : [0,1] \times [0,1] \to M$ by $h(s,t) = \gamma(t)$. There is a unique lift $\tilde{h} : [0,1] \times [0,1] \to X$ of h such that $\tilde{h}(0,t) = s \circ \gamma(t) = \tilde{h}(1,t)$ and $\tilde{h}(s,0) = g(s)$. This defines the new loop as $g^\gamma(s) = \tilde{h}(s,1)$. See [Cog11, Sec. 5.2] for full rigorous details.

Lemma 4.1. [Cog11, Sec. 5.1] For $g,h \in \pi_1(F, q_1)$ and $\gamma_1 \in \pi_1(M, p_1, p_2)$ and $\gamma_2 \in \pi_1(M, p_2, p_3)$:

$$
\begin{aligned}
g^{\gamma_2 \circ \gamma_1} &= (g^{\gamma_1})^{\gamma_2}, \\
(gh)^{\gamma_1} &= g^{\gamma_1} h^{\gamma_1}, \\
(g^{-1})^{\gamma_1} &= (g^{\gamma_1})^{-1}.
\end{aligned}
$$

Example 4.2. A simple non-trivial case arises when the fibration is trivial but the section is not constant. For instance, consider the fibration $\rho : S^1 \times F \to S^1$, where F is a unit disk $D = \{(x,y) \in \mathbb{R}^2 \mid |x|^2 + |y|^2 \leq 1\}$ with two punctures e.g. $F = D \setminus \{(0,1/2),(0,-1/2)\}$. For the loop $\gamma(t) = e^{2\pi i t} \in S^1$ in the base and the section $s(t) = (\gamma(t), e^{2\pi i t})$ section, then Figure 4.1 describes the γ-monodromy action on two closed paths $\alpha_1, \alpha_2 \in \pi_1(F,(0,1))$. We see that

$$
\begin{aligned}
\alpha_1^\gamma &= \alpha_1^{-1} \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1, \\
\alpha_2^\gamma &= \alpha_1^{-1} \alpha_2 \alpha_1.
\end{aligned}
$$

The next theorem explains how locally trivial fibrations can help us understand fundamental groups.

Theorem 4.3. [Cog11, Theorem 2.1] Suppose you have a locally trivial fibration $\rho : X \to M$ with section $s : M \to X$ and take a point $p \in M$. Let $F = \rho^{-1}(p)$ be the fibre over p. Then $\pi_1(X, s(p)) = \pi_1(F, s(p)) \rtimes \pi_1(M, p)$, where the action of $\pi_1(M, p)$ on $\pi_1(F, s(p))$ is given by monodromy.

Theorem 4.3 can help us compute the fundamental group of X by breaking down the group into lower dimensional components F and M. In fact we don't actually

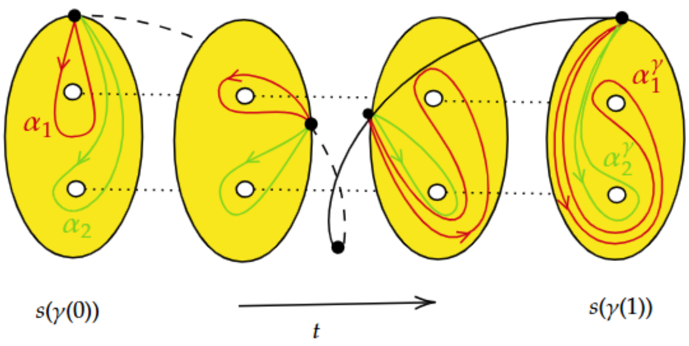


Figure 4.1: Monodromy of Example 4.2. The yellow circles represent fibres F. The trajectory of the path $s\circ\gamma$ is depicted by the black line connecting the yellow circles.

need a global section $s : M \to X$. It's sufficient to have a section when restricted to the supports of a representative set of generators of $\pi_1(M)$, so long as these are the generators in the group presentation implied by the semi-direct product.

Example 4.4 (Example 4.2 continued)**.** Theorem 4.3 tells us that

$$\begin{aligned}\pi_1(S^1\times F) &= \langle \gamma,\alpha_1,\alpha_2 \mid \gamma^{-1}\alpha_1\gamma=\alpha_1^\gamma, \gamma^{-1}\alpha_2\gamma=\alpha_2^\gamma\rangle,\\ &= \langle \gamma,\alpha_1,\alpha_2 \mid \gamma^{-1}\alpha_1\gamma=\alpha_1^{-1}\alpha_2^{-1}\alpha_1\alpha_2\alpha_1,\ \gamma^{-1}\alpha_2\gamma=\alpha_1^{-1}\alpha_2\alpha_1\rangle.\end{aligned}$$

We also clearly have another presentation from the fact that $F\times S^1$ is a direct product:

$$\pi_1(S^1\times F)=\langle \mu,\alpha_1,\alpha_2 \mid \mu^{-1}\alpha_1\mu=\alpha_1, \mu^{-1}\alpha_2\mu=\alpha_2\rangle.$$

The substitution $\gamma=\mu\alpha_2\alpha_1$ shows that the two presentations are equivalent.

We now go back to our curve complements $\mathbb{C}^2\setminus\mathscr{C}$, with our curve described by a reduced polynomial $f(x,y)\in\mathbb{C}[x,y]$. We will be considering linear projection onto $\mathbb{C}$, and note that via a linear coordinate change we can for simplicity just take this to be the projection onto the x-coordinate $\rho:\mathbb{C}^2_{x,y}\setminus\mathscr{C}\to\mathbb{C}_x$. For $x_0\in\mathbb{C}$, the fibre is

$$\rho^{-1}(x_0)\cong\mathbb{C}_y\setminus\{f(x_0,y)=0\}.$$

We note that we can write the polynomial in the following form

$$f(x,y)=f_d(x)y^d+f_{d-1}(x)y^{d-1}+\cdots+f_0(x),$$

where $f_i\in\mathbb{C}[x]$ and $f_d\neq 0$. This implies that for generic values, $f(x_0,y)=0$ has precisely d solutions. This is the case for all values except those in the sets

$$S'=\{\,x\in\mathbb{C}\mid \exists y_0\in\mathbb{C},\ f(x,y_0)=\frac{\partial f}{\partial y}(x,y_0)=0\,\}, \tag{4.1}$$

$$S_\infty=\{\,x\in\mathbb{C}\mid f_d(x)=0\}. \tag{4.2}$$

$S = S' \cup S_\infty$ is finite (say of size n), and note that the intersection $S' \cap S_\infty$ could be non-empty.

Theorem 4.5. Consider a plane algebraic curve $\mathscr{C} = V(f)$. Let $\rho : \mathbb{C}^2 \setminus \mathscr{C} \to \mathbb{C}$ be a linear projection. Let $S \subset \mathbb{C}$ be the critical set with $|S| = n$, and denote $\mathscr{H} = \rho^{-1}(S)$. The restriction

$$\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H})} : \mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H}) \to \mathbb{C} \setminus S,$$

is a locally trivial fibration. Moreover, for $p \in \mathbb{C} \setminus S$ and a choice of section $s : \mathbb{C} \setminus S \to \mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H})$, we have the following group presentation

$$\pi_1(\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H}), s(p)) = \langle g_1, \dots, g_d, \mu_1, \dots, \mu_n \mid \mu_j^{-1} g_i \mu_j = g_i^{\mu_j} \rangle,$$

where $g_1, \dots, g_d \in \pi_1(F := \rho^{-1}(p), s(p))$ and $\mu_1, \dots, \mu_n \in \pi_1(\mathbb{C} \setminus S, p)$ are both sets of free generators, $d \in \mathbb{Z}_{\geq 1}$, and the relations in the presentation hold for $i \in \{1, \dots, d\}$, $j \in \{1, \dots, n\}$.

Proof. Give $\mathbb{C}^2$ coordinates x, y and assume, without loss of generality, that ρ is the projection onto the x-coordinate. Suppose that f is degree d in the y-coordinate. By construction, the fibres of the restricted projection $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H})}$ are all isomorphic to the complex plane $\mathbb{C}$ with d points removed. By [Dim92, Lemma 3.3.5], $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H})}$ is a locally trivial fibration.

The statement on the presentation follows by applying Theorem 4.3 to $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H})}$ and the section s. □

Remark 4.6. In the statement of Theorem 4.5, we assumed the existence of a section of the locally trivial fibration $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H})}$. Many such sections exist, for example constant sections constructed in the following way. First, give $\mathbb{C}^2$ coordinates x, y and assume again, without loss of generality, that ρ is the projection onto the x-coordinate. Note that to apply Theorem 4.3, we only need a section on the support $Z \subset \mathbb{C} \setminus S$ of a set of generators of the free group $\mu_1, \dots, \mu_n \in \pi_1(\mathbb{C} \setminus S, p)$. Consider the restriction $\rho|_{\rho^{-1}(Z)} : \rho^{-1}(Z) \to Z$, and the projection onto the y-coordinate $Y = (\rho^{-1}(Z))_y \subset \mathbb{C}_y$. Y is equal to the complement of a 1-real dimensional closed subset of $\mathbb{C}_y$ which is the finite union of compact strands (i.e. either embedded intervals $[0,1]$ or circles) $(\rho^{-1}(\mu_i))_y$. It follows that Y is non-empty, and we can choose the section of $\rho|_{\rho^{-1}(Z)}$ to be one with constant y-coordinate for any choice of $y \in Y$. We will see in the rest of this chapter that we will want our choice of section to have particular properties.

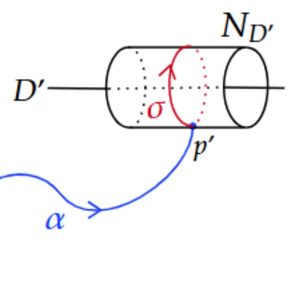
Theorem 4.5 gives us a presentation for $\pi_1(\mathbb{C}^2 \setminus (\mathscr{C} \cup \mathscr{H}))$ where $\mathscr{H} := \rho^{-1}(S)$. Our goal, however, is to compute a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathscr{C})$, which means that we need to somehow $\mathscr{H}$ back in. We can do this by trivialising the *meridians* of the irreducible lines in $\mathscr{H}$.

Definition 4.7. Let M be a complex manifold, D' be an irreducible component of a hypersurface $D \subset M$. A loop $\gamma \in \pi_1(M \setminus D, p)$ is a *meridian* of D' if it can be written as

$$\gamma = \alpha\sigma\alpha^{-1},$$

where:

- $\sigma \in \pi_1(\partial N_{D'}, p')$ where $N_{D'}$ is a local tubular neighbourhood $N_{D'} \subset M$ of D' near a smooth point,
- α is a path from p to p'.



Proposition 4.8. [Cog11, Proposition 1.2] Let $D \subset M$ be an irreducible hypersurface in M. Then the inclusion $M \setminus D \hookrightarrow M$ induces a surjective morphism $\pi_1(M \setminus D) \to \pi_1(M)$, whose kernel is $\langle\sigma\rangle$, the normal subgroup of $\pi_1(M \setminus D)$ generated by a meridian σ of D.

We could find out what the meridians of the lines in $\mathscr{H}$ were in terms of the generators in the presentation in Theorem 4.5, but it would be easier if they directly corresponded to generators so that we could just trivialise those. It turns out that we can do this by carefully choosing the constant section s, so that gluing $\mathscr{H}$ back in amounts to trivialising some of the μ_i, as we will explain in §4.1.2 and §4.1.3.

To find a presentation of $\pi_1(\mathbb{C}^2 \setminus \mathscr{C})$ we also need to know how to compute $g_i^{\mu_j}$. In §4.1.1 we discuss how braid monodromy can help with this. In particular we will see in §4.1.2 that for loops locally near critical points these braids can be combinatorially determined from finite data. In §4.1.3 we see that globalising is much trickier and may require computational tools.

4.1.1 Braid Monodromy

In the presentation in Proposition 4.5, we need to understand how to determine the monodromy action $g_i^{\mu_j}$. A helpful way to think about the monodromy of curves is in terms of *braid monodromy*. Consider our curve complement and projection $\rho : \mathbb{C}^2_{x,y} \setminus \mathscr{C} \to \mathbb{C}_x$ with a finite set of critical points $S \subset \mathbb{C}_x$.

Take $\gamma \in \pi_1(\mathbb{C}_x \setminus S, p)$. The fibre $F = \rho^{-1}(p)$ is isomorphic to the complex plane with d missing points. As we traverse γ, the missing points move through $M \subset \mathbb{C}_y \times \gamma([0,1]) \cong \mathbb{C}_y \times [0,1]$ trace out[1] a braid $\mathfrak{b}^\gamma \subset \mathbb{C}_y \times [0,1]$ on d-strands. This braid only depends on γ up to homotopy.

Theorem 4.9. [Bir69] There is an isomorphism between the group of braids on d-strands and the mapping class group of $\mathbb{C}$ fixing a set of d points:

$$\mathscr{M}(\mathbb{C}_d) \cong B_d.$$

We denote the braid group on d-strands by B_d, where braids are typically understood via generic 2D projections $\mathbb{C} \times [0,1] \to \mathbb{R} \times [0,1]$. We can also choose generating loops $g_1, \dots, g_d \in \pi_1(F)$ that are ordered compatibly with such a 2D projection, like in Figure 4.2.

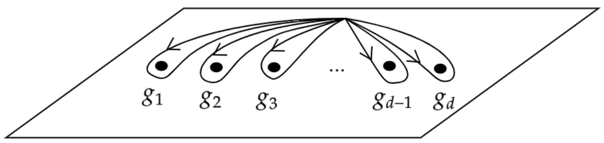


Figure 4.2: A 1D projection of generating loops $g_1, \dots, g_d \in \pi_1(F)$ which respects a choice of 2D braid projection.

This allows us to understand a *braid monodromy action* of $\mathfrak{b}^\gamma \in B_d$ on $g_i \in \pi_1(F)$. We denote the new loop by $g_i^{\mathfrak{b}^\gamma}$. Note that if we have the braid break up as $\mathfrak{b}^\gamma = \mathfrak{b} \cdot \mathfrak{b}'$, where $\mathfrak{b}, \mathfrak{b}' \in B_d$, then the braid monodromies compose associatively

$$g_i^{\mathfrak{b}^\gamma} = (g_i^{\mathfrak{b}})^{\mathfrak{b}'}.$$

Breaking down the braid $\mathfrak{b}^\gamma \in B_d$ into generators of the braid group can help with computations using Propositions 4.10 and 4.11.

Recall that the braid group B_d has generators $\mathfrak{b}_1, \dots, \mathfrak{b}_{d-1}$ as depicted in Figure 4.3.

[1] In fact for any path in the base, a braid is traced out, not just loops.

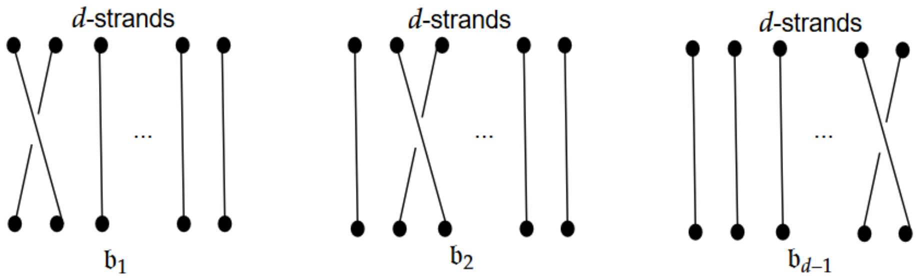


Figure 4.3: $\mathfrak{b}_1, \ldots, \mathfrak{b}_{d-1} \in B_d$ are the standard generators of the braid group.

Proposition 4.10 (Standard Braid monodromy, [Cog11])**.** The monodromy action of a standard basis of braids $\mathfrak{b}_1, \ldots, \mathfrak{b}_{d-1}$ (Figure 4.3) on standard loops $g_1, \ldots, g_d$ (Figure 4.2) is

$$g_j^{\mathfrak{b}_i} = \begin{cases} g_i^{-1} g_{i+1} g_i & i = j \\ g_i & j = i+1 \\ g_j & \text{otherwise.} \end{cases}$$

Proof. This is easy to see and essentially follows from Figure 4.4. □

Define $\langle ab \rangle^m$ to be the word $abab\ldots$ of length $m \in \mathbb{Z}_{\geq 0}$, e.g. $\langle ab \rangle^3 = aba$, and $\langle ab \rangle^{-m} = (\langle ab \rangle^m)^{-1} = \langle b^{-1} a^{-1} \rangle^m$.

Proposition 4.11 (Braid monodromy on two strands)**.** Let g_1 and g_2 be two standard loops and consider the monodromy action of $B = \mathfrak{b}^n$ where $\mathfrak{b}$ is the standard generator of B_2 the braid group on two strands. If n is even, then the monodromy map is

$$g_1^B = \langle g_2 g_1 \rangle^{-n} g_1 \langle g_2 g_1 \rangle^n,$$
$$g_2^B = \langle g_1 g_2 \rangle^{-n+1} g_2 \langle g_1 g_2 \rangle^{n-1}.$$

If n is odd,

$$g_1^B = \langle g_1 g_2 \rangle^{-n} g_2 \langle g_1 g_2 \rangle^n,$$
$$g_2^B = \langle g_2 g_1 \rangle^{-n+1} g_1 \langle g_2 g_1 \rangle^{n-1}.$$

Proof. By Proposition 4.10 and induction. □

We note that the monodromy action of γ is determined by the braid monodromy action of $\mathfrak{b}^\gamma$ if we pick the basepoint in the fibre (i.e. the section) so that it doesn't

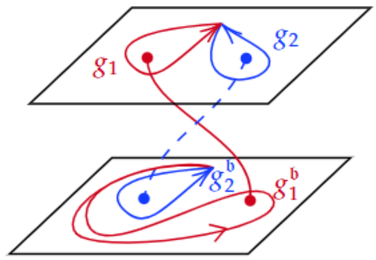


Figure 4.4: The action of the standard generator $\mathfrak{b} \in B_2$ on g_1, g_2.

interact with the movement of the missing points. This is the case if the section lies outside a ball containing the missing points as they move. What this does is ensure that the section itself does not contribute to the monodromy (as we saw it could in Example 4.2).

Remark 4.12. The braids $\mathfrak{b}^\gamma$ seem to be dependent on a choice of projection ρ and loop in the base γ. We will see however that our non-trivial monodromy actions often come from singularities of our curve $\mathscr{C}$, and different types of singularities have different uniquely associated braids[2]. For example, for the node, this unique braid is $\mathfrak{b}^2$, and for the cusp it is $\mathfrak{b}^3$. We will see examples in the next subsection.

4.1.2 Local Monodromy

Let us take a curve $\mathscr{C} = V(f) \subset \mathbb{C}^2_{x,y}$ where f is degree d in y, and consider the projection onto the x-coordinate

$$\rho : \mathbb{C}^2_{x,y} \setminus \mathscr{C} \to \mathbb{C}_x.$$

In this subsection we look locally, so we assume that we only have one critical point at that the origin $0 \in \mathbb{C}_x$. We will explain how to compute the monodromy action of a loop close to $x = 0$ using the braid monodromy perspective, as well as how to carefully choose a constant section in Theorem 4.5 so that we can easily use Proposition 4.8 to glue critical fibres back in.

Let's look at the critical fibre $H = \rho^{-1}(0)$. There are two cases:

1. $x \nmid f(x,y)$, so that H is isomorphic to $\mathbb{C}$ with $k < d$ points missing, and we have locally trivial fibration

$$\rho|_{\mathbb{C}^2_{x,y} \setminus (\mathscr{C} \cup H)} : \mathbb{C}^2_{x,y} \setminus (\mathscr{C} \cup H) \to \mathbb{C}^*_x.$$

[2]For a generic projection and arbitrarily small loop around the critical point.

2. $x \mid f(x,y)$, meaning $H \subset \mathbb{C}^2 \setminus \mathscr{C}$ is empty and ρ itself is already a locally trivial fibration[3].

We focus first on the case $x \nmid f(x,y)$. Let $p_1, p_2, \ldots, p_k \in \mathbb{C}_y$, $k < d$, denote the missing points in H. These solve $f(0, p_i) = 0$.

Take F to be a generic fibre of ρ, say over $x_0 \in \mathbb{C}_x^*$, i.e.,

$$F := \rho^{-1}(x_0) = \mathbb{C}_y \setminus \{q_1, \ldots q_d\}.$$

Consider a path γ connecting x_0 to 0. We track the monodromy movement of the missing points $q_1, \ldots, q_d$ as we traverse γ.

Either q_i will tend towards a missing point p_j in H or it will tend to infinity. Note that the latter case will only occur if $x = 0$ satisfies the condition in (4.2). Generally more than one missing point in F can tend to one missing point or infinity in H. Let $K \subset \{p_1, \ldots, p_k\} = M$ be the subset of missing points which have more than one q_i tending to them. These are precisely the y-points that satisfy the condition in (4.1).

Example 4.13. Let $\mathscr{C} \subset \mathbb{C}^2$ be the cuspidal cubic defined by $f(x,y) = x^3 - y^2$ and consider the projection ρ onto the x-coordinate. $x = 0$ is the only critical point, and $\rho^{-1}(0) = \mathbb{C}_y \setminus \{0\}$. All other fibres have two missing points and it is clear that as we move towards 0 in the base, those two points collide (because we can describe them both via $y = x^{\frac{3}{2}}$).

Example 4.14. Let $\mathscr{C} \subset \mathbb{C}^2$ be defined by $f(x,y) = x^3y^2 - 1$ and consider the projection ρ onto the x-coordinate. $x = 0$ is the only critical point and $\rho^{-1}(0) = \mathbb{C}_y$. We have that $y = x^{-\frac{3}{2}}$, so all other fibres have two missing points and it is clear that as we move towards 0 in the base, the two missing points tend to infinity.

In the examples in this thesis, we will see examples of $p_i \in K$ that are

1. irreducible[4] isolated singularities, specifically cusps e.g. Example 4.15,
2. reducible singularities where two smooth irreducible curves intersect with multiplicity. See Examples 4.16 and 4.20. These will mostly be multiplicity 1 (nodes) or multiplicity 2,

---

[3] We can clearly avoid this case altogether by choosing a generic projection ρ, but in some of our examples it is precisely the non-genericity of a projection that makes a good computational choice. See Example (b) in §5.2.

[4] A singularity of f is defined to be irreducible if f is irreducible in the ring of germs of holomorphic functions at the singularity, or equivalently if it is irreducible in the ring of formal power series.

3. smooth and irreducible points, but for which our projection is non-generic e.g. Example 4.18.

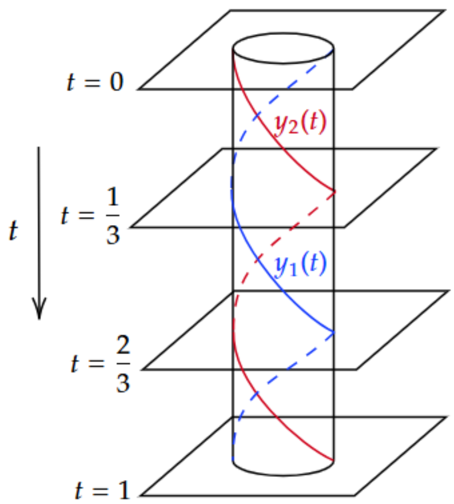
In particular, this means we will never see more than two q_j tending to one p_i.

Consider $\sigma \in \pi_1(\mathbb{C}_x^*)$ an arbitrarily small loop around 0 and now fix $x_0 = \sigma(0)$. The upshot of the following discussion is that using finite combinatorial data we can find a braid $\mathfrak{b}^\sigma$ which determines *the local monodromy* action of σ on some $g_i \in \pi_1(F)$. We will start by going through various examples to build some intuition.

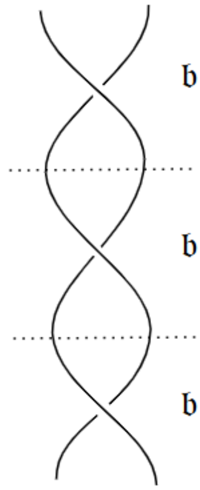
Example 4.15. (The Cusp) We consider $f(x,y) = y^2 - x^3$ defining the cuspidal cubic $\mathscr{C} \subset \mathbb{C}^2$ and the projection onto the x-coordinate $\rho : \mathbb{C}^2 \setminus \mathscr{C} \to \mathbb{C}_x$. Let $F = \rho^{-1}(1)$ and $\sigma(t) = e^{2\pi it} \in \mathbb{C}_x$, $t \in [0,1]$, be one loop of the unit circle in the base. By the description $y(x) = x^{3/2}$ we know that the solutions of $f(\sigma(t), y) = 0$ are

$$y_1(t) = e^{3\pi it},$$
$$y_2(t) = -e^{3\pi it}.$$

As we traverse $\sigma(t)$ we see the solutions y_1, y_2 trace out two strands that braid in the following way



We will represent the braid by a suitable plane projection:



We can break down the braid $\mathfrak{b}^\sigma = \mathfrak{b}^3$ where $\mathfrak{b}$ is the standard generator of B_2. Let $H = \rho^{-1}(0) \subset \mathbb{C}^2$ and note that

$$\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup H)} : \mathbb{C}^2 \setminus (\mathscr{C} \cup H) \to \mathbb{C}_x^*$$

is a locally trivial fibration. We take a constant section (and hence basepoint $p \in F$) far[5] from the movement of y_1 and y_2 and choose two loops $g_1, g_2 \in \pi_1(F, p)$ as per Figure 4.2. By Theorem 4.5 and using Proposition 4.8 to glue H back in and trivialise σ we have the fundamental group presentation

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle g_1, g_2 \mid g_1 = g_1^\sigma, g_2 = g_2^\sigma \rangle.$$

Using Proposition 4.11 we compute g_i^σ and find the well-known presentation for $\pi_1(\mathbb{C}^2 \setminus \mathscr{C})$ with the braid relation:

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle.$$

Example 4.16. (The Node) Let $f(x,y) = (y-x)(y+x)$ and $\mathscr{C} = V(f) \subset \mathbb{C}^2$. We have a nodal singularity at $(0,0)$ that is clearly not irreducible since the curve globally is reducible and has explicit global solutions

$$y = x,$$
$$y = -x.$$

Note that two components are smooth and meet at $(0,0)$ with multiplicity 1. Again we consider the projection onto the x-coordinate. For $\sigma(t) = e^{2\pi i t} \in \mathbb{C}_x$, $t \in [0,1]$

[5] This ensures that the braid $\mathfrak{b}^\sigma$ uniquely determines the monodromy action g_i^σ via Proposition 4.11, and that that lift of σ is a meridian of H and hence trivialisable when we glue H in.

the associated braid is $\mathfrak{b}^\sigma = \mathfrak{b}^2$ and using the same programme as Example 4.15 we get presentation:

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle g_1, g_2 \mid [g_1, g_2] = 1 \rangle.$$

Example 4.17. Let $f(x,y) = y^2 - x^2(x+1)$. The global curve is irreducible but we have a reducible singularity at the origin. We can see this because in the ring of holomorphic germs at the origin we can factorise $f(x,y) = (y - x\sqrt{x+1})(y + x\sqrt{x+1})$. The leading term in the series expansions is x, which tells us that the braid traced by the local monodromy at 0 is the same as that in Example 4.16. This is unsurprising since the singularity of g is of course analytically the same as that of the node[6]. Note that the fundamental group of the curve complement may still be different though, since the projection onto the x-coordinate has another critical point $x = -2/3$.

Example 4.18. Let $f(x,y) = y^2 - x$, $\mathscr{C} = V(f) \subset \mathbb{C}^2$ and consider the projection onto the x-coordinate. The curve $\mathscr{C}$ is clearly smooth but the projection has a critical point at $x = 0$. We can write the solutions in terms of y in terms of x as $y = \pm x^{1/2}$ and for $\sigma(t) = e^{2\pi i t} \in \mathbb{C}_x$, $t \in [0,1]$, the braid $\mathfrak{b}^\sigma$ is traced out by

$$\begin{aligned} y_1(t) &= e^{\pi i t}, \text{ and} \\ y_2(t) &= -e^{\pi i t}, \end{aligned}$$

so $\mathfrak{b}^\sigma = \mathfrak{b}$. We get fundamental group presentation

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle g_1, g_2 \mid g_1 = g_1^{-1} g_2 g_1,\ g_2 = g_1 \rangle \cong \mathbb{Z}.$$

Example 4.19. Let $f(x,y) = (y^2 - x)x$ and $\mathscr{C} = V(f)$. By Example 4.20 we know that using the projection onto the y-coordinate we have presentation

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle h_1, h_2 \mid (h_1 h_2)^2 = (h_2 h_1)^2 \rangle.$$

We however project onto x and using Example 4.18 find the presentation

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle \sigma, g_1, g_2 \mid \sigma^{-1} g_1 \sigma = g_1^{-1} g_2 g_1, \sigma^{-1} g_2 \sigma = g_1 \rangle,$$

where $\sigma(t) = e^{2\pi i t} \in \mathbb{C}_y$ can't be trivialised using Proposition 4.8 as $\{x = 0\} \subset \mathbb{C}^2$ is actually missing. Eliminating g_1 with the second relation and setting $h_1 = \sigma$ and $h_2 = g_2$, we see that the two group presentations match.

[6] We can see this immediately through the analytic coordinate change $u = x\sqrt{x+1}$ as $\sqrt{x+1}$ is a unit in the local ring.

Example 4.20. Let $f(x,y) = (y-x^{k_1})(y-x^{k_2})$, where $k_1, k_2 \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $k_1 < k_2$. Note that $k_2 = \infty$ corresponds to $y = 0$. Let $\mathscr{C} = V(f) \subset \mathbb{C}^2$. Again we project onto the x-coordinate and have critical point at $x = 0$ caused by a reducible singularity at $(0,0)$. We look at the loop in the base $\sigma(t) = re^{2\pi it} \in \mathbb{C}_x$, $t \in [0,1]$, with radius $0 < r < 1$. The braid $\mathfrak{b}^\sigma$ is traced out by

$$y_1(t) = r^{k_1} e^{2\pi itk_1}, \text{ and}$$
$$y_2(t) = r^{k_2} e^{2\pi itk_2}.$$

y_2 describes a smaller radius strand looping k_2-times around $0 \in \mathbb{C}_y$, while y_1 gives a bigger radius strand looping k_1-times around $0 \in \mathbb{C}_y$. We can 'straighten' this braid to see that $\mathfrak{b}^\sigma = \mathfrak{b}^{2k_1}$. Following the same programme as Example 4.15 we know we have fundamental group

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle g_1, g_2 \mid g_1 = g_1^{\mathfrak{b}^{2k_1}}, g_2 = g_2^{\mathfrak{b}^{2k_1}} \rangle.$$

Using Proposition 4.11 we have

$$g_1 = g_1^{\mathfrak{b}^{2k_1}} = (g_2 g_1^{-k_1}) g_1 (g_2 g_1)^{k_1},$$
$$g_2 = g_2^{\mathfrak{b}^{2k_1}} = g_1^{-1}(g_1 g_2)^{-(k_1-1)} g_2 (g_1 g_2)^{k_1-1} g_1.$$

It follows that the monodromy relations in the presentation for the fundamental group are summarised by the Artin braid relation of order k_1

$$(g_1 g_2)^{k_1} = (g_2 g_1)^{k_1}.$$

In the examples above we only ever had one missing point in the critical fibre and the solutions of y in terms x were obvious. We now explain how to handle any example. Recall we have a curve complement $\mathbb{C}^2 \setminus \mathscr{C}$ and a projection ρ onto the x-coordinate with one critical fibre $H = \rho^{-1}(0)$ that is isomorphic to $\mathbb{C}_y$ with missing points $M = \{p_1, \dots, p_k\}$. Recall also that $K \subset M$ is the set of the roots of $f(0,y) = 0$ with multiplicity greater than 1.

At the missing points $M \setminus K \subset \{p_1, \dots, p_k\}$, the holomorphic implicit function theorem says that locally our curve is given by y as a holomorphic function of x i.e. it can be represented by a convergent power series.

This local description does not hold for the missing points in K. One useful general tool to analyse curves locally at points are *Puiseux series*. For a comprehensive and leisurely exposition, we recommend [BK12, Part III, Sec. 8.3]. Puiseux series are generalisations of power series that allow for fractional exponents and they are often used to study curve singularities. The denominators in the exponents of Puiseux

Series are bounded and therefore we can find the smallest denominator $n \in \mathbb{Z}_{>0}$ so that all Puiseux series are of the form:

$$y(x) = \sum_{i=1}^{\infty} a_i x^{i/n}. \tag{4.3}$$

The Newton-Puiseux theorem [BK12, Part III, Sec. 8.3, Thm. 1] says that[7], given a complex algebraic plane curve $\mathscr{C} = V(f(x,y))$ and any point on $\mathscr{C}$, the solutions in y, viewed as functions of x, may be expanded as Puiseux series in x that are convergent to $\mathscr{C}$ in some small neighbourhood of the point. Without loss of generality (we can just change coordinates) we take the point to be the origin $(0,0)$.

The fractional exponents are needed because the solutions y to $f(x,y) = 0$ may not actually be functions of x, i.e. they may have multiple branches. In the form of (4.3), we expect to have n-branches, and we note that when a singularity is irreducible the expansion is unique.

Newton developed a very explicit algorithm to compute Puiseux series which is explained in [BK12]. In practice we can also use computational tools to compute the significant terms, which we will do in Examples (c) and (d) in Chapter 5. See Appendix B for the Mathematica code that was used.

The upshot is that much like in the examples we have already seen, each Puiseux Series determines a braid combinatorially. In fact the braid is determined by *Puiseux Pairs*, a finite set of pairs of integers which come from the exponents of the Puiseux expansions.

Suppose f is irreducible at the origin and we have the Puiseux expansion

$$y = \sum a_\kappa x^\kappa,$$

where $a_\kappa \neq 0, \kappa \in \mathbb{Q}_{>0}, \kappa \geq 1$. If κ are all integers no Puiseux pairs are defined. Otherwise there is a smallest κ_1 which is not an integer

$$\kappa_1 = \frac{n_1}{m_1},$$

with relatively prime m_1, n_1. The pair (m_1, n_1) is the first Puiseux pair. Some exponents to follow may be of the form $\frac{k}{m_1}$, but if not all of them are we shall come to a κ_2 which is not so representable. We write

$$\kappa_2 = \frac{n_2}{m_1 m_2}, \ \gcd(n_2, m_2) = 1, \ m_2 > 1,$$

[7]Under very mild conditions, that is that $f(0,y) \not\equiv 0$.

(if neccesary we multiply the fraction for κ_2 on the top and bottom by a divisor of m_1) and the pair (m_2, n_2) is the second Puiseux pair. We can continue in this way and this process eventually terminates so that we are left with a finite sequence $(m_1, n_1), \ldots, (m_g, n_g)$. Any two Puiseux series with the same Puiseux pairs determine the same braid $\mathfrak{b}^\sigma$ for a small loop σ around the critical point [BK12, Thm. 3, Sec. 8.3]. See page 410 for a good visualisation of these braids.

Example 4.21. Say we have Puiseux expansion

$$y = \sum_{i=1}^{\infty} a_i x^{i/2}, \tag{4.4}$$

where $a_3 \neq 0$. Then we only have one Puiseux pair $(2, 3)$, giving us the same braid $\mathfrak{b}^3$ we got for the cusp in Example 4.15. In fact is it easy to find an analytic change of coordinate to show that analytically locally such a Puiseux series must describe a cuspidal singularity with equation $u^2 = v^3$.

If our singularity is not irreducible, one needs to think about how the different braids interact. This depends on the multiplicity of intersection of the reducible components.

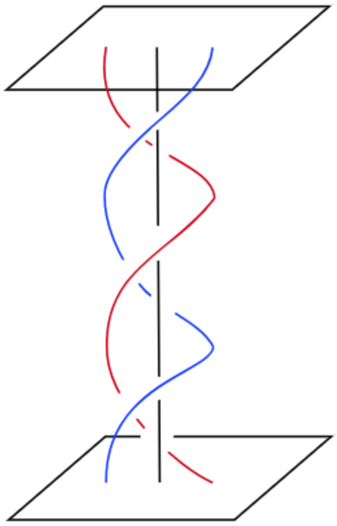
Example 4.22. Say we have two smooth irreducible curves with an isolated intersection at $(0, 0)$. Then for a generic projection we have power series expansions

$$y = \sum_{i}^{\infty} a_i x^i, \quad y = \sum_{i}^{\infty} b_i x^i.$$

Consider the difference between the two. The smallest power $k \in \mathbb{Z}_{\geq 1}$ such that $a_k - b_k$ is not zero is the multiplicity of the intersection, and by analogy with Example 4.20 have associated braid $\mathfrak{b}^{2k}$.

Given a $p_i \in M$, we now have a complete understanding of the movement as we move along σ of the points q_j that tend to p_i. What about the points that q_i that tend to infinity as $x \to 0$? The strands coming from the movements of these q_i wrap around the those coming from the other q_j tending to a p_k. We can Puiseux expand at $y = \infty$ by Puiseux expanding $(0, 0)$ for coordinate $z = 1/y$, and this expansion describes how the asymptotic strands braid around the rest. The leading term will have exponent $-k/d$ where k is the multiplicity of 0 as a root of f_d, where we recall the expression $f = y^d f_d(x) + \ldots f_0(x)$.

Example 4.23. Let $\mathscr{C}$ be the curve defined by the vanishing of $f(x,y) = y(x^3y^2 - 1)$ and consider the projection ρ onto the x-coordinate. We have a critical point at $x = 0$ because $f_3(x) = x^3$. $\rho^{-1}(0) = \mathbb{C}^*_y$, and for a loop in the base $\sigma(t) = e^{2\pi i t} \in \mathbb{C}_x$ we have braid



Theorem 4.24. Consider a plane algebraic curve $\mathscr{C} \subseteq \mathbb{C}^2$. Suppose we have a linear projection $\rho : \mathbb{C}^2 \setminus \mathscr{C} \to \mathbb{C}$ with one critical point $x_0 \in \mathbb{C}$ and denote the critical fibre by $H := \rho^{-1}(x_0)$. There exists a constant section of the locally trivial fibration $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup H)} : \mathbb{C}^2 \setminus (\mathscr{C} \cup H) \to \mathbb{C} \setminus \{x_0\}$ (Theorem 4.5) which yields the fundamental group presentations

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \begin{cases} \langle g_1, \dots, g_d \mid g_i = g_i^{\sigma} \rangle & \text{if } H \not\subseteq \mathscr{C} \\ \langle \sigma, g_1, \dots, g_d \mid \sigma^{-1} g_i \sigma = g_i^{\sigma} \rangle & \text{if } H \subseteq \mathscr{C}, \end{cases}$$

where $g_1, \dots, g_d \in \pi_1(F)$ are free generators for F a generic fibre of $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup H)}$, and $\sigma \in \pi_1(\mathbb{C} \setminus \{x_0\})$ is a loop around x_0 in the base.

Proof. Firstly, without loss of generality take ρ to be the projection onto the x-coordinate. We'd like to use Theorem 4.5 on the fibration $\rho|_{\mathbb{C}^2 \setminus (\mathscr{C} \cup H)}$ and therefore we have to pick a section s. Importantly, we want to make sure the lift $s \circ \sigma$ is a meridian of H, making it easy to use Proposition 4.8 to find a presentation for $\pi_1(\mathbb{C}^2 \setminus \mathscr{C})$. This depends on a careful choice of s, as we will describe.

For each missing point $p_i \in \mathbb{C}_y \subset \mathbb{C}^2$ in the critical fibre H, $V_i \subset \mathbb{C}^2$ be a small open neighbourhood around p_i such that its Puiseux expansion is convergent to f, which exists by the Newton-Puiseux Theorem. Let V_0 be a region where the Puiseux expansions at $x = 0, y = \infty$ are convergent, see Figure 4.5. Define the small open neighbourhood around $x = 0$:

$$U = \bigcap_{i=0}^{k} \rho(V_i).$$

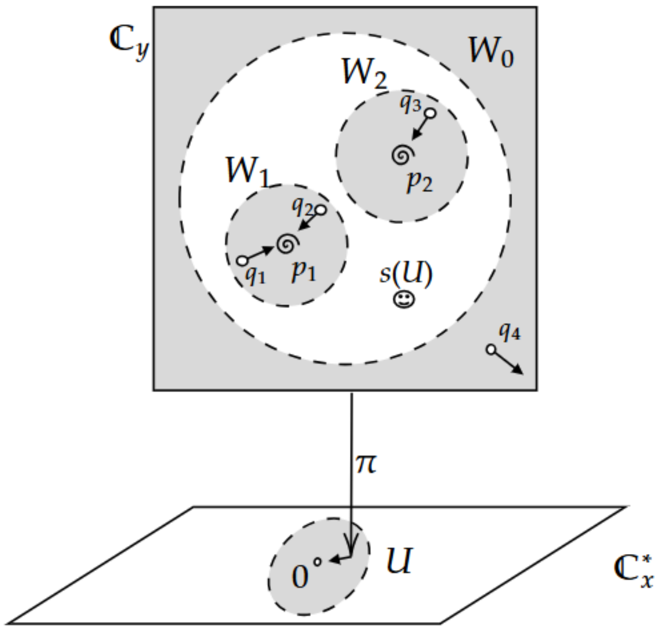


Figure 4.5: The missing points $q_i \in \mathbb{C}$ in the generic fibre over U, move to the missing points $p_i \in \mathbb{C}$ or infinity as $x \to 0$. The smiley face denotes a valid choice of constant section we could make.

Projecting onto the y-coordinate, the movements of the missing points in the fibres over U are restricted to the non-intersecting regions

$$W_i = (V_i)_y \subset \mathbb{C}_y,$$

where $i \in \{0, 1, \dots, k\}$. Let

$$W = \bigcup_{i=0}^{k} W_i,$$

and note that there is no movement in the fibres over U the non-empty region $\mathbb{C}_y \setminus W$.

We pick the loop around 0 $\sigma \in \pi_1(U) \cong \pi_1(\mathbb{C}^*)$ so that it lies in U. The braid $\mathfrak{b}^\sigma$ comprises k distinct braids $\mathfrak{b}^{i,\sigma}$ coming from each Puiseux series at p_i, as well as strands braiding around the k-braids coming from the moving points in V_0. See Figure 4.6. We pick a constant section $s : U \to \rho^{-1}(U)$ with fibre coordinate $p \in \mathbb{C}_y \setminus W$.

We start with the case $x \not| f(x,y)$. We use Theorem 4.5 to find a presentation

$$\pi_1(\mathbb{C}^2 \setminus (\mathscr{C} \cup H)) = \langle \sigma, g_1, \dots, g_d \mid \sigma^{-1} g_i \sigma = g_i^\sigma \rangle,$$

where $g_i \in \pi_1(F, p)$. We note that because of the way we chose our section, $s \circ \sigma$ is a meridian of H. By Proposition 4.8, we can glue H back into $\mathbb{C}^2 \setminus (\mathscr{C} \cup H)$ and

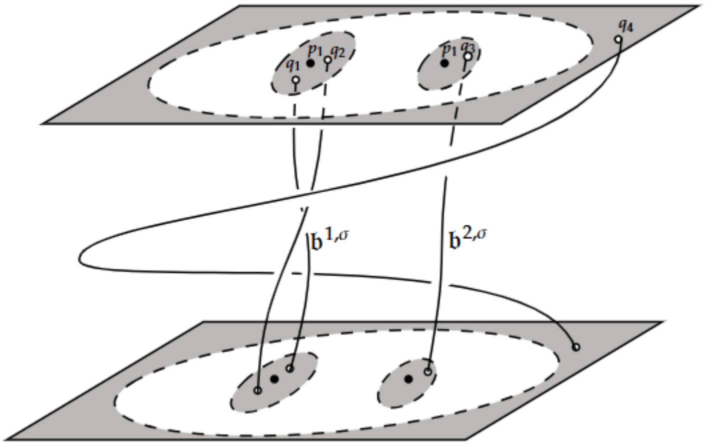


Figure 4.6: A picture of a local braid $\mathfrak{b}^{\sigma}$ in general.

trivialise σ in the group presentation. We find that

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle g_1, \dots, g_d \mid g_i = g_i^{\sigma} \rangle.$$

We now deal with the case $x \mid f(x,y)$, so that $\rho^{-1}(0) = H \subset \mathbb{C}^2 \setminus \mathscr{C}$ is empty. This just means that our projection with restricted codomain $\rho : \mathbb{C}^2 \setminus \mathscr{C} \to \mathbb{C}^*$ is a locally trivial fibration itself. We can still understand the geometry in the same way as for the case $x \nmid f(x,y)$ by analysing the polynomal $f(x,y)/x$ and similarly picking a section, but we get presentation

$$\pi_1(\mathbb{C}^2 \setminus \mathscr{C}) = \langle \sigma, g_1, \dots, g_d \mid \sigma^{-1} g_i \sigma = g_i^{\sigma} \rangle,$$

where we don't trivialise σ. $\square$

We have argued how when you have one critical point, the monodromy is entirely described combinatorially by Puiseux expansions. This is because having only one critical point allowed us to work locally. Next we will discuss the global case.

4.1.3 Global Monodromy

Theorem 4.25 (Global Monodromy)**.** For a plane algebraic curve $\mathscr{C} \subset \mathbb{C}^2$ and projection $\rho : \mathbb{C}^2 \setminus \mathscr{C} \to \mathbb{C}$, let $S = \{x_1, \dots, x_n\} \subset \mathbb{C}$ be the critical set, ordered so that:

1. for $1 \leq i \leq m \leq n$, $\rho^{-1}(x_i) = \emptyset \subseteq \mathbb{C}^2 \setminus \mathscr{C}$, and

2. for $m < j \leq n$, $\rho^{-1}(x_j) \cong \mathbb{C} \setminus \{d \text{ points}\} \subseteq \mathbb{C}^2 \setminus \mathscr{C}$.

Let $\mathscr{H} := \rho^{-1}(S)$ be the critical fibres. There exists a constant section s of the locally trivial fibration $\rho|_{\mathbb{C}^2\setminus(\mathscr{C}\cup\mathscr{H})}$ (Theorem 4.5) that yields the following fundamental group presention for any $p \in \mathbb{C}\setminus S$:

$$\pi_1(\mathbb{C}^2\setminus\mathscr{C}, s(p)) = \langle \mu_1,\dots,\mu_m, g_1,\dots,g_d \mid \mu_i^{-1} g_j \mu_i = g_j^{\mu_i},\ g_j = g_j^{\mu_k} \rangle, \tag{4.5}$$

where $g_1,\dots,g_d \in \pi_1(F, s(p))$ and $\mu_1,\dots,\mu_n \in \pi_1(\mathbb{C}\setminus S, p)$ are sets of free generators, the notation $g_r^{\mu_s}$ denotes the image of g_r under the μ_s monodromy action (see the beginning of §4.1), and the group relations hold for $1 \le j \le d$, $1 \le i \le m$, $1 < k \le n$.

Proof. Suppose without loss of generality that $\rho : \mathbb{C}^2\setminus\mathscr{C} \to \mathbb{C}$ is the projection onto the x-coordinate and has critical points $x_1,\dots,x_n \in S \subset \mathbb{C}_x$, such that $(x-x_i) \mid f(x,y)$ for $1 \le i \le m \le n$ and $(x-x_i) \nmid f(x,y)$ for $m < i \le n$. Let $H_i = \rho^{-1}(x_i)$ for $i = 1,\dots,n$ be the critical line fibres, so that

$$\mathscr{H} = \rho^{-1}(S) = \bigcup_{i=1}^{n} H_i \subset \mathbb{C}^2.$$

We choose a basepoint $p \in \mathbb{C}\setminus S$ in the base of the locally trivial fibration. $\pi_1(\mathbb{C}\setminus S, p) \cong \langle \mu_1,\dots\mu_n\rangle$ is the free group on n generators, but unlike in the local case, we can't necessarily take p and therefore μ_i to be close to each critical point x_i. For each x_i, we take $\mu_i \in \pi_1(\mathbb{C}\setminus\{x_1,\dots,x_n\})$ to be $\gamma_i\sigma_i\gamma_i^{-1}$, where γ_i is a path from p to a point $p_i \in \mathbb{C}\setminus S$ close to x_i, and σ_i is an arbitrarily small loop around x_i. See Figure 4.7.

We want to find a presentation for $\pi_1(\mathbb{C}^2\setminus\mathscr{C})$ as we did in the one critical point case (Theorem 4.24). To do that, we need to apply Theorem 4.5 to $\rho|_{\mathbb{C}^2\setminus(\mathscr{C}\cup\mathscr{H})}$ with a section

$$s : Z \to \rho^{-1}(Z),$$

where

$$Z := \operatorname{supp} \bigcup_i \mu_i \subset \mathbb{C}\setminus S.$$

Moreover, we want s to be such that the meridians of H_i for $m < i \le n$ are generators in the group presentation of $\pi_1(\mathbb{C}^2\setminus(\mathscr{C}\cup\mathscr{H}))$, so that gluing $\mathscr{H}$ back in corresponds to trivialising those generators.

We note

$$Z = \operatorname{supp}\bigcup_i \mu_i = \bigcup_i \operatorname{supp}\sigma_i \cup \bigcup_i \operatorname{supp}\gamma_i.$$

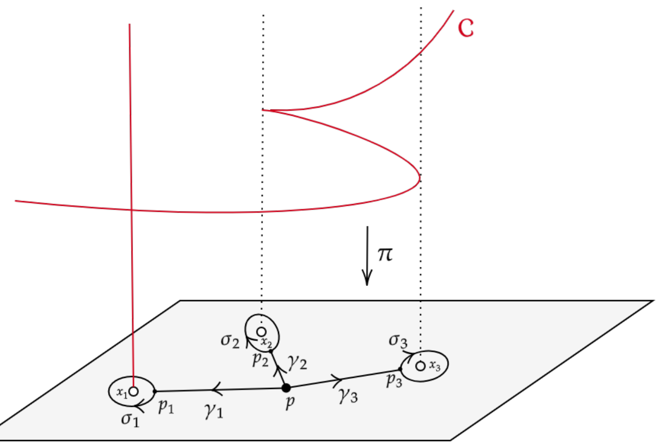


Figure 4.7: We have three critical points x_1, x_2, x_3 of ρ and curve $\mathscr{C}$ in red.

By the proof of Theorem 4.24, we know that for every critical point x_i there is an (arbitrarily small) open $W^i \subset \mathbb{C}_y$ so that for any section $s_i : \operatorname{supp}\sigma_i \to \rho^{-1}(\operatorname{supp}\sigma_i)$ with constant y-coordinate in $\mathbb{C}_y \setminus W^i$, $s_i \circ \sigma_i$ is a meridian of H_i.

Moreover, over $\bigcup_{1\le i\le m} \operatorname{supp}\sigma_i \cup \bigcup_i \operatorname{supp}\gamma_i$, the missing points in the fibres trace out a closed subset $X \subset \mathbb{C}^y$ which is isomorphic to a finite collection of compact 1-real dimensional stands (i.e. embedded intervals $[0,1]$ or circles).

$$Y := \mathbb{C}_y \setminus \bigcup_{i=m+1}^{n} W^i \cup X,$$

is non-empty and hence we choose $s : Z \to \rho^{-1}(Z)$ to be given by some constant in Y. We use Theorem 4.5 and then Proposition 4.8 to glue H_i back in for $m < i \le n$, where we note that $s \circ \mu_i = (s \circ \gamma_i)^{-1}(s \circ \sigma_i)(s \circ \gamma_i)$ is a meridian of $\mathscr{H}$ as $(s \cdot \sigma_i)$ is. The result follows. □

Let $F_i = \rho^{-1}(p_i)$ be the local fibre to x_i. The monodromy action of μ_i on $\rho^{-1}(F)$ can be decomposed as the action of γ_i on $\pi_1(F) \to \pi_1(F_i)$ and the action of σ_i and $\pi_1(F_i)$. The latter is the local monodromy at x_i, which we know is completely determined by the local topological type of the curve (§4.1.2).

The global monodromy depends the monodromy action of γ_i too, in particular on the position of the other critical points and the global geometry of the curve. Whether or not there is a finite set of global data on the curve that determines the global monodromy is still unknown.

It is the patching up of the local monodromy into global monodromy that makes computing fundamental groups really hard. In general it requires numerically solving $f = 0$ in the fibres over γ_i to a sufficiently high degree of accuracy. This explains one reason we haven't been able to prove the mirror symmetry monodromy conjecture by trying to understand the fundamental group directly. As far as we know, we need to compute fundamental groups of curve complements in rather ad-hoc ways in general.

In the Examples in Chapter 5 we will have to globalise monodromy action, and it will be quite fiddly and delicate.

Remark 4.26. The Zariski-Van Kampen method is implemented in Sagemath. The program works by randomly picking a projection, computing critical points, randomly picking a basepoint and loops around the critical points, and then uses numerical approximation techniques. The code may not always finish, but if it does, the output will be a group presentation which is guaranteed to be correct. Using this code is problematic for us because the monodromy conjecture is very sensitive to the way in which we match group generators in the fundamental group to autoequivalences. The program just outputs a presentation with some generators but does not give enough information on what the generators actually represent. For this reason, we have to compute the fundamental groups ourselves.

Remark 4.27. We now have two perspectives on the fundamental group of $FIPS$. Given the degree of the discriminant polynomial, we could use the Zariski-Van Kampen method to determine the number of generators in a presentation. Additionally, by using our insight on the derived autoequivalences, we could predict how many generators there should be after perhaps elimination via relations. This could for example be used to predict the minimal number of generators of a presentation for curves coming from discriminants. Exploring this relationship could be an interesting research direction.

Remark 4.28. We note that, in principle, knowing how to compute the fundamental groups of curve complements can help us compute $\pi_1(FIPS)$ for rank $r > 2$ cases too. By the Lefschetz Hyperplane Theorem[8] we can intersect $\mathbb{C}^r \setminus \nabla$ with successive

[8] There are various versions of the Lefschetz Hyperplane Theorem e.g. [Ham83], [HT85], [Gor+88], including for smooth DM stacks [Hal10].

(generic) hyperplane sections until we reach a plane curve complement $\mathbb{C}^2 \setminus \mathscr{C}$ such that $\pi_1(\mathbb{C}^r \setminus \nabla) \cong \pi_1(\mathbb{C}^2 \setminus \mathscr{C})$.

4.2 Orbifold Fundamental Groups

In this section, we explain what we mean by the *orbifold fundamental group*[9] of $FIPS$. We recall again that by equation (3.7), $FIPS$ in our examples is not just a curve complement but also has a smooth DM stack structure:

$$FIPS = \left[\, (\mathbb{C}^2 \setminus \bar{\nabla}) \,/\, G \,\right].$$

As such, we will see that $\pi_1(FIPS)$ will depend not only on $\pi_1(\mathbb{C}^2 \setminus \bar{\nabla})$, which we now know how to compute by Chapter 4, but also on the finite group action by G.

We will restrict to the nice case of smooth DM stacks (Remark 2.3) that are given globally by one orbifold chart. That is we will be considering the smooth DM stack $[X/G]$ where $X = \operatorname{Spec} R$ be a smooth affine algebraic variety and G be a finite group acting on X.

Consider first a gerbe BG. It is useful to keep in mind that however we define the fundamental group for an orbifold, we want $\pi_1(BG) \cong G$, so that the fundamental group is somehow tracking stabilisers.

Let us denote the obvious quotient map by

$$p : X \to [X/G].$$

Note that the stack structure of $[X/G]$ means p can be understood as a finite cover with fibres isomorphic to G as a set. We will be trying to find a definition for loops in the stack $[X/G]$ by lifting to the geometric space X.

Consider now $[\mathbb{C}/\mathbb{Z}_5]$ with $\mathbb{Z}_5$ acting with weight 1, and let $p : \mathbb{C} \to [\mathbb{C}/\mathbb{Z}_5]$ be the quotient map. For any point $x \in \mathbb{C} \setminus \{0\}$, the $\mathbb{Z}_5$ action identifies the distinct points $\operatorname{orb}(x) = \{x, \omega x, \omega^2 x, \omega^3 x, \omega^4 x\} \subset \mathbb{C}$ where ω is a primitive 5th root of unity. The only point where this is not true is the origin, which is a fixed point and hence in $[\mathbb{C}/\mathbb{Z}_5]$ is a gerbe $B\mathbb{Z}_5$.

For $x \in \mathbb{C}^*$, a path between any two points in $\operatorname{orb}(x)$ should be a loop in $[\mathbb{C}/\mathbb{Z}_5]$ based at $[x] := p(x)$. Additionally, in a definition of $\pi_1([\mathbb{C}/\mathbb{Z}_5], [x])$ we want to observe the following properties

- The line from say x to ωx is the same as the line $\omega^2 x$ to $\omega^3 x$ (as loops in $[\mathbb{C}/\mathbb{Z}_5]$ based at $[x] = p(x)$).

[9] We will be using the term orbifold and smooth DM stack interchangeably.

- Contractible loops at the points in $\mathrm{orb}(x)$ are trivial in $[\mathbb{C}/\mathbb{Z}_5]$. Therefore, a path tracing out the pentagon should be trivial, because we can retract it to a point.

But note that in general we can't just see loops in $[X/G]$ in terms of paths in X. To see why, remember that we want $\pi_1(BG) \cong G$, but simply path lifting to Spec $\mathbb{C}$ seems to imply that $\pi_1(BG)$ is trivial. For this reason, with each path from say x to y in X, we also have to track an element $g \in G$ such that $g \cdot x = y$. The following definition unifies the ideas above.

Definition 4.29 (The Orbifold Fundamental Group)**.** An element of the *orbifold fundamental group* $\pi_1([X/G],[x])$ is a pair (λ, g) where $g \in G$, and λ is a path in X starting at some $x_0 \in$ orb(x) and ending at $g \cdot x_0$. Moreover we identify two pairs $(\lambda_1, g_1) \sim (\lambda_2, g_2)$ if

- there exists $h \in G$ such that $h \cdot \lambda_1 = \lambda_2$ as homotopic paths in X with fixed endpoints, and
- $g_1 = g_2$.

Note that for any two pairs (λ_1, g_1) and (λ_2, g_2) there exists $h \in G$ such that $h \cdot \lambda_2(0) = \lambda_1(1)$. This defines their composition as $((h \cdot \lambda_2) \circ \lambda_1, g_2 g_1)$.

Remark 4.30. There are various ways to define orbifold fundamental groups. One quite general and clean way is to define it as the group of deck transformations of a generalisation of a universal cover. This is the definition used in most standard orbifold references [Thu22], but it's not the easiest to work with for computations. Fortunately for us, because of the simplicity of our stacks, we have access to the much more geometrically intuitive path perspective.

Proposition 4.31. Consider the smooth DM stack $[X/G]$ with X connected. For any $x \in X$, we have the short exact sequence:

$$1 \to \pi_1(X,x) \to \pi_1([X/G], p(x)) \to G \to 1. \tag{4.6}$$

Proof. $\pi_1([X/G], p(x))$ is defined as a quotient of a subset of $\pi_1(X, \mathrm{orb}(x)) \times G$ and there is obviously a well-defined surjective map $\psi : \pi_1([X/G], p(x)) \to G$ onto the G factor.

$(\gamma, g) \in \ker\, \psi$ if $\psi(\gamma, g) = g = e$. But that implies that γ is a loop in X based at some $x_0 \in \mathrm{orb}\,(x)$. By the equivalence relation, this is a subgroup of $\pi_1([X/G], p(x))$ that is isomorphic to $\pi_1(X, x_0) \cong \pi_1(X,x)$. □

Corollary 4.32. For X simply connected, $\pi_1([X/G]) \cong G$.

Remark 4.33. We can also understand Proposition 4.31 via covering spaces. Classically, given a pointed fibration $f:(Y,y)\to(X,x)$ of algebraic spaces, we have the following long exact sequence

$$\cdots \to \pi_1(Y_x,y) \to \pi_1(Y,y) \to \pi_1(X,x) \to \pi_0(Y_x,y) \to \pi_0(Y,y) \to \pi_0(X,x) \to \{*\}.$$

where $Y_x = f^{-1}(x) \hookrightarrow Y \xrightarrow{f} X$. Thinking of covering $p: X \to [X/G]$ as a fibration $G \hookrightarrow X \xrightarrow{p} [X/G]$, which we can as per Theorem A.12 in [Noo04] or Corollary 6.4 in [Zoo01], this simplifies to the short exact sequence (4.6).

The following lemma will be very useful in our computations, in particular in Examples (b) and (d).

Lemma 4.34. Assume X is connected and the action of G on X has a fixed point. The short exact sequence (4.6) splits

$$\pi_1([X/G],p(x)) = \pi_1(X,x) \rtimes G,$$

where $x \in X$.

Proof. The fixed point is a gerbe BG in $[X/G]$ and hence we have a map $G \to \pi_1([X/G])$ via the inclusion, which gives us the splitting. □

Example 4.35. Consider $\mathscr{X} = [\mathbb{C}^*/\mathbb{Z}_2]$. The action is free so we are just taking a naive quotient and $\mathscr{X}$ is homeomorphic to the non-stacky $\mathbb{C}^*$. It follows that $\pi_1(\mathscr{X}) \cong \mathbb{Z}$, and we see that the short exact sequence (4.6) does not split.

Example 4.36. Consider $\mathscr{X} = [\mathbb{C}/\mathbb{Z}_2]$. $\mathbb{C}$ is contractible so the short exact sequence clearly splits and $\pi_1(\mathscr{X}) \cong \mathbb{Z}_2$.

Example 4.37. Consider $\mathscr{X} = [\mathbb{C}\setminus\{\pm 1\}/\mathbb{Z}_2]$. The short exact sequence (4.6) reads:

$$1 \to \langle x,y\rangle \to \pi_1(\mathscr{X},0) \to \mathbb{Z}_2 \to 1,$$

where x and y are loops around 1 and -1 respectively, based at the origin. Under $\mathbb{Z}_2$, x and y are identified, and we have splitting $\pi_1(\mathscr{X},0) = \langle x,y\rangle \rtimes \mathbb{Z}_2 = \langle x,y,\alpha \mid \alpha^{-1}x\alpha = y, \alpha^2 = 1\rangle = \langle x,\alpha \mid \alpha^2 = 1\rangle$.

We now explain a stacky version of Proposition 4.8, which allowed us to remove and glue back in hypersurfaces. A codimension 1 substack of the smooth DM stack $\mathscr{X} = [X/G]$ is given by $\mathscr{D} = [D/G] \subseteq \mathscr{X}$ where $D \subset X$ is a codimension 1 G-invariant subvariety. We call $\mathscr{D}$ irreducible if D is irreducible.

Proposition 4.38. Suppose we have a smooth DM stack $\mathscr{X} = [X/G]$ where G is finite. Let $\mathscr{D} = [D/G] \subset \mathscr{X}$ be an irreducible codimension 1 substack. Then the inclusion $\mathscr{X} \setminus \mathscr{D} \hookrightarrow \mathscr{X}$ induces a surjective morphism $\pi_1(\mathscr{X} \setminus \mathscr{D}) \to \pi_1(\mathscr{X})$ whose kernel is $p_*(\langle\sigma\rangle)$, where $\langle\sigma\rangle$ is the normal subgroup of $\pi_1(X \setminus D)$ generated by a meridian of D and $p : X \to \mathscr{X}$ is the covering map.

Proof. If a loop in $\mathscr{X} \setminus \mathscr{D}$ is trivial in $\mathscr{X}$, it means that there is a lift of it $X \setminus D$ that is trivial in X. By Proposition 4.8, this lift is in the normal subgroup generated by $\langle\sigma\rangle \subset \pi_1(X)$, where σ is a meridian σ of D. So $\langle\sigma\rangle \subset \pi_1(X \setminus D)$ surjects under the quotient map $p_* : \pi_1(X \setminus D) \to \pi_1(\mathscr{X} \setminus \mathscr{D})$ onto the kernel of $\pi_1(\mathscr{X} \setminus \mathscr{D}) \to \pi_1(\mathscr{X})$. □

Proposition 4.38 will be useful as it will allow us to remove stacky loci and then compute fundamental groups of non-stacky spaces. For example, we could have used it to compute Example 4.37 and we will use it in Example (a).

Example 4.39. A consequence of Proposition 4.38 is that the orbifold fundamental group of the orbifold Riemann surface $\Sigma_{g,n,k}$ of genus g and n orbifold points p_i of order k_i is given by

$$\pi_1(\Sigma_{g,n,k}) = \langle \alpha_i, \beta_i, \sigma_j, 1 \leq i \leq g, 1 \leq j \leq n \mid \sigma_1 \ldots \sigma_n[\alpha_i, \beta_i] = 1, \sigma_j^{k_j} = 1 \rangle.$$

4.3 At Large Radius

In this section we describe some relations in $\pi_1(FIPS)$ that hold in general. This allows us to conjecture some relations between the autoequivalences associated to a specific choice of wall W.

Recall from §3.2 a wall W in the secondary fan separating two quotients X and X' has a corresponding rational toric stacky boundary curve $C_W \subset \mathfrak{F}$ in the secondary toric stack, connecting large radius limits c_X and $c_{X'}$ (Definition 3.10). We assume here that C_W is non-stacky and is contained in the discriminant locus[10]. The discriminant locus $\nabla \subset \mathfrak{F}$, intersects C_W in a single point (Lemma 3.21), and each irreducible component (equation 3.6) $\nabla_\Gamma \subset \mathfrak{F}$ will intersect C_W with some multiplicity. This intersection point is a reducible singularity, and therefore contributes relations to $\pi_1(FIPS)$.

We could try to use our braid monodromy techniques from §4.1.1 to compute the relations, but this quickly becomes very tedious, especially in higher rank problems. Instead, a better approach is to repeatedly apply blow-ups to the singularity

[10] We think that it would be possible to generalise to the stacky case but things are less clear.

to separate components. The key idea is that the fundamental group of the blow-up without the exceptional divisor is the same as the original fundamental group, and that blowing up will separate components and resolve the singularities until they are ordinary. This is a standard technique for computing fundamental groups of curve complements, and we refer the reader to [ACO12] for a detailed reference.

We start with the rank 2 case, where the discriminant locus $\nabla \subset \mathfrak{F}$ is a curve. Note that the components of the discriminant correspond to minimal faces, which correspond to relevant subspaces (Proposition 3.16), which correspond to factors in the SOD for Z_W (equation (2.19)). Since the Higgs GIT problem for W (Definition 2.10) is rank 1, wall-crossing will only yield at most two distinct components. This means that excluding any components that are toric boundary curves, there are at most two other irreducible components, the principal component and another one, which we denote by ∇_0 and ∇_1, respectively. Additionally, we know that the curve C_W intersects ∇_0 at a smooth point[11] with some multiplicity $m \in \mathbb{Z}_{\geq 0}$. ∇_1 intersects C_W at the same point (Lemma 3.21) with multiplicity 0 or 1. The multiplicity statement follows by Theorem 3.22, and because the factor corresponding to ∇_1 in the SOD (2.19) is the minimal GIT quotient of the Higgs GIT problem for W, which is either empty and therefore does not appear in the SOD, or appears once under wall-crossing [Kit19, Sec. 6.2].

Say ∇_1 does not intersect C_W. Then we have a reducible singularity of multiplicity m where ∇_0 and C_W meet. By Example 4.22, the following relation should hold in $\pi_1(FIPS)$:

$$(\gamma g_0)^m = (g_0 \gamma)^m.$$

where γ is a meridian of C_W and g_0 a meridian of ∇_0. If ∇_1 intersects C_W (necessarily with multiplicity 1), then we apply blow-ups at the intersection point to separate components. One blow-up separates ∇_1, and yields the relation

$$[\gamma_{E^1}, g_1] = 1,$$

where g_1 is a meridian of ∇_1 and $\gamma_{E^1} = \gamma g_1 g_0$ is a meridian of the exceptional divisor[12]. At the i-th blow up for $0 \leq i \leq m-1$, the exceptional divisor has meridian (up to conjugation) given by

$$\gamma_{E^i} = \gamma_{E^1} (\gamma g_0)^{i-1},$$

[11] By Remark 6.6 in [Kit19] we know that the intersection occurs at smooth points of the discriminant components in rank 2 problems. We would guess this to be true in higher rank but do not know how to prove it.

[12] In general the meridian of the exceptional divisor is the product of meridians of the irreducible components at the blow-up point.

and we obtain an additional relation $[\gamma_{E^{i-1}}, \gamma_{E^i}] = 1$. Exactly at the $i = m$th blow up, the original singularity is resolved into two ordinary singularities. This implies the $\pi_1(FIPS)$ relations:

$$[\gamma_{E^m}, g_0] = 1, [\gamma_{E^m}, \gamma] = 1.$$

Let $\mathscr{L} \in \operatorname{Pic} X$ be the canonical line-bundle corresponding to C_W by Lemma 3.15. Let T_{F_0} and T_{F_1} be two of the factors of the spherical twist T_F from crossing W, corresponding to ∇_0 and ∇_1 respectively (§2.4.2). Following Conjecture 3.17, we expect the matchings

$$\begin{aligned} \gamma &= \otimes\mathscr{L}, \\ g_0 &= T_{F_0}, \\ g_1 &= T_{F_1}. \end{aligned}$$

Substituting these assignments into the relations above, we can conjecture the following relations between the autoequivalences.

Conjecture 4.40. Consider a rank 2 GIT problem with quotient X and adjacent wall W. If ∇_1 does not intersect C_W, then

$$((\otimes\mathscr{L}) \circ T_{F_0})^m = (T_{F_0} \circ (\otimes\mathscr{L}))^m.$$

If ∇_1 does intersect C_W, then

$$\begin{aligned} T_{F_1} \circ T_{F_0} \circ F_1 &= (\otimes\mathscr{L}^{-1}) \circ F_1, \\ F_0 &= T_{(\otimes\mathscr{L})\circ F} \circ F_0. \end{aligned}$$

The relations above are satisfied in our examples in Chapter 5 (the only examples with a C_W that's non-stacky are Examples (a) and (c)). We highlight that in rank 2 the conjectured relations should be a direct corollary of the near-large-radius representation proved in [Kit19, Sec. 6]. Confirming this would require careful checking that our equivalence matching is the same, which we have not done.

The same blow-up technique can extend to higher rank, although explicit closed formulas become unwieldy. Nevertheless, one can still obtain a constructive presentation. Let $U_W \subset FIPS$ be a small analytic neighbourhood around the intersection point of C_W with the divisor ∇. A generic two-dimensional slice of U_W is isomorphic to the complement in $\mathbb{C}^2$ of a line, corresponding to C_W, and curves $C_1, \ldots, C_k$, corresponding to irreducible discriminant components. All the C_i intersect C_W at the same point, say with multiplicity m_i (Lemma 3.21). The Lefschetz Hyperplane Theorem (see Remark 4.28) tells us that the fundamental group of the complement of these curves is isomorphic to $\pi_1(U_W)$.

So we compute

$$\pi_1(\mathbb{C}^2 \setminus (C_W \cup \bigcup_{i=1}^{k} C_i)).$$

During the first m_1 blow-ups, the meridian of each exceptional divisor E_i^0, for $1 \leq i \leq m_1$, is

$$\gamma_{E_0^i} = (\gamma_0 \gamma_1 \dots \gamma_k)^i.$$

At precisely the m_1-th blow up, the component(s) with the same multiplicity $m_1 = m_2 = \cdots = m_r$ separate creating relations

$$[\gamma_{E_0^{m_1}}, \gamma_j] = 1, \tag{4.7}$$

$j = 1, \dots, r$. After the next blow ups, the additional exceptional divisors E_1^i will have meridians of the type:

$$\gamma_{E_1^i} = \gamma_{E_0^{m_1}} (\gamma_0 \gamma_{r+1} \dots \gamma_k)^i,$$

for $1 \leq i \leq m_2$, and eventually we reach ordinary singularities again and new relations of the form (4.7) for the separated components. This procedure ends and we find many relations that should hold in $\pi_1(FIPS)$.

Following Conjecture 3.17, each of the γ_i corresponds to a specific autoequivalence associated to the wall W. Therefore, the fundamental group relations we found above should imply relations between these autoequivalences. In some specific examples, like those below, nice relations can be found.

Example 4.41. If $m_1 = m_2 = \cdots = m_k = m$ then from the construction above one obtains

$$[(\gamma_0 \gamma_1 \dots \gamma_k)^m, \gamma_i] = 1,$$

which can be simplified as

$$(\gamma_0 \gamma_1 \dots \gamma_k)^m = (\gamma_1 \gamma_2 \dots \gamma_k \gamma_0)^m = (\gamma_2 \gamma_3 \dots \gamma_k \gamma_0 \gamma_1)^m = \cdots = (\gamma_k \gamma_0 \gamma_1 \dots \gamma_{k-1})^m.$$

Example 4.42. Say you have k components intersecting at an ordinary singularity, e.g. $(y - a_1 x)(y - a_2 x) \dots (y - a_r x)$ where a_i are distinct real numbers and $0 < a_1 < a_2 < \cdots < a_r$ and each component has a meridian $\gamma_1, \dots, \gamma_r$, then by doing one blow up we see that the relations at the singular point are described as:

$$[\gamma, \gamma_i] = 1,$$

where $\gamma = \gamma_1 \gamma_2 \dots \gamma_r$. This relative simplicity helps explain why the fundamental groups of hyperplane complements are comparatively tractable.

Note that these relations are not all the relations that we expect to see in a presentation for $\pi_1(FIPS)$. They are just some relations coming from large radius regions, but $\nabla \subset FIPS$ can have singularities away from these, implying additional relations.

Chapter 5

Toric Calabi-Yau 3-folds of Picard rank 2

In this chapter we will be proving Conjecture 3.17 for toric Calabi-Yau threefolds[1] of Picard rank 2. This will prove the Main Theorem in §1.3.

Concretely, this means considering linear toric Calabi-Yau GIT problems of the form $(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^5$. The rank of the GIT problem is implied to be 2 because of the rank of the Picard group, and we know the dimension is 5 because the quotients have to be three-dimensional. Our problem is therefore given by the GIT complex

$$0 \to L \xrightarrow{Q^\vee} (\mathbb{Z}^n)^\vee \xrightarrow{A^\vee} N \to 0, \tag{5.1}$$

with Q the weight map, ${}^6\vee = \operatorname{coker} Q^\vee$, and $L \cong \mathbb{Z}^2$. We also assume $N \cong \mathbb{Z}^3$, i.e. that it is torsion free, as otherwise none of our quotients will be a non-stacky toric variety.

For a choice of lattice basis on N we have a matrix representation of $A^\vee$ and a primary polygon $P \subset \mathbb{R}^3$ (Definition 2.7), which is the convex hull of the rays $a_i = A^\vee(e_i) \in \mathbb{Z}^3$. Because of our Calabi-Yau assumption (§2.2), we can choose generators for N so that the first column of $A = (A^\vee)^\vee$ is a list of 1s, and we see P living in the rank 2 lattice embedded in $\mathbb{Z}^3$ where the first integer coordinate is 1. Imposing the first column of A to be 1 does not uniquely determine generators for N. More precisely, these matrix representations of A are unique up to right multiplication by matrices in $GL_3(\mathbb{Z})$ given in block form as

$$B = \left(\begin{array}{c|cc} 1 & t_1 & t_2 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{M}} \\ 0 & & \end{array}\right)$$

[1] Some quotients will be smooth 3D DM stacks rather than varieties.

where $t_1, t_2 \in \mathbb{Z}$ and $M \in GL_2(\mathbb{Z})$. Thus the primary polygon $P \subset \mathbb{Z}^2$ is only unique up to an action of $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$[2] given by

$$\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, M \right) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Because A^T is surjective, the primary polytope P has exactly 5 integer points in it. This means that toric Calabi-Yau threefolds of Picard rank 2 come from convex lattice polygons in $\mathbb{Z}^2$ with exactly 5 integer points, up to the $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$ action. There are only six such classes of polygons [LZ11], which we show in Figure 5.1.

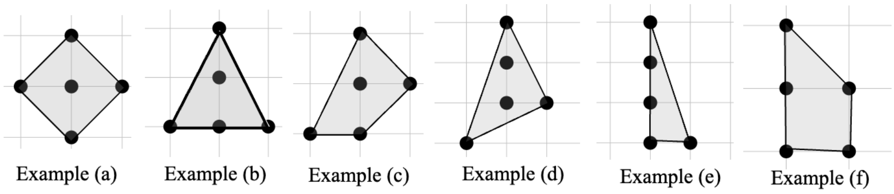


Figure 5.1: The primary polygons of toric Calabi-Yau threefolds of Picard rank 2.

Conjecture 3.17 was proved for Examples (e) and (f) by Donovan-Segal in [DS15] as part of a bigger class of examples. We will be showing the Conjecture holds Examples (a), (b), (c), and (d) one by one, but in only one GIT quotient. This will imply the more general Main Conjecture in §1.2 for all GIT quotients.

For each GIT problem we will follow a similar structure to the background in Chapters 2, 3, 4, where we go through the toric geometry and variation of GIT, compute *FIPS*, and work out the fundamental group. The last thing we do is match autoequivalences to fundamental group generators and show that the relations hold. The remarkable point is that these relations, though not difficult to show, are highly non-obvious a priori.

Remark 5.1. To improve readability, in the examples we will denote the weights by $\alpha, \beta, \gamma, \delta, \varepsilon \in L^\vee$ instead of the usual q_i notation. By abuse of notation we will also identify the weight symbols with coordinates on $\mathbb{C}^5$. Lastly we denote the rays by $a, b, c, d, e \in N$ instead of by a_i.

[2] Multiplication in $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$ is given by

$$\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, M \right) \cdot \left(\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, N \right) = \left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + N \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, MN \right).$$

5.1 Example (a)

In this section we prove Conjecture 3.17 in the case of Example (a), whose primary polygon is shown in Figure 5.2. In §5.1.1 we start by going through the toric geometry and study the autoequivalences on our chosen GIT quotient

$$X = \left[\mathscr{O}(-1)^{\oplus 2}{}_{\mathbb{P}^1}/\mathbb{Z}_2\right].$$

In §5.1.2, we compute $FIPS$, and in §5.1.3 we prove the fundamental group presentation

$$\begin{aligned}\pi_1(FIPS) = \langle g_1, g_2, g_3 \mid (g_3g_2)^2 = (g_2g_3)^2,\ [g_1,g_2] = 1,\\ (g_1g_3)^2 = (g_3g_1)^2,\ g_2{}^2 = 1\rangle.\end{aligned}$$

Following the programme of the conjecture, in §5.1.4 we match autoequivalences to fundamental group generators:

$$\begin{aligned} g_1 &= \otimes\mathscr{O}(1,0),\\ g_2 &= \otimes\mathscr{O}(0,1),\\ g_3 &= T_{\mathscr{O}_{\mathbb{P}^1}}, \end{aligned}$$

where $\operatorname{Pic} X \cong \mathbb{Z}\oplus\mathbb{Z}_2$, and we show that the fundamental group relations are satisfied by these autoequivalences.

5.1.1 Toric Geometry

We examine the geometry of the GIT quotients and study the autoequivalences in Example (a), leaning on the background from §2.1.

Choosing lattice bases, we can take representations for the matrices packaging the data of our linear toric CY GIT problem (see (5.1)) to be the following:

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array} \qquad Q = \begin{array}{c} \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \varepsilon \end{array} \end{array}$$

Plotting the weights of the action we see our secondary fan as in Figure 5.3. With reference to Figure 5.3, we see that for the chamber corresponding to GIT quotient X, the unstable locus is

$$U = V(\mathscr{I}_X) = V(\beta\alpha, \gamma\alpha) = \{\alpha = 0\}\cup\{\beta = \gamma = 0\},$$

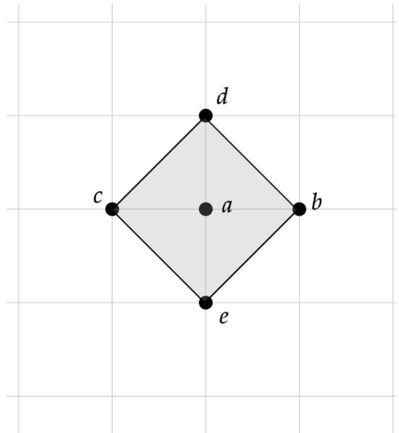


Figure 5.2: The primary polytope of Example (a).

and so

$$\begin{aligned} X &= \frac{\mathbb{C}^5 \backslash \{\alpha = 0\} \cup \{\beta = \gamma = 0\}}{(\mathbb{C}^*)^2} \\ &\cong \left[\mathscr{O}(-1)^{\oplus 2}{}_{\mathbb{P}^1_{\beta:\gamma}} / \mathbb{Z}_2 \right], \end{aligned}$$

where we scaled out the α coordinate, δ and ε are coordinates on the fibres, and $\mathbb{Z}_2$ acts on both the coordinates of the fibres of X with weight 1, so that we have a stacky $\mathbb{P}^1$ (the zero section of the vector bundle). We have Picard group

$$\text{Pic } X \cong \mathbb{Z} \oplus \mathbb{Z}_2, \tag{5.2}$$

via equation (2.5) for the Picard group of GIT quotients, where the torsion free part comes from $\mathscr{O}(-1)^{\oplus 2}{}_{\mathbb{P}^1_{\beta:\gamma}}$ and $\mathbb{Z}_2$ comes from a choice of representation of the action of $\mathbb{Z}_2$. We represent[3] our line bundles by $\mathscr{O}(n,m)$ where $(n,m) \in \mathbb{Z} \oplus \mathbb{Z}_2$.

There are two walls W_1 and W_2 adjacent to our chamber, into chambers corresponding to GIT quotients X_1 and X_2 respectively. The unstable loci are

$$\begin{aligned} U_1 &= V(\delta\beta, \delta\gamma, \varepsilon\beta, \varepsilon\gamma) = \{\delta = \varepsilon = 0\} \cup \{\beta = \gamma = 0\}, \\ U_2 &= V(\delta\alpha, \varepsilon\alpha) = \{\alpha = 0\} \cup \{\delta = \varepsilon = 0\}, \end{aligned}$$

[3] The reader may find it helpful to think of coherent sheaves on X as finitely generated $\mathbb{Z} \oplus \mathbb{Z}_2$ graded $\mathbb{C}[\beta, \gamma, \delta, \varepsilon]$ modules where δ, ε are in graded piece $(-1,-1)$ and β, γ in graded piece $(1,0)$. Note however that the morphisms are not simply graded-module morphisms.

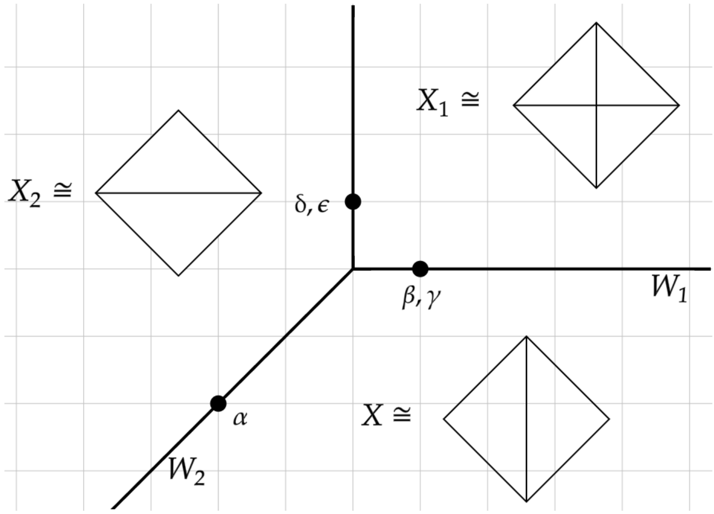


Figure 5.3: The secondary fan of Example (a). X, X_1, X_2 denote the GIT quotients corresponding to the chambers. W_1 and W_2 label the walls adjacent to chamber X. In each chamber we also have the triangulation of the primary polygon corresponding to that GIT quotient.

and hence

$$\begin{aligned} X_1 &\cong \mathscr{O}(-2,-2)_{\mathbb{P}^1_{\beta:\gamma} \times \mathbb{P}^1_{\delta:\varepsilon}}, \\ X_2 &\cong \left[\mathscr{O}(-1)^{\oplus 2}{}_{\mathbb{P}^1_{\delta:\varepsilon}} / \mathbb{Z}_2 \right]. \end{aligned}$$

We investigate the spherical twists coming from wall-crossing, described in §2.4.2, specifically Theorem 2.25. Looking at the GIT quotient X_1 on the other side of W_1, we remove $\{\delta = \varepsilon = 0\}$ instead of $\{\alpha = 0\}$ for X. In X, $\{\delta = \varepsilon = 0\}$ is just the stacky zero section of $\left[\mathbb{P}^1_{\beta:\gamma}/\mathbb{Z}_2\right]$, and we have spherical functor $F_1 = i_* \pi_1^*$

$$\begin{tikzcd} \left[\mathbb{P}^1_{\beta:\gamma}/\mathbb{Z}_2\right] \arrow[r, "i_1"] \arrow[d, "\pi_1"] & X \\ Z_{W_1} = \mathbb{P}^1 & \end{tikzcd}$$

where i is the inclusion and π_1 forgets the $\mathbb{Z}_2$ action[4] and is a $B\mathbb{Z}_2$-bundle.

Looking at X_2 on the other side of W_2, we see that again we remove $\{\delta = \varepsilon = 0\}$

[4] As a map of stacks π_1 is the identity on $\mathbb{P}^1$ and the trivial map $\mathbb{Z}_2 \to \{e\}$ on groups.

(instead of $\{\beta=\gamma=0\}$ for X). Again $\{\delta=\varepsilon=0\}$ is just the stacky zero section $\left[\mathbb{P}^1_{\beta:\gamma}/\mathbb{Z}_2\right]$ and we have spherical functor $F_2 = i_*\pi_2^*$:

$$\begin{tikzcd} \left[\mathbb{P}^1_{\beta:\gamma}/\mathbb{Z}_2\right] \arrow[r, "i_2"] \arrow[d, "\pi_2"] & X \\ Z_{W_2} = [pt/\mathbb{Z}_2] & \end{tikzcd}$$

It looks like we have two spherical functors F_1 and F_2 coming from the walls W_1 and W_2, but note that we have semi-orthogonal decompositions (SOD) $D^b(Z_{W_1}) \cong D^b(\mathbb{P}^1) \cong \langle \mathscr{O}, \mathscr{O}(1)\rangle \cong \langle pt, pt\rangle$ and $D^b(Z_{W_2}) \cong D^b([pt/\mathbb{Z}_2]) \cong \langle \mathscr{O}_{pt}, \mathscr{O}_{pt}(1)\rangle \cong \langle pt, pt\rangle$.

Consider a restriction of the spherical functors F_1 or F_2 with source categories $D^b(Z_{W_1})$ or $D^b(Z_{W_2})$ to any of the $D^b(pt)$ SOD factors. In each case we obtain a spherical functor $F = i_*\pi^*$ where π^* is the composition of projection π_1^* or π_2^* with the inclusion of the $D^b(pt)$ factor. The spherical twist about a spherical functor F with source category $D^b(pt)$ is in fact the spherical twist about the spherical object $F\mathscr{O}_{pt}$ (Example 2.22), and will either be $\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}(0,1)$ or $\mathscr{O}_{\mathbb{P}^1}(1,0)$.

5.1.2 *FIPS*

We analyse the geometry of $FIPS$, referring to background from §3.1. The primary polygon (Figure 5.2) has only one minimal face, the primary polygon itself. By equation (3.6) and Remark 3.4 this implies that the discriminant locus is given by

$$\nabla = V(bcde) \cup \nabla_A,$$

So let's compute what the principal component ∇_A is. This example is particularly nice because we can compute a defining equation by hand, which is not the case when your primary polytope has more integer points or is not as symmetric, for example. The Hori-Vafa superpotential $W_{a,b,c,d,e} : (\mathbb{C}^*)^3 \to \mathbb{C}$ is the Laurent polynomial

$$W_{a,b,c,d,e}(x,y,z) = x\left(a + by + cy^{-1} + dz + ez^{-1}\right),$$

where $(a,b,c,d,e) \in \mathbb{C}^5$. Before we compute the discriminant, we notice that we can scale out various coefficients via the $(\mathbb{C}^*)^3$ action on $\mathbb{C}^5 \backslash \nabla$ with weights prescribed by the matrix A^T. We scale c,d,e-coordinates to 1 and therefore identify $FIPS$ as the smooth DM stack

$$FIPS = \left[\mathbb{C}^2_{a,b} \backslash V(b) \cup \bar{\nabla}_A \Big/_{1,0} \mathbb{Z}_2\right],$$

where $\bar{\nabla}_A = V(f(a,b))$ is the image of ∇_A under the $(\mathbb{C}^*)^3$-quotient. To compute a defining polynomial f, we need to find the a,b for which

$$W_{a,b}(x,y,z) = x(a+by+y^{-1}+z+z^{-1}),$$

has critical values, in other words the a,b for which the partial differential equations have a common zero:

$$\frac{\partial W_{a,b}}{\partial x} = a+by+y^{-1}+z+z^{-1} = 0, \tag{5.3}$$

$$\frac{\partial W_{a,b}}{\partial y} = b-y^{-2} = 0, \tag{5.4}$$

$$\frac{\partial W_{a,b}}{\partial z} = 1-z^{-2} = 0. \tag{5.5}$$

Equation (5.4) and (5.5) imply $y = \pm\frac{1}{\sqrt{b}}$ and $z = \pm 1$, and plugging this into (5.3) we get

$$a \pm 2\sqrt{b} \pm 2 = 0.$$

We can do some simple resclaing to make the computations easier. Dividing through by 2 and rescaling a according will not change the topology of $FIPS$. By slight abuse of notation, we continue to denote by a the rescale coordinate by a factor of 2. So we have

$$a \pm \sqrt{b} \pm 1 = 0.$$

Succesively squaring our the square-roots, we find that $f(a,b) = (a^2+b)^2 - 4a^2b - 2(a^2+b)+1$ cuts out $\bar{\nabla}_A$.

5.1.3 Fundamental Group

We now compute the orbifold fundamental group of

$$FIPS = \left[\mathbb{C}^2_{a,b} \backslash V(bf(a,b)) \Big/_{1,0} \mathbb{Z}_2\right].$$

All the necessary background to understand the following computations can be found in Chapter 4. We make a choice of basepoint $b_X \in FIPS$ near the origin, which is the large radius limit X in the secondary toric stack $\mathfrak{F}$. By Proposition 4.38 we have that

$$\pi_1(FIPS) = \frac{\pi_1(FIPS \backslash \{a=0\})}{\eta}, \tag{5.6}$$

where $\eta = p_*(\mu)$ and μ is a meridian around $\{a=0\} \subset \mathbb{C}^2_{a,b}$. So we compute the fundamental group of $FIPS\backslash\{a=0\}$. We have an isomorphism

$$\pi_1(FIPS\backslash\{a=0\}) \cong \pi_1(\mathbb{C}^2 \backslash V(\tilde{a}bg(\tilde{a},b))),$$

via the homeomorphism $(a,b) \mapsto (a^2,b)$, where $\tilde{a} = a^2$ and

$$g(\tilde{a},b) = (\tilde{a}+b)^2 - 4\tilde{a}b - 2(\tilde{a}+b) + 1.$$

Let $Y := \mathbb{C}^2 \backslash V(\tilde{a}bg(\tilde{a},b))$. We will use the theory of computing fundamental groups of curve complements in §4.1 to compute $\pi_1(Y)$. We first make the coordinate change

$$u = \tilde{a}+b, \ \ v = \tilde{a}-b.$$

Substituting into g we get the simpler polynomial

$$h(u,v) = 1 - 2u + v^2.$$

and we consider the projection onto the v-coordinate

$$\rho : \mathbb{C}^2_{u,v} \backslash V((u-v)(u+v)h(u,v)) \to \mathbb{C}_v.$$

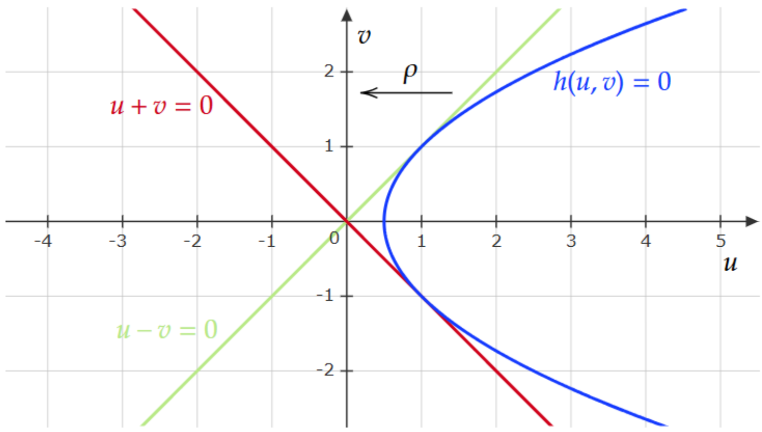


Figure 5.4: Real picture of Y in u,v-coordinates and the projection ρ.

We can immediately write the irreducible components of the curve as the solutions for u in terms of v

$$u = v, \ \ u = -v, \ \ u = \frac{v^2+1}{2}. \tag{5.7}$$

We have critical points $-1,0,1 \in \mathbb{C}_v$ and hence ρ is a locally trivial fibration outside of the three special fibers

- $L_1 = \rho^{-1}(-1)$,
- $L_2 = \rho^{-1}(0)$,

- $L_3 = \rho^{-1}(1)$.

We restrict ρ to $Y' = Y \backslash (L_1 \cup L_2 \cup L_3)$ so that $\rho : Y' \to \mathbb{C}_v \backslash \{-1,0,1\}$ is a locally trivial fibration. Each fibre of ρ is now homeomorphic to $\mathbb{C} \backslash \{3pts\}$ and we denote the base $\mathbb{C}_v \backslash \{-1,0,1\}$ by B.

Based on the positions of the critical points we choose a fibre $F = \rho^{-1}(v_0)$, where $-1 < v_0 < 0$. We make this choice because $v_0 \in B$ is 'close' to two of the critical points $-1, 0 \in B$, and being as close to as many critical points as possible will make globalising our local monodromy understanding easier.

We consider freely generating loops $\sigma_1, \sigma_2, \sigma_3 \in \pi_1(B, v_0)$ around -1, 0, 1 respectively. Following §4.1.3, we pick a constant (in y-coordinates) section $B \to Y'$ given by $u_0 \in \mathbb{C}_u$ which is 'far enough away' from the movement of the missing points solving (5.7).

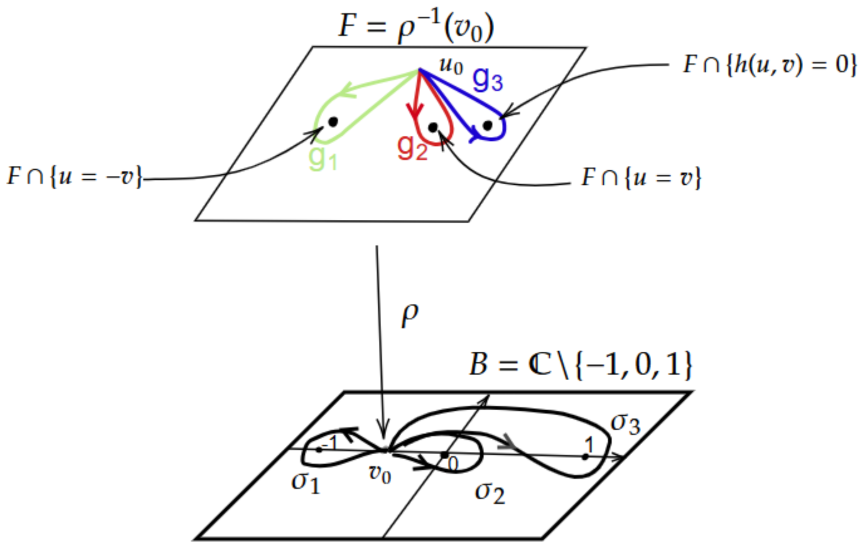


Figure 5.5: Loops from the fibration $\rho : Y' \to B$. $\sigma_1, \sigma_2, \sigma_3 \in \pi_1(B, v_0)$ loop the critical points -1, 0, and 1. $g_1, g_2, g_3 \in \pi_1(F, u_0)$ loop the missing points in the fibre coming from $\{u = -v\}$, $\{u = v\}$, and $\{h(u,v) = 0\}$.

Let $g_1, g_2, g_3 \in \pi_1(F, u_0)$ be generating loops as per Figure 5.5. By Theorem 4.25 we have group presentation for the fundamental group of Y

$$\pi_1(Y, (u_0, v_0)) = \langle g_1, g_2, g_3 \,|\, \forall i, j \in \{1,2,3\},\ {g_j}^{\sigma_i} = g_j \rangle.$$

We now need to understand ${g_j}^{\sigma_i}$. Using the globalising programme from §4.1.3 we

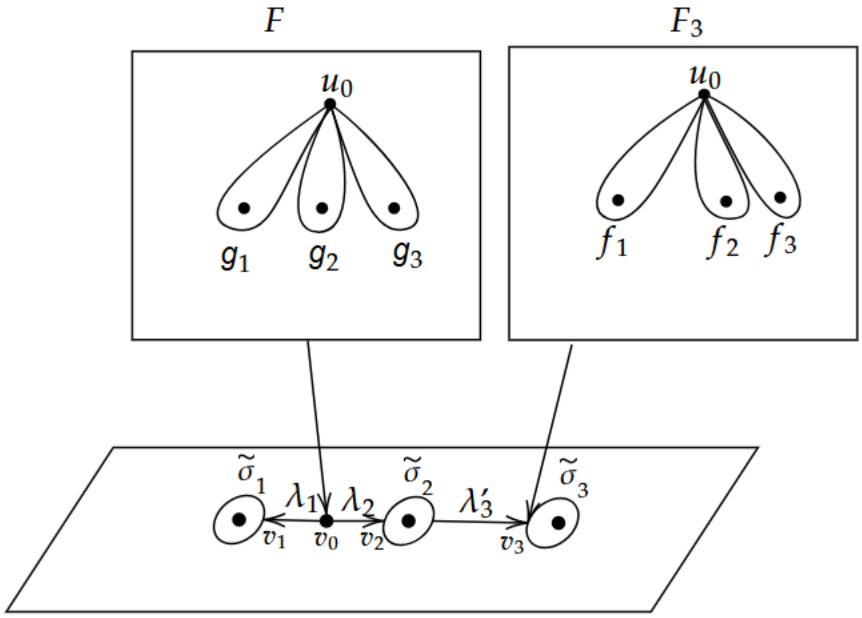


Figure 5.6: The breakdown of the paths $\sigma_1, \sigma_2, \sigma_3 \in \pi_1(B, v_0)$.

break up each loop as

$$\sigma_1 = \lambda_1^{-1}\tilde{\sigma}_1\lambda_1,$$
$$\sigma_2 = \lambda_2^{-1}\tilde{\sigma}_2\lambda_2,$$
$$\sigma_3 = \lambda_3^{-1}\tilde{\sigma}_3\lambda_3,$$

where each λ_i is a path that takes $v_0 \in B$ to a point $v_i \in B$ arbitrarily close to the critical point, and $\tilde{\sigma}_i$ are small loops around each critical point (small enough so that the Puiseux series in the critical fibre are convergent, see §4.1.2). We can pick λ_1, λ_2 to lie entirely on the real line, and we can break down λ_3 as

$$\lambda_3 = \lambda_3'\tau_2\lambda_2$$

where

$$\tau_2(t) = \tilde{\sigma}_2(t/2),\ t \in [0,1]$$

is the path that traverses half of $\tilde{\sigma}_2$ and λ_3' is a path on the real line connecting $\tau_2(1)$ to v_3.

The small loops $\tilde{\sigma}_i$ determine the local monodromy. To globalise rigorously we also need analyse the monodromy action $\pi_1(F) \to \pi_1(F_i)$ of each path λ_i, where $F_i = \pi^{-1}(v_i)$.

For σ_1 and σ_2, we can see from the real picture of Y that the monodromy of λ_1 and λ_2 is trivial, in that the braids they trace are trivial.

This means the braids which determine the monodromy of σ_1 and σ_2 from F are given by the local braids from F_1 and F_2, i.e.

$$\mathfrak{b}^{\sigma_1} = \mathfrak{b}^{\tilde{\sigma}_1}, \ \mathfrak{b}^{\sigma_2} = \mathfrak{b}^{\tilde{\sigma}_2}.$$

So for σ_1 and σ_2 we really just need to look locally at the critical points.

We can see that the critical point $-1 \in B$ is the result of the intersection of $V(h)$ and $u = -v$ at $(1,-1)$. We do a coordinate change $\bar{u} = u+1$ and $\bar{v} = v-1$, so that the intersection is shifted to the origin. Using equation (5.7) we see that at $(\bar{u},\bar{v}) = (0,0)$ we have Puiseux series:

$$\begin{aligned} \bar{u} &= -\bar{v}, \\ \bar{u} &= \frac{\bar{v}^2}{2} - \bar{v}. \end{aligned}$$

By Example 4.22, we find that the multiplicity of the intersection is 2. Referring to the standard braid generators (Figure 4.3), by Example 4.20 we find that $\mathfrak{b}^{\sigma_1} = (\mathfrak{b}_2)^4$ so that the monodromy relations $g_i = g_i^{\sigma_1}, i \in \{1,2,3\}$ are summarised by the Artin braid relation

$$(g_2 g_3)^2 = (g_3 g_2)^2.$$

We can see the associated braid and movement of the $g_i \in \pi_1(F,p)$ along it in Figure 5.7.

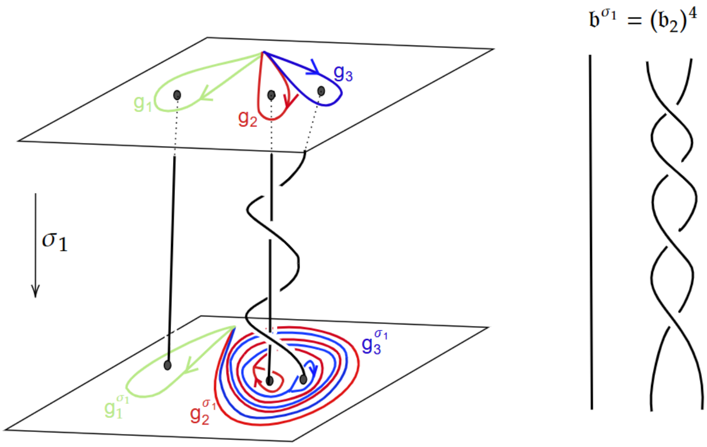


Figure 5.7: The braid monodromy action of σ_1 on g_1, g_2, g_3 (left) and the local braid associated to σ_1 (right).

The critical point $0 \in B$ comes from $u = v$ and $u = -v$ intersecting traversely and hence we have $\mathfrak{b}^{\sigma_1} = (\mathfrak{b}_1)^2$ from fibre F. It follows that the relations $g_i = g_i^{\sigma_2}, i \in$

$\{1,2,3\}$ amount to commutation relation

$$[g_1, g_2] = 1.$$

We now need to compute the monodromy of σ_3, which is slightly more delicate because the fibre F is further away from the critical point $1 \in B$ and we need to 'move past' $0 \in B$. This makes the monodromy action $\pi_1(F,u_0) \to \pi_1(F_3,u_0)$ of λ_3 non-trivial.

By the real picture again we can determine that the braid monodromy of λ_3' is trivial. This means that the braid $\mathfrak{b}^{\lambda_3}$ connecting F to F_3 is given by the braid $\mathfrak{b}^{\tau_2}$. Since τ_2 just traverses $\tilde{\sigma}_2$ half-way we have that $\mathfrak{b}^{\tau_2} = \mathfrak{b}_1$.

Consider generators $f_1, f_2, f_3 \in \pi_1(F_3,u_0)$ as depicted in Figure 5.8. The loops f_i in terms of g_i are given by

$$\begin{aligned}
f_1^{\lambda_3^{-1}} &= f_1^{\mathfrak{b}_1^{-1}} = g_2,\\
f_2^{\lambda_3^{-1}} &= f_2^{\mathfrak{b}_1^{-1}} = g_2 g_1 g_2^{-1},\\
f_3^{\lambda_3^{-1}} &= f_3^{\mathfrak{b}_1^{-1}} = g_3.
\end{aligned}$$

Since g_1 and g_2 commute, in fact $f_2^{\lambda_3^{-1}} = g_1$. Now we need to understand the local

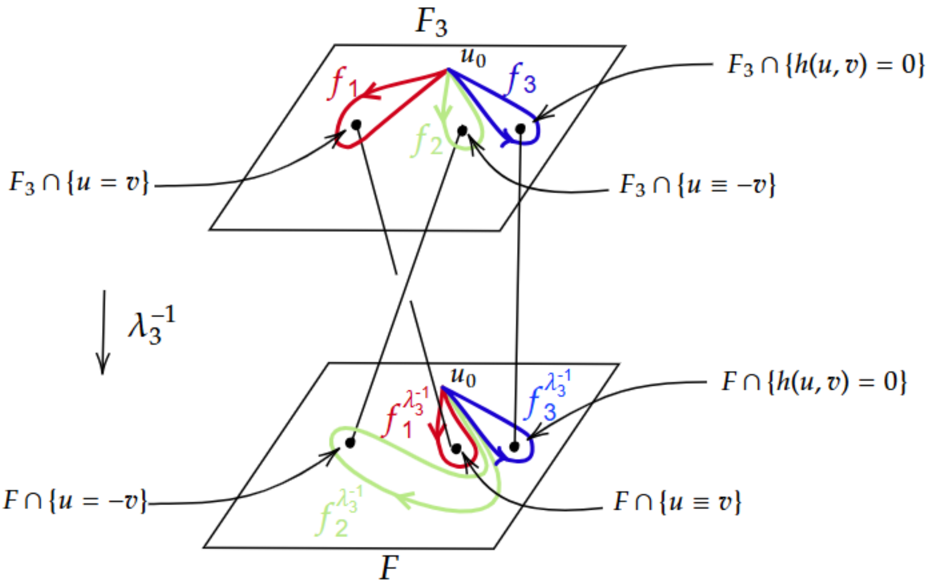


Figure 5.8: λ_3^{-1} monodromy action $\pi_1(F_3,u_0) \to \pi_1(F,u_0)$.

braid for $\tilde{\sigma}_3$ from the fibre F_3. The local braid is again ${\mathfrak{b}_2}^4$ (just like for σ_1, which makes sense given the symmetry) but acts on the loops f_i. The resulting monodromy relation is

$$(f_2 f_3)^2 = (f_3 f_2)^2,$$

and using the monodromy of λ_3^{-1} we find that the following summarises the monodromy relations $g_i = g_i^{\sigma_3}$:

$$(g_1 g_3)^2 = (g_3 g_1)^2.$$

We can finally write the fundamental group presentation for Y

$$\pi_1(Y,(u_0,v_0)) = \langle g_1, g_2, g_3 | g_1 g_2 = g_2 g_1, (g_1 g_3)^2 = (g_3 g_1)^2, (g_2 g_3)^2 = (g_3 g_2)^2 \rangle.$$

This is also known as the even Artin group $\mathbb{A}(2,4,4)$. To get the orbifold fundamental group of $\pi_1(FIPS)$ recall from (5.6) we have to quotient by η, which in our group presentation is ${g_2}^2$. Finally, we have the desired fundamental group presentation

$$\begin{aligned} \pi_1(FIPS) = \langle g_1, g_2, g_3 \mid (g_3 g_2)^2 = (g_2 g_3)^2,\ [g_1, g_2] = 1, \\ (g_1 g_3)^2 = (g_3 g_1)^2,\ {g_2}^2 = 1 \rangle. \end{aligned} \tag{5.8}$$

5.1.4 Relations between Functors

The mirror symmetry monodromy conjecture predicts an action of $\pi_1(FIPS)$ on the bounded derived category of the GIT quotient

$$X = \left[\mathscr{O}(-1)^{\oplus 2}{}_{\mathbb{P}^1} / \mathbb{Z}_2 \right].$$

More specifically, by Conjecture 3.17 we want a meridian around the principal discriminant component to be represented by spherically twisting the spherical object $S = \mathscr{O}_{\mathbb{P}^1}$. We also want meridians around $\{b = 0\}$ and $\{a = 0\}$ to correspond canonically to line bundle twisting by $\mathscr{O}(1,0), \mathscr{O}(0,1) \in \operatorname{Pic} X$.

Accordingly, we match autoequivalences of $D^b(X)$ with the generators of the group presentation (5.8) for $\pi_1(FIPS)$ in the following way

$$\begin{aligned} g_1 &= \otimes \mathscr{O}(1,0), \\ g_2 &= \otimes \mathscr{O}(0,1), \\ g_3 &= T_S. \end{aligned}$$

Theorem 5.2. Conjecture 3.17 is true for GIT quotient X in Example (a).

Proof. We need to verify that the assigned autoequivalences satisfy the group relations in the fundamental group presentation (5.8).

The relations $[g_1, g_2] = 1$ and ${g_2}^2 = 1$ follow since tensoring by line bundles is commutative and we know that $\mathscr{O}(0,1)$ is 2-torsion. The other two relations can be rewritten:

$$g_3 g_2 g_3 {g_2}^{-1} {g_3}^{-1} = {g_2}^{-1} g_3 g_2,$$
$$g_3 g_1 g_3 {g_1}^{-1} {g_3}^{-1} = {g_1}^{-1} g_3 g_1.$$

Substituting in our autoequivalences and using Lemma 2.24, the relations read:

$$T_{T_S \mathscr{O}_{\mathbb{P}^1}(0,1)} = T_{\mathscr{O}_{\mathbb{P}^1}(0,1)},$$
$$T_{T_S \mathscr{O}_{\mathbb{P}^1}(1,0)} = T_{\mathscr{O}_{\mathbb{P}^1}(-1,0)},$$

where we used that $\mathscr{O}(0,1)^{-1} = \mathscr{O}(0,1)$. Showing the the spherical objects are the same (up to a shift) is enough to show that the spherical twists are equal by property (1.) in Lemma 2.16, i.e. it's enough to show

$$T_S \mathscr{O}_{\mathbb{P}^1}(0,1) = \mathscr{O}_{\mathbb{P}^1}(0,1), \tag{5.9}$$
$$T_S \mathscr{O}_{\mathbb{P}^1}(1,0) = \mathscr{O}_{\mathbb{P}^1}(-1,0), \tag{5.10}$$

then we are done. Let us start by computing $T_S \mathscr{O}_{\mathbb{P}^1}(0,1)$. We note that[5]

$$\mathrm{RHom}(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}(m_1,m_2)) = \mathrm{R}\Gamma\left(\mathrm{R}\mathscr{H}om(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1})(m_1,m_2)\right).$$

So we compute $\mathrm{R}\mathscr{H}om(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1})$, then twist it accordingly, and lastly take derived global sections (sheaf cohomology). To compute the derived functor we start by Koszul resolution of $\mathscr{O}_{\mathbb{P}^1}$:

$$0 \to \mathscr{O}(2,0) \xrightarrow{(-\varepsilon,\delta)} \mathscr{O}(1,1)^2 \xrightarrow{(\delta,\varepsilon)^T} \mathscr{O} \to \mathscr{O}_{\mathbb{P}^1} \to 0.$$

Replacing $\mathscr{O}_{\mathbb{P}^1}$ with the quasi-isomorphic Koszul resolution complex and taking $\mathscr{H}om(\ ,\mathscr{O}_{\mathbb{P}^1})$ we get

$$\mathrm{R}\mathscr{H}om(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}) = \left[0 \leftarrow \mathscr{O}_{\mathbb{P}^1}(-2,0) \xleftarrow{0} \mathscr{O}_{\mathbb{P}^1}(-1,1)^2 \xleftarrow{0} \mathscr{O}_{\mathbb{P}^1} \leftarrow 0\right]. \tag{5.11}$$

We twist by $\mathscr{O}(0,1)$:

$$0 \leftarrow \mathscr{O}_{\mathbb{P}^1}(-2,1) \xleftarrow{0} \mathscr{O}_{\mathbb{P}^1}(-1,0)^2 \xleftarrow{0} \mathscr{O}_{\mathbb{P}^1}(0,1) \leftarrow 0.$$

[5] In the second equality we use that the local-to-global spectral sequence terminates on the second page, which is easy to argue.

We get zero derived global sections in all degrees. Firstly, $\Gamma(\mathscr{O}_{\mathbb{P}^1}(-2,1)) = 0$ because there is only one extension (of degree 1) of $\mathscr{O}_{\mathbb{P}^1}(-2)$ in $\mathbb{P}^1$, and in $[\mathbb{P}^1/\mathbb{Z}_2]$ it is not $\mathbb{Z}_2$ invariant:

$$0 \to \mathscr{O}_{\mathbb{P}^1}(-2,1) \xrightarrow{(-\gamma,\beta)} \mathscr{O}_{\mathbb{P}^1}(-1,1)^2 \xrightarrow{(\beta,\gamma)^T} \mathscr{O}_{\mathbb{P}^1}(0,1) \to 0.$$

$\Gamma(\mathscr{O}_{\mathbb{P}^1}(-1,0)) = 0$ as well, which we can deduce because $\mathscr{O}_{\mathbb{P}^1}(-1)$ in $\mathbb{P}^1$ has no sheaf cohomology. Lastly, for $\mathscr{O}_{\mathbb{P}^1}(0,1)$, the derived global sections come from genuine sections of $\mathscr{O}_{\mathbb{P}^1}$ in $\mathbb{P}^1$ i.e. degree 0 polynomials in β,γ variables (constants), but these are $\mathbb{Z}_2$ invariant. So $\mathrm{RHom}\,(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}(0,1)) = 0$ and we have

$$T_{\mathscr{O}_{\mathbb{P}^1}} \mathscr{O}_{\mathbb{P}^1}(0,1) = C(0 \to \mathscr{O}_{\mathbb{P}^1}(0,1)) = \mathscr{O}_{\mathbb{P}^1}(0,1),$$

which is precisely the relation (5.9). To check the relation (5.10) we twist (5.11) by $\mathscr{O}(1,0)$:

$$0 \leftarrow \mathscr{O}_{\mathbb{P}^1}(-1,0) \xleftarrow{0} \mathscr{O}_{\mathbb{P}^1}(0,1)^2 \xleftarrow{0} \mathscr{O}_{\mathbb{P}^1}(1,0) \xleftarrow{0} 0.$$

Taking global sections we get $\mathbb{C}^2_{\beta,\gamma}$ in degree 0. So

$$T_{\mathscr{O}_{\mathbb{P}^1}} \mathscr{O}_{\mathbb{P}^1}(1,0) = C({\mathscr{O}_{\mathbb{P}^1}}^2 \to \mathscr{O}_{\mathbb{P}^1}(1,0)) = \mathscr{O}_{\mathbb{P}^1}(-1,0),$$

using the Euler sequence. This proves relation (5.10) and completes the verification of Conjecture 3.17 for Example (a). □

5.2 Example (b)

In this section we prove Conjecture 3.17 in the case of Example (b), whose primary polygon is shown in Figure 5.9. In §5.2.1 we start by going through through the toric geometry and study the autoequivalences on our chosen GIT quotient

$$X \cong \left[\mathbb{A}^3 \,/_{2,1,1}\, \mathbb{Z}_4\right].$$

In §5.2.2, we compute $FIPS$, and in §5.2.3 we prove the fundamental group presentation

$$\begin{aligned}\pi_1(FIPS,0) = \langle \omega, \sigma, g_1, g_2, g_3, g_4 \mid \omega^4 = 1,\ [\sigma, \omega^2] = 1,\ \omega^{-1} g_i \omega = g_{i+1},\\ \sigma^{-1} g_2 \sigma = g_3{}^{-1} g_2 g_3,\ \sigma^{-1} g_3 \sigma = g_1 \rangle,\end{aligned}$$

where $i \in \{1,2,3,4\}$. Following the programme of the conjecture, in §5.2.4 we match autoequivalences to fundamental group generators:

$$\begin{aligned}\sigma &= T_F,\\ \omega &= \mathcal{O}(1) \in \operatorname{Pic} X \cong \mathbb{Z}_4,\\ g_1 = T_{\mathcal{O}_0},\ g_2 = T_{\mathcal{O}_0(1)},&\ g_3 = T_{\mathcal{O}_0(2)},\ g_4 = T_{\mathcal{O}_0(3)}.\end{aligned}$$

where the spherical functor $F = i_* \pi^* : D^b([\mathbb{A}^1/\mathbb{Z}_2]) \to D^b(X)$ is given by

$$\begin{tikzcd} \left[\mathbb{A}^1 \,/_2\, \mathbb{Z}_4\right] \arrow[r, "i"] \arrow[d, "\pi"] & X \\ \left[\mathbb{A}^1 \,/\, \mathbb{Z}_2\right] & \end{tikzcd}$$

and we show that the fundamental group relations are satisfied by these autoequivalences.

5.2.1 Toric Geometry

We examine the geometry of the GIT quotients and study the autoequivalences in Example (b), leaning on the background from Section 2.1.

Choosing lattice bases, we can take representations for the matrices packaging the data of our linear toric CY GIT problem (see (5.1)) to be the following:

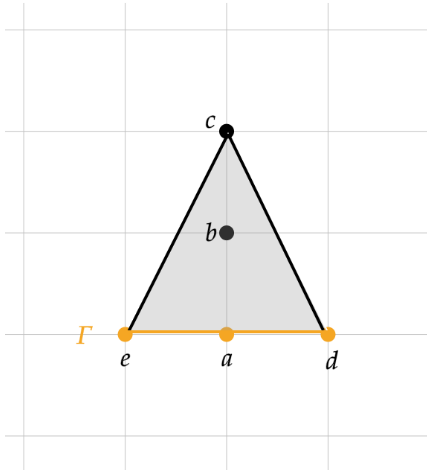


Figure 5.9: The primary polytope of Example (b). We have a minimal face Γ containing rays a,d,e.

$$A = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \qquad Q = \begin{array}{ccccc} & & & & \\ \end{array}\begin{array}{c} \begin{pmatrix} -2 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \varepsilon \end{array} \end{array}$$

Plotting the weights $\alpha,\beta,\gamma,\delta,\varepsilon \in \mathbb{Z}^2$, we see our secondary fan as in Figure 5.10. With reference to Figure 5.10, we see that for the chamber corresponding to GIT quotient X, the unstable locus is

$$U = V(\alpha\beta) = \{\alpha = 0\} \cup \{\beta = 0\}.$$

We can use one parameter subgroup $\lambda = (0,1) \in L^\vee$ to scale α to 1 since α is non-zero. This scaling is non-stacky, since λ acts on $\mathbb{C}^5$ with weights $(1,-2,1,0,0)$ and the weight on the α coordinate is 1. Similarly we can use one parameter subgroup $\mu = (1,2)$ to scale β to 1, where μ acts on $\mathbb{C}^5$ with weights $(0,-4,2,1,1)$. This scaling is stacky, with stabiliser group $\mathbb{Z}_4$. We get the affine quotient stack

$$X \cong \left[\mathbb{A}^3_{\gamma,\delta,\varepsilon} \Big/_{2,1,1} \mathbb{Z}_4\right].$$

We note that

$$\operatorname{Pic} X \cong \mathbb{Z}_4.$$

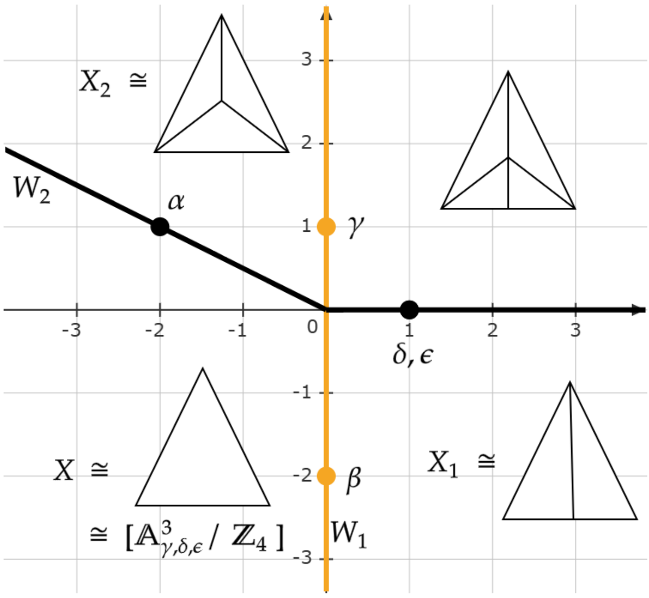


Figure 5.10: The secondary fan of Example (b). X, X_1, X_2 denote the GIT quotients corresponding to the chambers. W_1 and W_2 label the walls adjacent to chamber X. In each chamber we also have the primary polygon triangulation corresponding to each GIT quotients. In orange we can also see the secondary fan of Higgs GIT quotient Q_Γ associated to minimal face Γ.

We have two walls W_1 and W_2 adjacent to our chamber, into chambers corresponding to GIT quotients X_1 and X_2 respectively. For each one we have a spherical functor, see Theorem 2.25. The unstable locus giving GIT quotient X_1 is

$$U_1 = V(\beta\delta, \beta\varepsilon) = \{\beta = 0\} \cup \{\delta = \varepsilon = 0\},$$

so compared to the unstable locus of X, we remove $\{\delta = \varepsilon = 0\}$ instead of $\{\alpha = 0\}$. In X, $\{\delta = \varepsilon = 0\}$ is $[\mathbb{A}^1_\gamma /_2 \mathbb{Z}_4]$, where by this notation we mean that $\mathbb{Z}_4$ is acting with weight 2. We have an inclusion i and projection π

$$\begin{array}{ccc} \left[\mathbb{A}^1_\gamma /_2 \mathbb{Z}_4\right] & \xrightarrow{i} & X \\ \downarrow{\scriptstyle \pi} & & \\ Z_{W_1} = \left[\mathbb{A}^1 / \mathbb{Z}_2\right] & & \end{array}$$

We define the spherical functor

$$F := i_*\pi^* : D^b(Z_{W_1}) \to D^b(X),$$

where all the functors are derived.

The unstable locus giving GIT quotient X_2 is

$$U_2 = V(\alpha\gamma, \alpha\delta, \alpha\varepsilon) = \{\alpha = 0\} \cup \{\gamma = \delta = \varepsilon = 0\},$$

so compared to X we remove $\{\gamma = \delta = \varepsilon = 0\}$ instead of $\{\beta = 0\}$. Because $Z_{W_2} \cong pt$, our spherical twist will be the classical twist about a spherical object (Example 2.22). The spherical sheaf is supported on $\{\gamma = \delta = \varepsilon = 0\}$, hence the autoequivalence associated to W_2 is the spherical twist $T_{\mathscr{O}_0 \otimes \mathscr{L}}$, where $\mathscr{O}_0 \otimes \mathscr{L}$ is a line bundle $\mathscr{L}$ twist of the skyscraper sheaf $\mathscr{O}_0$ of the origin in X.

Notice in Figure 5.9 our primary polytope $P \subset \mathbb{Z}^2$ has minimal faces $\Gamma \subset A$ and itself. The secondary fan of Q_Γ is depicted in orange in Figure 5.10 (the vertical axis). We have that $Z_A \cong pt$ and $Z_\Gamma \cong \mathbb{A}^1$, which will be factors in the semi-orthogonal decompositions of Z_{W_1} and Z_{W_2} (§2.3). By wall-crossing $D^b(Z_{W_2}) \cong \langle pt \rangle$ and $D^b(Z_{W_1}) \cong \langle \mathbb{A}^1, pt \rangle$. The latter decomposition implies gives restrictions $F_{\mathbb{A}^1} : D^b(\mathbb{A}^1) \to$ and $F_{pt} : D^b(pt) \to D^b(X)$ of F to the two factors, where we note that the spherical twist of F_{pt} is also a spherical twist about a line-bundle twist of $\mathscr{O}_0 \in D^b(X)$.

5.2.2 *FIPS*

As in Example (a), we can compute the discriminant locus $\nabla = V(E_A) \subset \mathbb{C}^5$ by hand. By equation (3.6) in §3.1 we have two main discriminant components $\nabla_A, \nabla_\Gamma \subset \mathbb{C}^5$, along with $\{cde = 0\}$ from the vertices of the primary polygon (Remark 3.4). By looking at the superpotential

$$W = x(a + by + cy^2 + dz + ez^{-1}),$$

and doing a quick computation, we find

$$\begin{aligned} \nabla_A &= V((4ac - b^2)^2 - 64c^2de), \\ \nabla_\Gamma &= V(a^2 - 4de). \end{aligned}$$

So, we have that

$$\begin{aligned} FIPS &= \left[(\mathbb{C}^5 \backslash \nabla)/(\mathbb{C}^*)^3 \right] \\ &= \left[\frac{\mathbb{C}^5 \backslash \{cde = 0\} \cup \nabla_A \cup \nabla_\Gamma}{(\mathbb{C}^*)^3} \right]. \end{aligned}$$

We want to focus on the GIT quotient X. By analysing the geometry of the secondary stack $\mathfrak{F}$ we see that if we work in a, b coordinates, then the the origin corresponds to the large radius limit of X. We therefore scale out c, d, e with the $(\mathbb{C}^*)^3$

action, which is possible since c,d,e are non-zero. Specifically scaling c to $1/4$, and both d and e to 1 we see $FIPS$ in a,b coordinates as the smooth DM stack:

$$\begin{aligned} FIPS &= \left[\mathbb{C}^2_{a,b}\backslash(\{(a-b^2)^2=4\}\cup\{a^2=4\})\,/_{2,1}\,\mathbb{Z}_4\,\right] \\ &= \left[\mathbb{C}^2_{a,b}\backslash(\{b^2=a\pm 2\}\cup\{a=\pm 2\})\,/_{2,1}\,\mathbb{Z}_4\,\right], \end{aligned}$$

where $\mathbb{Z}_4$ acts on the coordinates a,b with weights $2,1$ respectively.

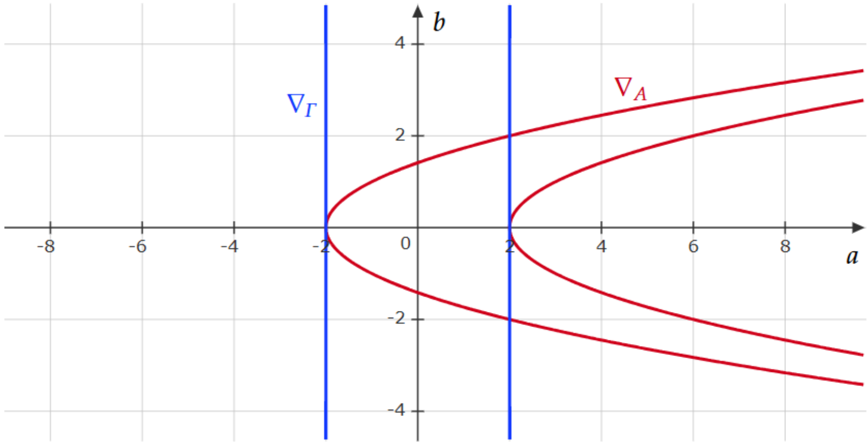


Figure 5.11: A real picture of $\mathbb{C}^2_{a,b}\backslash\{b^2=a\pm 2\}\cup\{a=\pm 2\}$.

5.2.3 Fundamental Group

We now compute the orbifold fundamental group of $FIPS$. We choose our basepoint to be at the origin, which corresponds to the large radius limit of our quotient X.

Define $Y := \mathbb{C}^2_{a,b}\backslash(\{b^2=a\pm 2\}\cup\{a=\pm 2\})$, whose real picture can be seen in Figure 5.11. The action of $\mathbb{Z}_4$ on Y has a fixed point at the origin and via the splitting of the short exact sequence in Proposition 4.31 we have

$$\pi_1(FIPS,0)=\pi_1(Y,0)\rtimes\mathbb{Z}_4. \tag{5.12}$$

Hence we start by computing $\pi_1(Y,0)$.

Remark 5.3. We first tried to take a similar approach to Example (a) to deal with the $\mathbb{Z}_4$ action by taking a map to the surface $V(u^2-vw)\subset\mathbb{C}^3$ via invariants: $(a,b)\to(a^2,ab^2,b^4)$. We found a presentation for the fundamental group of the image, but our issue was with matching autoequivalences to the generators. In Example (a), it was clear how loops in image of the invariant map corresponded to the meridians around discriminant curves due to the simplicity of the map, but here we found that not to be the case. So we chose an approach where the functor matching was going to be more clear.

We will be using the Van-Kampen method by choosing the projection $\rho : Y \to \mathbb{C}_a \backslash \{a = \pm 2\}$ onto the a coordinate. This projection is in fact a locally trivial fibration, with each fibre isomorphic to $\mathbb{C}$ with 4 points missing. We take generic fibre

$$F = \rho^{-1}(0) \cong \mathbb{C}_b \backslash \{\pm\sqrt{2}, \pm\sqrt{2}i\}.$$

With reference to Figure 5.12, we consider $g_1, g_2, g_3, g_4 \in \pi_1(F, 0)$ and $\sigma, \tau \in \pi_1(\mathbb{C}_a \backslash \{a = \pm 2\}, 0)$. By Theorem 4.25

$$\pi_1(Y, 0) = \langle \sigma, \tau, g_i \mid g_i^\sigma = \sigma^{-1} g_i \sigma,\ g_i^\tau = \tau^{-1} g_i \tau \rangle, \tag{5.13}$$

for $i \in \{1, 2, 3, 4\}$. So we need to work out g_i^σ and g_i^τ. We break up σ and τ to

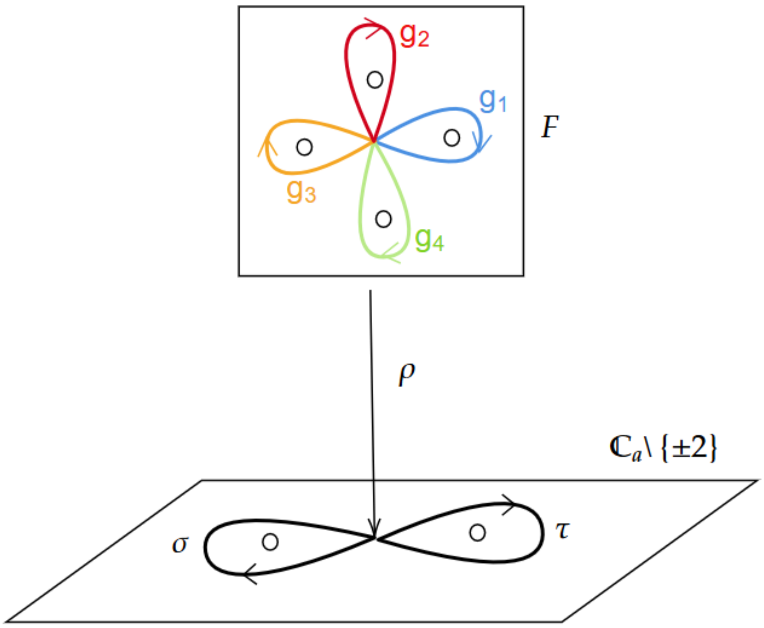


Figure 5.12: Loops generating $\pi_1(Y, 0)$.

consider the local monodromy and patching paths separately as per §4.1.3

$$\sigma = \lambda_-^{-1} \tilde{\sigma} \lambda_-,$$
$$\tau = \lambda_+^{-1} \tilde{\tau} \lambda_+.$$

where λ_+ and λ_- are paths that traverse the real line from $0 \in \mathbb{C}_a \backslash \{\pm 2\}$ to points $a_-, a_+ \in (-2, 2) \subset \mathbb{C}_a \backslash \{\pm 2\}$ arbitrarily close to -2 and 2 respectively, and $\tilde{\sigma}$ and $\tilde{\tau}$ are two small loops around -2 and 2 respectively (small enough so that the Puiseux series in the critical fibre are convergent, see §4.1.2). Let $F_\pm = \rho^{-1}(a_\pm)$.

We start by analysing the monodromy of τ. Since the equations describing the missing points are $b = \pm\sqrt{a \pm 2}$, we see that traversing λ_+ just radially dilates the missing points. This implies that that monodromy action is trivial, in that monodromy of τ is determined by the local monodromy $\tilde{\tau}$.

So we just look locally at the critical point $a = 2$. We note that g_2 and g_4 are meridians of $b^2 = a-2$. Therefore as we take the limit of a tending to 2, the two missing points around which g_2 and g_4 loop tend to 0. This is because ρ is a non-generic projection of $b^2 = a-2$. We can describe those two missing points by $b = \sqrt{a-2}$ and hence as we traverse a local loop close around $a = 2$, the two missing points rotate 180 degrees around 0. This gives us the monodromy action of τ on g_i, and by symmetry we also have the monodromy action of σ on g_i. See Figure 5.13 for a visualisation.

We find the σ and τ monodromy relations:

$$\sigma^{-1}g_2\sigma = g_3{}^{-1}g_2g_3,\ \sigma^{-1}g_4\sigma = g_1{}^{-1}g_4g_1,\ \sigma^{-1}g_3\sigma = g_1,\ \sigma^{-1}g_1\sigma = g_3,$$
$$\tau^{-1}g_2\tau = g_4,\ \tau^{-1}g_4\tau = g_2,\ \tau^{-1}g_3\tau = g_4{}^{-1}g_3g_4,\ \tau^{-1}g_1\tau = g_2{}^{-1}g_1g_2.$$

Remark 5.4. Note that in the use of the Zariski Van-Kampen Theorem we implicitly chose the section of ρ to be the zero-section. Normally we want to choose a section that does not interact with the movement of the missing points in the fibres (see §4.1.3), for example because this ensures that loop lifts are meridians. The upshot of this choice however is that it'll be easier to see the semi-direct product structure (5.12) of $\pi_1(FIPS)$, since our basepoint (0,0) is the stacky point.

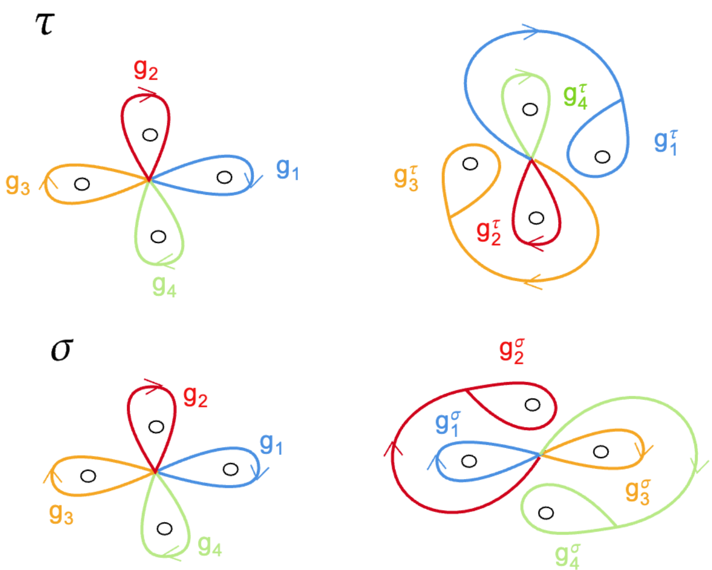


Figure 5.13: The monodromy actions of σ and τ.

By Equation (5.12), $\pi_1(FIPS)$ also has an additional generator ω of order 4, along with the orbifold relations:

$$\omega^{-1}\sigma\omega = \tau,\ \omega^{-1}\tau\omega = \sigma,\ \omega^{-1}g_i\omega = g_{i+1},$$

where $i = 1,2,3,4$ and the g_i suffix is read modulo 4. Using the orbifold relations, we find that the τ and σ monodromy relations imply each other: for $n = 0,1$ and $i, j = 1,2,3,4$

$$\tau^{-1}g_i\tau = {g_j}^{-n}g_k{g_j}^{n} \iff \sigma^{-1}g_{i-1}\sigma = {g_{j-1}}^{-n}g_{k-1}{g_{j-1}}^{n}.$$

with subscripts $i-1$ or $j-1$ corresponding to their representative modulo 4. It follows we can eliminate the τ via orbifold relation $\tau = \omega^{-1}\sigma\omega$ and that that leaves us with the remaining orbifold relations and the σ-monodromy relations. Additionally we can use leftover orbifold relations $\omega^4, \omega^{-2}\sigma\omega^2$ and $\omega^{-2}g_1\omega^2 = g_3$ to show equivalences in the relations $\sigma^{-1}g_2\sigma = {g_3}^{-1}g_2g_3 \iff \sigma^{-1}g_4\sigma = {g_1}^{-1}g_4g_1$ and $\sigma^{-1}g_3\sigma = g_1 \iff \sigma^{-1}g_1\sigma = g_3$. In summary, we are left with fundamental group presentation

$$\pi_1(FIPS, 0) = \langle \omega, \sigma, g_1, g_2, g_3, g_4 \mid \omega^4 = 1,\ [\sigma, \omega^2] = 1,\ \omega^{-1}g_i\omega = g_{i+1}, \\ \sigma^{-1}g_2\sigma = {g_3}^{-1}g_2g_3,\ \sigma^{-1}g_3\sigma = g_1\rangle, \quad (5.14)$$

where the g_i suffixes are read modulo 4.

5.2.4 Relations between Functors

The mirror symmetry monodromy conjecture predicts an action of $\pi_1(FIPS)$ on the bounded derived category of the GIT quotient

$$X = \left[\mathbb{A}^3 /_{2,1,1} \mathbb{Z}_4\right].$$

More specifically, by Conjecture 3.17 we want to prove the following representation

$$\sigma = T_F, \text{because } \sigma \text{ is the composition of a meridian of } \nabla_A \text{ and } \nabla_\Gamma,$$
$$\omega = \mathscr{O}(-1) \in \text{Pic } X \cong \mathbb{Z}_4, \text{since } \omega \text{ has order 4 in } \pi_1(FIPS),$$
$$g_1 = T_{\mathscr{O}_0}, \text{because } g_1 \text{ is a meridian of } \nabla_A,$$
$$g_2 = T_{\mathscr{O}_0(1)},\ g_3 = T_{\mathscr{O}_0(2)},\ g_4 = T_{\mathscr{O}_0(3)}.$$

Theorem 5.5. Conjecture 3.17 is true for GIT quotient X in Example (b).

Proof. We need to show that the autoequivalence assignments satisfy the relations in the fundamental group presentation in equation (5.14). $\omega^4 = 1$ is satisfied because $\mathscr{O}(-1) \in \text{Pic}X$ is 4-torsion, and the relations $\omega^{-1}g_i\omega = g_{i+1}$ follow automatically from Lemma 2.24. Using Lemma 2.24 again, the relations

$\sigma^{-1}g_2\sigma = g_3{}^{-1}g_2g_3$ and $\sigma^{-1}g_3\sigma = g_1$ can be read as

$$T_{(T_F)^{-1}\mathscr{O}_0(1)} = T_{(T_{\mathscr{O}_0(2)})^{-1}\mathscr{O}_0(1)},$$
$$T_{(T_F)^{-1}\mathscr{O}_0(2)} = T_{\mathscr{O}_0}.$$

Hence to prove the theorem it is enough to show that:

$$(\otimes\mathscr{O}(2))\circ T_F = T_F\circ(\otimes\mathscr{O}(2)), \tag{5.15}$$
$$(T_F)^{-1}\mathscr{O}_0(1) = (T_{\mathscr{O}_0(2)})^{-1}\mathscr{O}_0(1), \tag{5.16}$$
$$\mathscr{O}_0(2) = T_F\mathscr{O}_0, \tag{5.17}$$

where the last two equalities are only required to hold up to a shift, by property (1.), Lemma 2.16.

We start with relation (5.15). It is enough to show that RF commutes with $\otimes\mathscr{O}(2)$. Pullbacks commute with line bundle tensoring, so we just need to show that our pushforwards i_* and π_* commute with tensoring by $\mathscr{O}(2)$. For i_* this is obvious via the projection formula, and that $\mathscr{O}_{\mathbb{A}^1}(2) = i^*(\mathscr{O}_X(2))$. We also note that $\pi_* : D^b([\mathbb{A}^1/_2\mathbb{Z}_4]) \to D^b([\mathbb{A}^1/\mathbb{Z}_2])$ simply takes $\frac{\mathbb{Z}_4}{\mathbb{Z}_2}$ invariants so that for any $\mathscr{E}\in D^b([\mathbb{A}^1/_2\mathbb{Z}_4])$

$$\pi_*(\mathscr{E}(2)) = \pi_*\mathscr{E}(1).$$

Relation (5.15) follows directly from these observations.

For (5.16), we notice that the source category of F has semi-orthogonal decomposition

$$D^b([\mathbb{A}^1/\mathbb{Z}_2]) = \langle \mathscr{O}_{\mathbb{A}^1}(1), \mathscr{O}_0(1)\rangle.$$

By Theorem 2.27, we have $T_F = T_{F_{\mathbb{A}^1}}\circ T_{\mathscr{O}_0(2)}$ where $F_{\mathbb{A}^1} : D^b(\mathbb{A}^1)\to D^b(X)$ is defined as the restriction of F to the component generated by $\mathscr{O}_{\mathbb{A}^1}(1)$. It follows therefore that we just need to prove $T_{F_{\mathbb{A}^1}}\mathscr{O}_0(1) = \mathscr{O}_0(1)$, which is equivalent to showing that $F_{\mathbb{A}^1}R_{\mathbb{A}^1}\mathscr{O}_0(1) = 0$, where $R_{\mathbb{A}^1} = \mathfrak{p}_{\mathbb{A}^1}\circ R$ and $\mathfrak{p}_{\mathbb{A}^1} : D^b(X)\to D^b(\mathbb{A}^1)$ is the right adjoint to the inclusion $\langle\mathscr{O}_{\mathbb{A}^1}(1)\rangle \to D^b(X)$ [Huy06, p.17].

We have Koszul Resolution of $\mathscr{O}_0(k)$ in X:

$$\cdots\to 0\to\mathscr{O}(k)\to\mathscr{O}(k-3)^{\oplus 2}\oplus\mathscr{O}(k-2)\to\mathscr{O}(k-1)^{\oplus 2}\oplus\mathscr{O}(k-2)\to\mathscr{O}(k)\to 0\to\dots$$

Pulling back to $[\mathbb{A}^1_\gamma/_2\mathbb{Z}_4]$ and taking cohomology we compute

$$i^*\mathscr{O}_0(k) = \mathscr{O}_0(k-2)[2]\oplus\mathscr{O}_0(k-1)^2[1]\oplus\mathscr{O}_0(k).$$

Hence[6]

$$\begin{aligned} R\mathscr{O}_0(1) &= \pi_* i^! \mathscr{O}_0(1) \\ &= \pi_* i^* \mathscr{O}_0(3)[-2] \\ &= \mathscr{O}_0(1)^2[-1]. \end{aligned}$$

Relation (5.16) then follows because $\mathfrak{p}_{\mathbb{A}^1}\mathscr{O}_0(1)^2[-1] = 0$, i.e. the projection of $\mathscr{O}_0(1)^2[-1]$ onto the $\langle \mathscr{O}_{\mathbb{A}^1}(1)\rangle$ factor is 0. This is because $\mathscr{O}_0(1)^2[-1] \in \langle \mathscr{O}_0(1)\rangle$.

Relation (5.17) is slightly more delicate. Again we compute

$$\begin{aligned} R\mathscr{O}_0 &= \pi_* i^! \mathscr{O}_0 \\ &= \pi_* i^* \mathscr{O}_0(2)[-2] \\ &= \mathscr{O}_0 \oplus \mathscr{O}_0(1)[-2] \end{aligned}$$

So we have the cone

$$T_F\mathscr{O}_0 = C(\mathscr{O}_0 \oplus \mathscr{O}_0(2)[-2] \to \mathscr{O}_0).$$

We want to argue that the restriction of the cone map $\mathscr{O}_0 \to \mathscr{O}_0$ is non-zero. This follows because the map is an evaluation map, and thus includes all non-zero maps $\mathscr{O}_0 \to \mathscr{O}_0$ by definition. Since we clearly have one non-zero map, the identity, $T_F\mathscr{O}_0 = \mathscr{O}_0(2)$ up to a shift. This completes the verification of Conjecture 3.17 for Example (b). $\square$

[6]The canonical sheaves are $\omega_X = \mathscr{O}_X$ and $\omega_{[\mathbb{A}^1\,/_2\,\mathbb{Z}_4]} = \mathscr{O}_{[\mathbb{A}^1\,/\,\mathbb{Z}_4]}(2)$, so $i^!\mathscr{E} = i^*\mathscr{E}(2)[-2]$ for any $\mathscr{E} \in D^b(X)$.

5.3 Example (c)

In this section we prove Conjecture 3.17 in the case of Example (c), whose primary polygon is shown in Figure 5.14. We start by going through the toric geometry and study the autoequivalences on our chosen GIT quotient X in §5.3.1. In §5.3.2 we compute $FIPS$ and §5.3.3 we prove the fundamental group presentation

$$\pi_1(FIPS) = \langle \sigma_1, g_1, g_2, g_3 \mid (g_1 g_2)^3 = \sigma_1,\ [\sigma_1, g_1] = 1,\ [\sigma_1, g_2] = 1$$
$$[g_2, g_3] = 1,\ g_1 g_3 g_1 = g_3 g_1 g_3 \rangle.$$

Following the programme of the conjecture, in §5.3.4 we match autoequivalences to fundamental group generators in the following way

$$\begin{aligned}
\sigma_1 &= \otimes \mathscr{O}(\{\beta = 0\}), \\
g_1 &= T_{\mathbb{P}^2}, \\
g_2 &= \otimes \mathscr{O}(\{\varepsilon = 0\}), \\
g_3 &= T_{\mathbb{P}^1},
\end{aligned}$$

and we show that the fundamental group relations are satisfied by these autoequivalences.

5.3.1 Toric Geometry

We examine the geometry of the GIT quotients and study the autoequivalences in Example (c), leaning on the background from Section 2.1.

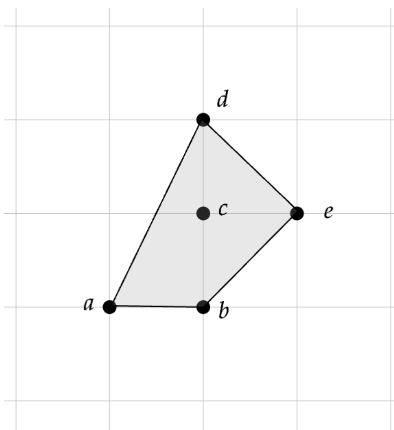


Figure 5.14: The primary polytope of Example (c).

Choosing lattice bases, we can take representations for the matrices packaging the data of our linear toric CY GIT problem (see (5.1)) to be the following:

$$A = \begin{array}{c@{}c} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array} \qquad Q = \begin{array}{c} \begin{pmatrix} 1 & 0 & -3 & 1 & 1 \\ 0 & 1 & -2 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \varepsilon \end{array} \end{array}$$

Plotting the weights $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{Z}^2$ we see our secondary fan as in Figure 5.15.

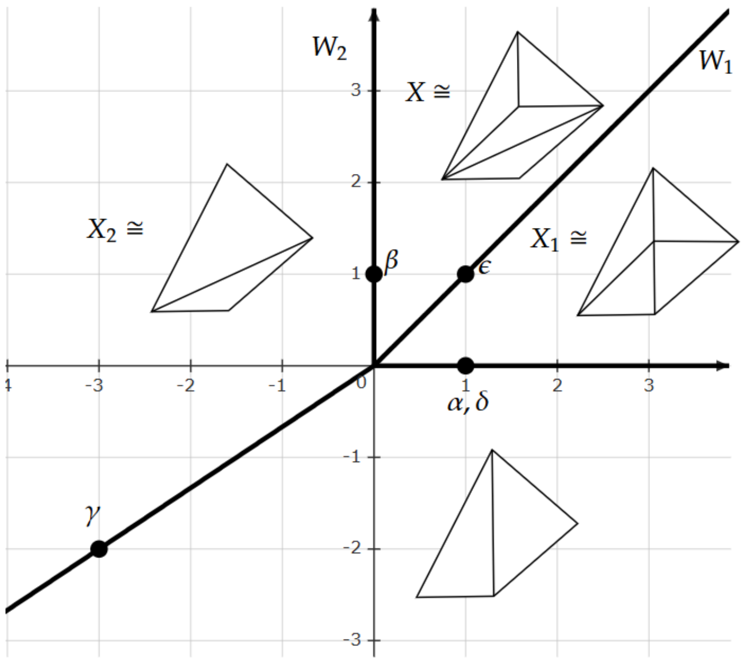


Figure 5.15: The secondary fan of Example (c). X, X_1, X_2 denote the GIT quotients corresponding to the chambers. W_1 and W_2 label the walls adjacent to chamber X. In each chamber we also have the triangulation corresponding to that GIT quotient.

With reference to Figure 5.15, we see that for the chamber corresponding to GIT quotient X, the unstable locus is

$$U = V(\beta\varepsilon, \beta\alpha, \beta\delta, \gamma\varepsilon) = \{\beta = \varepsilon = 0\} \cup \{\beta = \gamma = 0\} \cup \{\alpha = \delta = \varepsilon = 0\}.$$

The toric fan for X is given by the cone on the triangulation of the primary polygon in the chamber for X in Figure 5.15, and we see that X is a smooth toric variety (not a stack). We have

$$\operatorname{Pic} X \cong \mathbb{Z}^2,$$

with divisors $D_1 = \{\beta = 0\}$ and $D_2 = \{\varepsilon = 0\}$ as generators. There are two walls W_1 and W_2 adjacent to our chamber, into chambers corresponding to GIT quotients X_1 and X_2 respectively. To understand the spherical twists we get from crossing the walls (as defined in Theorem 2.25), first note that Z_{W_1} and Z_{W_2} are isomorphic to $pt = \operatorname{Spec}\mathbb{C}$. This makes sense because the primary polygon only has trivial faces, that is itself and its vertices.

So the spherical twists arising from the wall crossings W_1 and W_2 are simply spherical twists about spherical objects S_1 and S_2 which are supported on the unstable loci $U_1, U_2 \subset X$ associated to GIT quotients X_1 and X_2 respectively. These are

$$\begin{aligned} U_1 &= V(\alpha\varepsilon, \delta\varepsilon, \alpha\beta, \delta\beta) = \{\alpha = \delta = 0\} \cup \{\beta = \varepsilon = 0\}, \\ U_2 &= V(\gamma\beta, \gamma\varepsilon) = \{\gamma = 0\} \cup \{\beta = \varepsilon = 0\}. \end{aligned}$$

Therefore, S_1 is supported on $\{\alpha = \delta = 0\} \cong \mathbb{P}^1 \subset X$ and S_2 is supported on $\{\gamma = 0\} \cong \mathbb{P}^2 \subset X$. For a reader with experience with toric VGIT, this may make sense by looking at the triangulations, where crossing W_1 locally just loops like an Atiyah flop and W_2 looks like the GIT problem given by weight matrix (1,1,1,-3), which flops a $\mathbb{P}^2$ to a $B\mathbb{Z}_3$.

5.3.2 *FIPS*

In this example, unfortunately, we cannot brute-force compute E_A. We use Macaulay 2 “Dmodules” and “HolonomicSystems” packages to compute the principal A-determinant:

$$E_A = a^2bd^2e^2(-bc^4 + ac^3d + 8b^2c^2e - 36abcde + 27a^2d^2e - 16b^3e^2),$$

where we determined the multiplicity of the a, b, d and e factors in E_A because the extremal monomial terms of E_A are in one-to-one correspondence with the (coherent) triangulations of the primary polytope P (Theorem 3.8). As usual we have

$$FIPS = \left[\mathbb{C}^5 \backslash V(E_A) \big/_A (\mathbb{C}^*)^3\right].$$

Scaling out a, b, d coordinates we find that $FIPS$ is has no stacky points.

5.3.3 Fundamental Group

Our basepoint choice for $\pi_1(FIPS)$ will be near the large radius limit of X. Therefore, we want to work in b, e coordinates. We can use the $(\mathbb{C}^*)^3$ action via A to scale a, d to 1, but since we don’t remove $\{c = 0\}$, we can’t scale the c coordinate out directly.

Fortunately, by Proposition 4.8, we know that $\pi_1(FIPS\backslash\{c=0\}) \to \pi_1(FIPS)$ is surjective, and its kernel is $\langle\eta\rangle$, the normal subgroup generated by a meridian of the divisor $\{c=0\}$. So

$$\pi_1(FIPS) \cong \frac{\pi_1(FIPS\backslash\{c=0\})}{\langle\eta\rangle}. \tag{5.18}$$

We define $Y = FIPS\backslash\{c=0\}$ and start by computing $\pi_1(Y)$. Because $c \neq 0$, we can now scale the c-coordinate out and get:

$$Y \cong \mathbb{C}^2_{b,e}\backslash V(be(-b+1+8b^2e-36be+27e-16b^3e^2))$$

with no stackiness. We consider the projection $\rho : Y \to \mathbb{C}^*_b$ onto the b-coordinate.

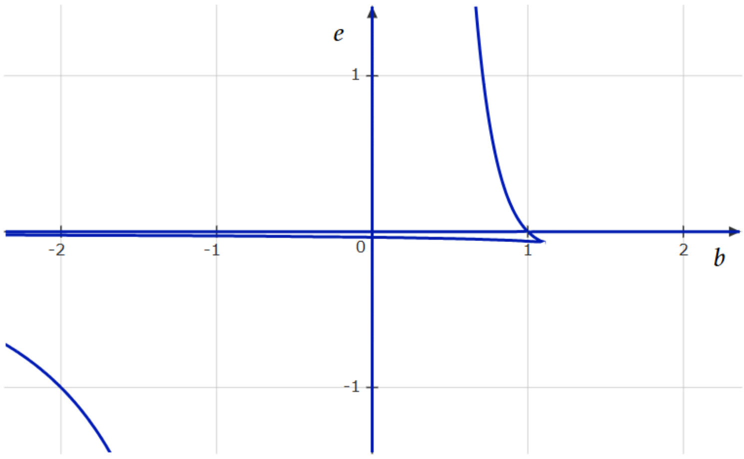


Figure 5.16: Real picture of Y.

Let $f(b,e) = be(-b+1+8b^2e-36be+27e-16b^3e^2)$. A generic fibre $\rho^{-1}(b_0)$ will be isomorphic to $\mathbb{C}$ with three points missing because f is degree 3 in e. There are however two critical points in $\mathbb{C}^*_b$

$$b=1,\ b=9/8.$$

Let $B = \mathbb{C}^*_b\backslash\{1,9/8\}$. Choose b_0 between 0 and 1 and consider generic fibre

$$F = \rho^{-1}(b_0).$$

We choose generators in the base $\sigma_1,\sigma_2,\sigma_3 \in \pi_1(B,b_0)$ around $0,1,9/8$. We pick a constant section of ρ, given by $e_0 \in \mathbb{C}_e$, as per Theorem 4.25. We take generators $g_1,g_2,g_3 \in \pi_1(F,e_0)$ which are loops as depicted in Figure 5.17. By Theorem 4.25 we have fundamental group presentation

$$\pi_1(Y,(b_0,e_0)) \cong \left\langle \sigma_1,g_1,g_2,g_3 \mid \sigma_1^{-1}g_i\sigma_1 = g_i^{\sigma_1}, g_i = g_i^{\sigma_2}, g_i = g_i^{\sigma_3} \right\rangle, \tag{5.19}$$

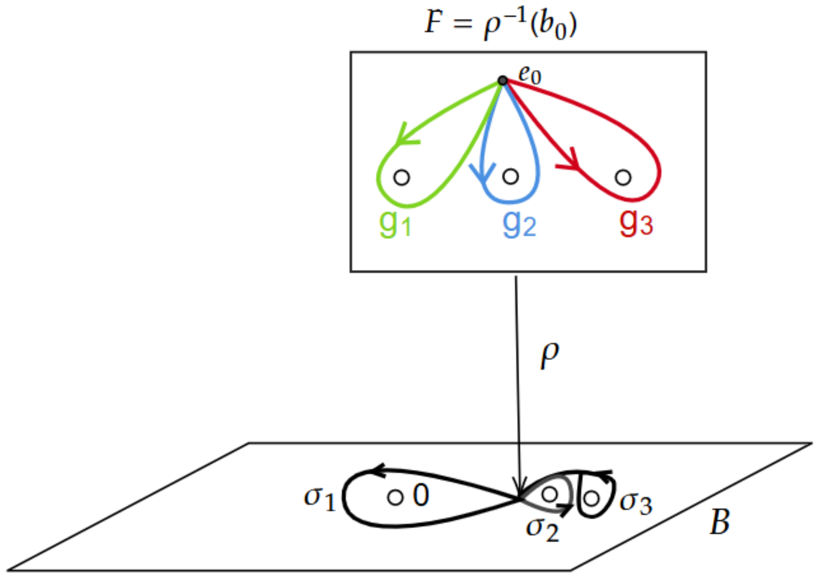


Figure 5.17: A more complex picture of Y, where we see the generators of $\pi_1(Y)$.

for $i = 1,2,3$.

We now need to understand ${g_j}^{\sigma_i}$. Using the globalising programme from §4.1.3 we break up each loop as

$$\begin{aligned}
\sigma_1 &= \lambda_1^{-1}\tilde{\sigma}_1\lambda_1, \\
\sigma_2 &= \lambda_2^{-1}\tilde{\sigma}_2\lambda_2, \\
\sigma_3 &= \lambda_3^{-1}\tilde{\sigma}_3\lambda_3,
\end{aligned}$$

where λ_i are paths that take $b_0 \in B$ to a point $b_i \in B$ arbitrarily close to each critical point, and $\tilde{\sigma}_i$ are small loops around each critical point (small enough so that the Puiseux series in the critical fibre are convergent, see §4.1.2). We pick λ_1, λ_2 to lie entirely on the real line, and we break-down λ_3 as

$$\lambda_3 = \lambda_3'\tau_2\lambda_2,$$

where

$$\tau_2(t) = \tilde{\sigma}_2(t/2),\ t \in [0,1],$$

is the path that traverses half of $\tilde{\sigma}_2$ and λ_3' is a path on the real line connecting $\tau_2(1)$ to b_3.

The small loops $\tilde{\sigma}_i$ determines the local monodromy. To globalise rigorously we also need analyse the monodromy action $\pi_1(F) \to \pi_1(F_i)$ of each λ_i, where $F_i = \pi^{-1}(b_i)$.

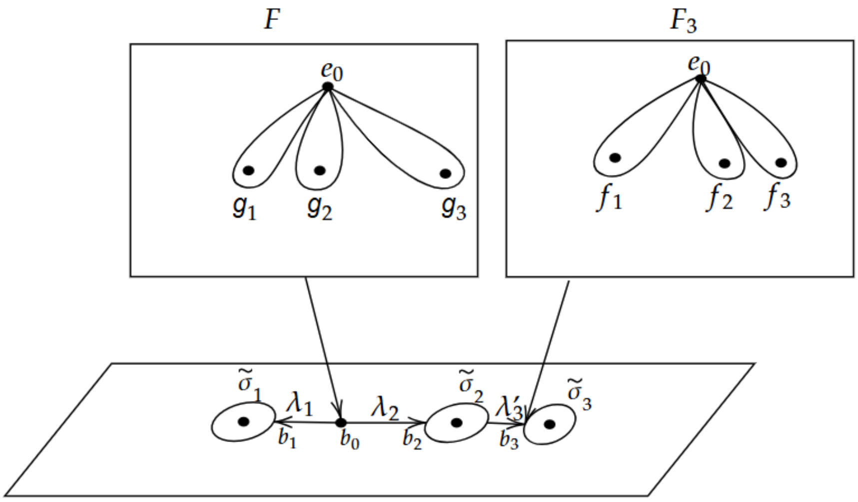


Figure 5.18: The breakdown of the paths $\sigma_1, \sigma_2, \sigma_3 \in \pi_1(B, b_0)$.

For σ_1 and σ_2, we can see from the real picture of Y that the monodromy of λ_1 and λ_2 is trivial.

This means the braids which determine the monodromy of σ_1 and σ_2 from F are canonically identified with the local braids from F_1 and F_2, i.e.

$$\mathfrak{b}^{\sigma_1} = \mathfrak{b}^{\tilde{\sigma}_1}, \ \mathfrak{b}^{\sigma_2} = \mathfrak{b}^{\tilde{\sigma}_2}.$$

So for σ_1 and σ_2 we really just need to look locally at the critical points, see §4.1.2 for detailed background. We start with $\tilde{\sigma}_1$. $\frac{f(b,e)}{b}(0,e) = e(1+27e)$ has two solutions, $e = 0$ and $e = -1/27$. As $b \to 0$, the three missing points of F will tend either to $e = 0, -1/27$, or ∞, and we can understand precisely how via Puiseux expanding at each of these points. Using Mathematica (code in Appendix B) we find that the Puiseux Series expansions of $V(f/b)$ over $b = 0$:

$$\begin{aligned}
e(b) &= 0, \\
e(b) &= -\frac{1}{27} - \frac{b}{81} - \frac{4b^2}{729} - \frac{56b^3}{19683} + O(b^4), \\
e(b) &= \frac{27b^{-3}}{16} - \frac{9b^{-2}}{4} + \frac{b^{-1}}{2} + \frac{1}{27} + \frac{b}{81} + O(b^2).
\end{aligned}$$

Exploiting the real picture in Figure 5.16 we can see that as b tends to 0, the missing point around which g_2 loops tends to $e = 0$, the missing point around which g_1 loops tends to $e = -1/27$, and the missing point around which g_3 loops tends to ∞. Using the Puiseux Expansions, we find that as we traverse $\tilde{\sigma}_1$ the missing points tending to $e = -1/27$ and $e = 0$ don't braid amongst themselves, but the missing point tending to ∞ braids around them both 3 times. This describes $\mathfrak{b}^{\tilde{\sigma}}$ in full.

Our choice of constant section $e_0 \in \mathbb{C}_e$ from Theorem 4.25 is such that the monodromy from the braid $\mathfrak{b}^{\tilde{\sigma}}$ leaves g_1 and g_2 unchanged, e.g. lies in the white region of Figure 4.6. On g_3, every rotation of missing point corresponding to g_3 around the other two points acts via conjugation by $g_1 g_2$. We hence find the following σ_1 monodromy relations

$$\begin{aligned}
\sigma_1^{-1} g_1 \sigma_1 &= g_1^{\sigma_1} = g_1, \\
\sigma_1^{-1} g_2 \sigma_1 &= g_2^{\sigma_1} = g_2, \\
\sigma_1^{-1} g_3 \sigma_1 &= g_3^{\sigma_1} = (g_1 g_2)^{-3} g_3 (g_1 g_2)^3.
\end{aligned}$$

Let us now look at the critical fibre over $1 \in \mathbb{C}_b^*$. We have two points missing, 0 and $-1/16$. Again, as $b \to 1$, the three missing points of F will tend either to $e = 0, -1/16$, or ∞. Again using Mathematica (Appendix B) we find that the Puiseux Series expansions of $V(f)$ over $b = 1$:

$$\begin{aligned}
e(b) &= 0, \\
e(b) &= -\frac{1}{16} - \frac{b-1}{16} - \frac{(b-1)^2}{8} - 3\frac{(b-1)^3}{8} - O((b-1)^4), \\
e(b) &= -(b-1) + 4(b-1)^2 - 8(b-1)^3 + 16(b-1)^4 + O((b-1)^5).
\end{aligned}$$

Exploiting the real picture in Figure 5.16 we can see that as b tends to 1, the missing point around which g_1 loops tends to $-1/16$ and the missing points around which g_2 and g_3 loop tend to $e = 0$.

From the Puiseux expansions we see that $b = 1, e = 0$ is a nodal singularity, i.e. a reducible singulaty with smooth components intersecting with multiplicity 1. Following Examples 4.22 and 4.20 ,we find we can summarise the relations $g_i = g_i^{\sigma_2}$ for $i \in \{1,2,3\}$ with the single commutation relation

$$[g_2, g_3] = 1.$$

We now need to compute the monodromy of σ_3, which is slightly more delicate because the fibre F is further away from the critical point $1 \in B$ and we need to 'move past' $0 \in B$. This makes the globalising monodromy action $\pi_1(F, e_0) \to \pi_1(F_3, e_0)$ of λ_3 non-trivial.

By the real picture again we can determine that the braid monodromy of λ_3' is trivial. This means that the braid $\mathfrak{b}^{\lambda_3}$ connecting F to F_3 is given by the braid $\mathfrak{b}^{\tau_2}$. Since τ_2 just traverses $\tilde{\sigma}_2$ half-way we have that $\mathfrak{b}^{\tau_2} = \mathfrak{b}_2$.

This is exactly what we saw in Example (a), and taking generators $f_1, f_2, f_3 \in \pi_1(F_3, e_0)$

as depicted in Figure 5.18

$$
\begin{aligned}
f_1^{\lambda_3^{-1}} &= f_1^{\mathfrak{b}_2^{-1}} = g_1,\\
f_2^{\lambda_3^{-1}} &= f_2^{\mathfrak{b}_2^{-1}} = g_2 g_3 g_2^{-1} = g_3,\\
f_3\lambda_3^{-1} &= f_3^{\mathfrak{b}_2^{-1}} = g_2,
\end{aligned}
$$

because g_2 and g_3 commute.

Now we want understand local braid for $\tilde{\sigma}_3$ from fibre F_3. Over $9/8$ we have two missing points, that is 0 and $-2/27$. Exploiting the real picture in Figure 5.16 and our understanding of $\tilde{\sigma}$ we deduce that the missing points in F_3 that f_1 and f_2 loop around tend to $-2/27$, while the other tends to 0 Using computation tools again we find that over $9/8$ we have Puiseux Expansions (see Appendix B for the code):

$$
\begin{aligned}
e(b) &= 0,\\
e(b) &= -\frac{2}{27} - \frac{16(b-9/8)}{81} - \frac{256\sqrt{2}(b-9/8)^{3/2}}{729} + O((b-9/8)^2).
\end{aligned}
$$

So at $(9/8, -2/27)$ there is a cuspidal singularity (Example 4.21) and by Example 4.15 the resulting non-trivial monodromy relation is

$$f_1 f_2 f_1 = f_2 f_1 f_2.$$

Hence, using the monodromy action of λ_3^{-1}, we find that the braid relation

$$g_1 g_3 g_1 = g_3 g_1 g_3$$

summarises the monodromy relations $g_i = g_i^{\sigma_3}$. This completes our group presentation for $\pi_1(Y)$.

Lastly, recall by equation (5.18) that to compute $\pi_1(FIPS)$ we need an additional a relation corresponding to a meridian η of $\{c = 0\}$. So we need to find a loop in terms of our generators σ_1, g_1, g_2, g_3 which is homotopic to a meridian η. To do this, we look at the chart of $FIPS$ with b,c-coordinates and analyse the gluing map between charts. Scaling e to 1 we have

$$Y_{b,c} \cong \left[\mathbb{C}^2_{b,c} \backslash \{bc(-bc^4 + c^3 + 8b^2c^2 - 36bc + 27 - 16b^3) = 0\} \big/_{2,1} \mathbb{Z}_3\right],$$

where by $Y_{b,c}$ we mean Y in b,c coordinates and by $Y_{b,e}$ we mean Y in b,e coordinates.

We have transition maps

$$Y_{b,e} \cong Y_{b,c}$$
$$(b,e) \overset{\psi}{\mapsto} (b\sqrt[3]{e}, \frac{1}{\sqrt[3]{e}})$$
$$(bc, \frac{1}{c^3}) \leftarrow\!\shortmid (b,c).$$

Take a meridian of $\{c=0\}$, for example $l(t) = (0.1, 0.1e^{2\pi it}) \in Y_{b,c}$, $0 \leq t \leq 1$. We see that $\psi^{-1}(l(t)) = (0.01e^{2\pi it}, 1000e^{-6\pi it}) \in Y_{b,e}$ traces out $(g_1g_2)^{-3}\sigma_1$, which we therefore take to be η.

Indeed $(g_1g_2)^3 = \sigma_1$ implies the σ_1-monodromy relation $\sigma_1^{-1}g_3\sigma_1 = g_3^{\sigma_1} = (g_1g_2)^{-3}g_3(g_1g_2)^3$. So finally we get the group presentation

$$\begin{aligned}\pi_1(FIPS,(b_0,e_0)) = \langle \sigma_1, g_1, g_2, g_3 \mid (g_1g_2)^3 = \sigma_1,\ [\sigma_1, g_1] = 1,\ [\sigma_1, g_2] = 1,\\ [g_2, g_3] = 1,\ g_1g_3g_1 = g_3g_1g_3\rangle.\end{aligned} \tag{5.20}$$

5.3.4 Relations between Functors

The mirror symmetry monodromy conjecture predicts an action of $\pi_1(FIPS)$ on the bounded derived category of the GIT quotient X. More specifically, by Conjecture 3.17 we want to prove the following representation

$$\begin{aligned}\sigma_1 &= \otimes\mathscr{O}(D_1),\\ g_1 &= T_{S_2} = T_{\mathbb{P}^2},\\ g_2 &= \otimes\mathscr{O}(D_2),\\ g_3 &= T_{S_1} = T_{\mathbb{P}^1},\end{aligned}$$

where we recall that the spherical object S_1 on X is the sheaf supported on $\{\alpha = \delta = 0\} \cong \mathbb{P}^1$, S_2 is supported on $\{\gamma = 0\} \cong \mathbb{P}^2$, $D_1 = \{\beta = 0\}$, and $D_2 = \{\varepsilon = 0\}$.

Theorem 5.6. Conjecture 3.17 is true for GIT quotient X in Example (c).

Proof. To prove that $\pi_1(FIPS)$ acts on $D^b(X)$ we have to show that our autoequivalences satisfy the relations in the presentation (5.20).

Relation $[\sigma_1, g_2] = 1$ follows simply because Pic X is commutative.

We can rewrite relation $[\sigma_1, g_1] = 1$ so that it reads

$$T_{\mathscr{O}_{\mathbb{P}^2}(D_1)} = T_{\mathscr{O}_{\mathbb{P}^2}}.$$

The relation follows because $D_1 = \{\beta = 0\} \in \mathrm{Div}X$ is not supported on $\mathbb{P}^2 = \{\gamma = 0\} \subset X$ and hence $\mathscr{O}_{\mathbb{P}^2}(D_1) \cong \mathscr{O}_{\mathbb{P}^2}$.

Similarly $[g_2, g_3] = 1$ reads

$$T_{\mathscr{O}_{\mathbb{P}^1}(D_2)} = T_{\mathscr{O}_{\mathbb{P}^1}},$$

and is implied similarly by the fact that $\mathscr{O}_{\mathbb{P}^1}(D_2) \cong \mathscr{O}_{\mathbb{P}^1}$.

By directly substituting, the braid relation $g_1 g_3 g_1 = g_3 g_1 g_3$ reads as

$$T_{\mathbb{P}^2} \circ T_{\mathbb{P}^1} \circ T_{\mathbb{P}^2} = T_{\mathbb{P}^1} \circ T_{\mathbb{P}^2} \circ T_{\mathbb{P}^1},$$

and to show it holds we just need to show that $\dim \mathrm{Hom}(S_2, S_1) = 1$ by Proposition 2.18. First note that $\mathbb{P}^2 = \{\gamma = 0\} \subset X$ and $\mathbb{P}^1 = \{\alpha = \delta = 0\} \subset X$ intersect transversely at a single point $p \in \mathbb{P}^1 \cap \mathbb{P}^2 \subseteq X$. We Koszul resolve $S_2 = \mathscr{O}_{\mathbb{P}^2}$ to get

$$0 \to \mathscr{O}(-\mathbb{P}^2) \xrightarrow{\gamma} \mathscr{O} \to 0,$$

and apply $\mathrm{Hom}(\ \ , S_1)$ to obtain:

$$0 \leftarrow \mathscr{O}_{\mathbb{P}^1}(p) \xleftarrow{\gamma} \mathscr{O}_{\mathbb{P}^1} \leftarrow 0.$$

Taking cohomology we find that $\mathrm{Ext}^1(S_2, S_1) = \Gamma(\mathscr{O}_p) = \mathbb{C}$, and $\mathrm{Ext}^i(S_2, S_1) = 0$ for $i \neq 1$. Hence $\dim \mathrm{Hom}(S_2, S_1) = 1$ and the braid relation is satisfied.

The trickiest relation to check is $(g_1 g_2)^3 = \sigma_1$, where direct substitution gives us

$$\left(T_{\mathbb{P}^2} \circ (\otimes \mathscr{O}(D_2))\right)^3 = \otimes \mathscr{O}(D_1). \tag{5.21}$$

We start by arguing that the autoequivalence

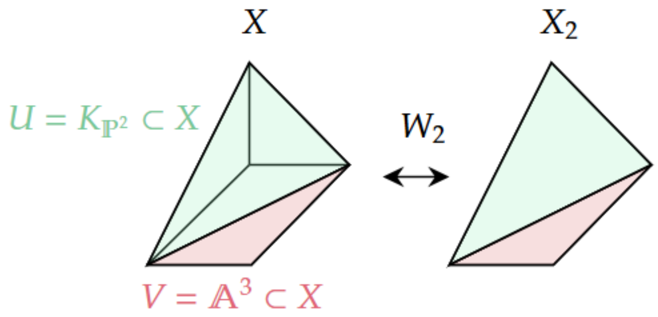
$$\Psi := \left(T_{\mathbb{P}^2} \circ (\otimes \mathscr{O}(D_2))\right)^3,$$

must be a line bundle twist.

We recall that the spherical twist autoequivalence $T_{\mathbb{P}^2} : D^b(X) \to D^b(X)$ arises from the following wall-crossing where $U \cong K^2_{\mathbb{P}}$ and $V \cong \mathbb{A}^3$ are open subsets of X whose union is all of X. We note that $T_{\mathbb{P}^2}$, and therefore $\Psi : D^b(X) \to D^b(X)$, intertwines over the identity $\mathrm{Id}_V : D^b(V) \to D^b(V)$, i.e. the following diagram commutes

$$\begin{tikzcd} D^b(X) \arrow[r, "\Psi"] \arrow[d] & D^b(X) \arrow[d] \\ D^b(V) \arrow[r, "\mathrm{Id}_V"] & D^b(V) \end{tikzcd}$$

where the vertical maps $D^b(X) \to D^b(V)$ are both the pullback of the inclusion $V \hookrightarrow X$. Similarly, Ψ intertwines over an autoequivalence of $\Phi : D^b(U) \to D^b(U)$ given by restriction, i.e. the following diagram commutes



$$\begin{tikzcd} D^b(X) \arrow[r, "\Psi"] \arrow[d] & D^b(X) \arrow[d] \\ D^b(U) \arrow[r, "\Phi"] & D^b(U) \end{tikzcd}$$

where again the vertical maps $D^b(X) \to D^b(U)$ are both the pullback of the inclusion $U \hookrightarrow X$ and the autoequivalence Φ is given by

$$\Phi := (T_{\mathscr{O}_{\mathbb{P}^2}|_U} \circ (\otimes \mathscr{O}_U(1)))^3.$$

We note that $U = K_{\mathbb{P}^2}$ is GIT quotient of the rank 1 linear toric CY GIT problem $\mathbb{C}^* \curvearrowright \mathbb{C}^4$ with weights $1,1,1,-3$, with the other GIT quotient being the DM stack $[\mathbb{C}^3/\mathbb{Z}_3]$. In rank 1, the monodromy conjecture has been proven, see for example [Kit19, Sec. 2.1] for full details. In particular this means that $T_{\mathscr{O}_{\mathbb{P}^2}|_U} \circ (\otimes \mathscr{O}_U(1))$ corresponds to twisting by a generator of Pic $[\mathbb{C}^3/\mathbb{Z}_3]$ under one of the derived equivalences $D^b(K_{\mathbb{P}^2}) \cong D^b([\mathbb{C}^3/\mathbb{Z}_3])$. By the fact that Pic $[\mathbb{C}^3/\mathbb{Z}_3] \cong \mathbb{Z}_3$, it follows that

$$\Phi = \mathrm{Id}_U.$$

Consider now the Fourier-Mukai kernel of Ψ:

$$\mathscr{K} \in D^b(X \times X).$$

By composition formula for Fourier-Mukai kernels [Huy06, Ch. 5], the intertwining diagrams over the identities of $D^b(U)$ and $D^b(V)$ imply

$$\begin{aligned} \mathscr{K}|_{X\times V} &= \mathscr{O}_{\Delta_V}, \\ \mathscr{K}|_{X\times U} &= \mathscr{O}_{\Delta_U}, \end{aligned}$$

where $\mathscr{O}_{\Delta_V} \in D^b(X \times V)$ and $\mathscr{O}_{\Delta_U} \in D^b(X \times U)$ are the structure sheaves of the diagonals $\Delta_V \subseteq V \times V \subseteq X \times V$ and $\Delta_U \subseteq U \times U \subseteq X \times U$. Via gluing of sheaves, this implies that $\mathscr{K}$ is isomorphic to a line bundle on the diagonal $\Delta_X \subset X \times X$, which proves that Ψ is a line bundle twist.

To determine which line bundle, we just need to compute where $\mathscr{O}$ gets sent under

$$(T_{\mathbb{P}^2} \circ (\otimes \mathscr{O}(D_2)))^3 .$$

We start by computing $T_{\mathbb{P}^2}\mathscr{O}(D_2)$. By Serre duality and the fact that the global sections of $\mathscr{O}(-1)$ on $\mathbb{P}^2$ are trivial we have

$$\mathrm{Hom}^i(\mathscr{O}_{\mathbb{P}^2}, \mathscr{O}(D_2)) \cong H^{3-i}(\mathscr{O}_{\mathbb{P}^2}(-D_2)) = 0.$$

This means

$$T_{\mathbb{P}^2}\mathscr{O}(D_2) = \mathscr{O}(D_2).$$

So we have

$$(T_{\mathbb{P}^2} \circ (\otimes \mathscr{O}(D_2)))^3 \mathscr{O} = (T_{\mathbb{P}^2} \circ (\otimes \mathscr{O}(D_2)))\, T_{\mathbb{P}^2}\mathscr{O}(2D_2).$$

By the same reasoning $\mathrm{Hom}^i(\mathscr{O}_{\mathbb{P}^2}, \mathscr{O}(2D_2)) = 0$

$$T_{\mathbb{P}^2}\mathscr{O}(2D_2) = \mathscr{O}(2D_2).$$

So

$$(T_{\mathbb{P}^2} \circ (\otimes \mathscr{O}(D_2)))^3 \mathscr{O} = T_{\mathbb{P}^2}\mathscr{O}(3D_2).$$

We note now that $\cong H^{3-i}(\mathscr{O}_{\mathbb{P}^2}(-3D_2)) = \mathbb{C}$ for $i = 1$. So

$$T_{\mathbb{P}^2}\mathscr{O}(3D_2) = C(\mathscr{O}_{\mathbb{P}^2}[-1] \to \mathscr{O}(3D_2)).$$

Note that dim $\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^2}, \mathscr{O}(3D_2)) = 1$. We have the short exact sequence

$$0 \to \mathscr{O}(-\{\gamma = 0\}) \to \mathscr{O} \to \mathscr{O}_{\mathbb{P}^2} \to 0.$$

Since $3D_2 = -\mathbb{P}^2 + D_1$, we can twist by $\mathscr{O}(D_1)$ to get

$$0 \to \mathscr{O}(3D_2) \to \mathscr{O}(D_1) \to \mathscr{O}_{\mathbb{P}^2} \to 0,$$

where we used that $\mathscr{O}_{\mathbb{P}^2}(D_1) = \mathscr{O}_{\mathbb{P}^2}$ since D_1 and $\mathbb{P}^2$ don't intersect. This is our extension, proving relation 5.21 and completing the verification of Conjecture 3.17 for Example (c).

$\square$

5.4 Example (d)

In this section we prove Conjecture 3.17 in the case of Example (d), whose primary polygon is shown in Figure 5.19. In §5.4.1 we start by going through the toric geometry and study the autoequivalences on our chosen GIT quotient

$$X = \left[\mathbb{A}^3 \,/_{1,2,2}\, \mathbb{Z}_5\right].$$

In §5.4.2 we compute $FIPS$ and §5.4.3 we prove the fundamental group presentation

$$\pi_1(FIPS) = \langle \omega, g_1, g_2, g_3, g_4, g_5 \mid g_2g_5g_2 = g_5g_2g_5,\ [g_5^{-1}g_2g_5, g_1] = 1,$$
$$\omega g_i \omega^{-1} = g_{i+3},\ \omega^5 = 1\rangle,$$

with $i \in 1, \dots, 5$ and indices taken modulo 5. Following the programme of the conjecture, in §5.4.4 we match autoequivalences to fundamental group generators

$$g_1 = T_{\mathscr{O}_0(2)},\ g_2 = T_{\mathscr{O}_0},\ g_3 = T_{\mathscr{O}_0(3)},\ g_4 = T_{\mathscr{O}_0(1)},\ g_5 = T_{\mathscr{O}_0(4)},\ \omega = \mathscr{O}(4),$$

and we show that the fundamental group relations are satisfied by these autoequivalences.

5.4.1 Toric Geometry

We examine the geometry of the GIT quotients and study the autoequivalences in Example (d), leaning on the background from Section 2.1.

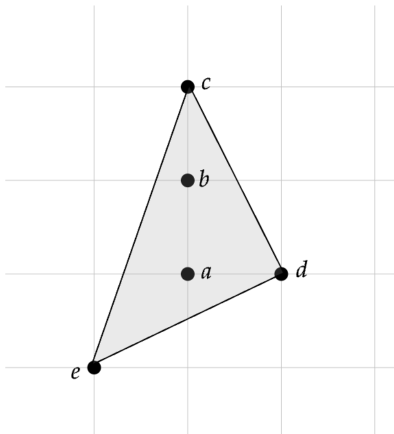


Figure 5.19: The primary polytope of Example (d).

Choosing lattice bases, we can take representations for the matrices packaging the data of our linear toric CY GIT problem (see (5.1)) to be the following:

$$A = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \qquad Q = \begin{array}{ccccc} \begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \end{pmatrix} \\ \alpha \quad \beta \quad \gamma \quad \delta \quad \varepsilon \end{array}$$

Plotting the weights $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{Z}^2$ we see our secondary fan as in Figure 5.20.

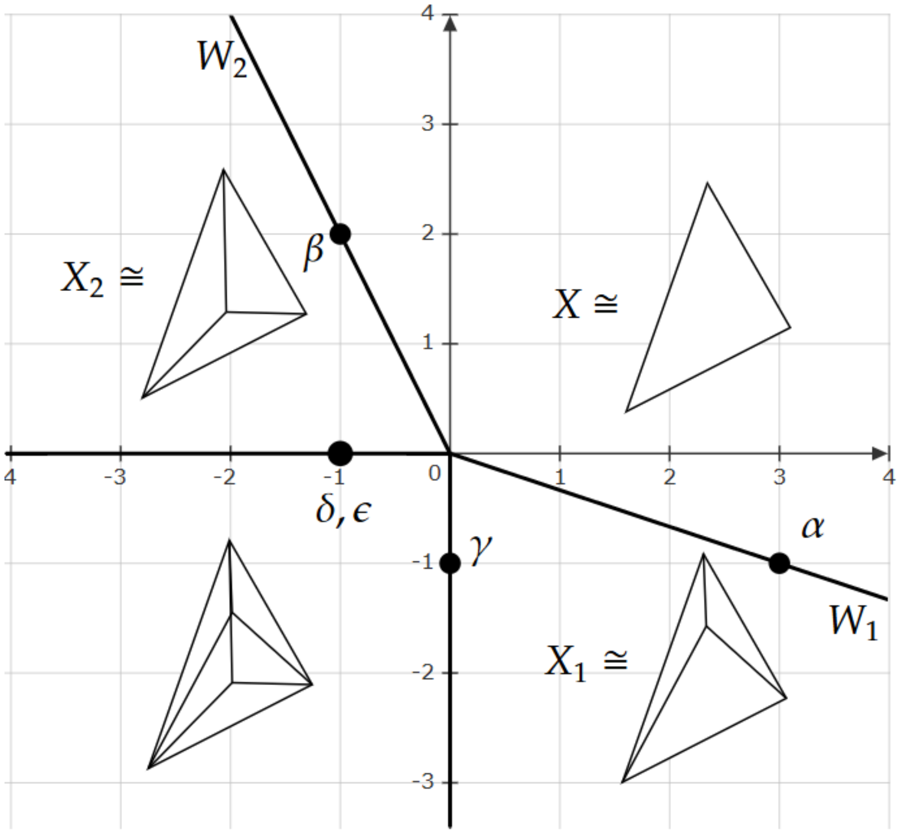


Figure 5.20: The secondary fan of Example (d). X, X_1, X_2 denote the GIT quotients corresponding to the chambers.

With reference to Figure 5.20, we see that for the chamber corresponding to GIT quotient X, the unstable locus is

$$U = V(\alpha\beta) = \{\alpha = 0\} \cup \{\beta = 0\}.$$

Since $\beta \neq 0$, we can use one parameter subgroup $\mu = (1,0) \in L^\vee$ to scale β to 1. This scaling is free as μ acts on $\mathbb{C}^5$ with weights $(3,-1,0,-1,-1)$ and the weight on the β coordinate is -1. Similarly we can use one parameter subgroup $\nu = (2,1)$

to scale α to 1, where ν acts on $\mathbb{C}^5$ with weights $(5,0,-1,-2,-2)$. This scaling is stacky, with stabiliser group $\mathbb{Z}_5$. We get the affine quotient stack

$$X \cong \left[\mathbb{A}^3_{\gamma,\delta,\varepsilon} \big/_{1,2,2} \mathbb{Z}_5\right],$$

and we have that

$$\text{Pic } X \cong \mathbb{Z}_5.$$

We have two walls W_1 and W_2 adjacent to our chamber, into chambers corresponding to GIT quotients X_1 and X_2 respectively. To understand the spherical twists we get from crossing the walls (as defined in Theorem 2.25), first note that Z_{W_1} and Z_{W_2} are isomorphic to $pt = \text{Spec } \mathbb{C}$. So the spherical twists arising from the wall crossings W_1 and W_2 are simply spherical twists about spherical objects S_1 and S_2, which are supported on the unstable loci of X_1 and X_2. The unstable locus U_i of each X_i can be computed to be

$$U_1 = V(\alpha\gamma, \alpha\delta, \alpha\varepsilon) = \{\alpha = 0\} \cup \{\gamma = \delta = \varepsilon = 0\},$$
$$U_2 = V(\beta\gamma, \beta\delta, \beta\varepsilon) = \{\beta = 0\} \cup \{\gamma = \delta = \varepsilon = 0\}.$$

This tells us that both S_1 and S_2 are supported on the origin of X, and hence $T_{\mathscr{O}_0}$ and $\mathscr{O}(1) \in \text{Pic } X$ generate the autoequivalences of interest.

5.4.2 *FIPS*

Using Macaulay 2 and the "Dmodules" and "HolonomicSystems" packages, we compute the principal A-determinant.

$$E_A = c^2d^3e^2(a^3b^4 - 8a^4b^2c + 16a^5c^2 + 27b^5de - 225ab^3cde + 500a^2bc^2de + 3125c^3d^2e^2).$$

We were able to know what the powers of the c, d and e factors of E_A by Theorem 3.8, which says that the extremal terms of E_A are given by (coherent) triangulations of the primary polytope. Recall the definition

$$FIPS := \left[\mathbb{C}^5 \backslash \nabla \big/_A (\mathbb{C}^*)^3\right],$$

where $\nabla = V(E_A)$. Because our basepoint choice when computing $\pi_1(FIPS)$ will be near the large radius limit of X, we want to work in a, b coordinates. We can use the $(\mathbb{C}^*)^3$ action via A to scale c, d, e to 1, and are left with a residual $\mathbb{Z}_5$ action:

$$FIPS \cong \left[\mathbb{C}^2_{a,b} \backslash V(f) \big/_{1,3} \mathbb{Z}_5\right],$$

where $f(a,b) = a^3b^4 - 8a^4b^2 + 16a^5 + 27b^5 - 225ab^3 + 500a^2b + 3125$. Take $Y = \mathbb{C}^2 \backslash V(f)$.

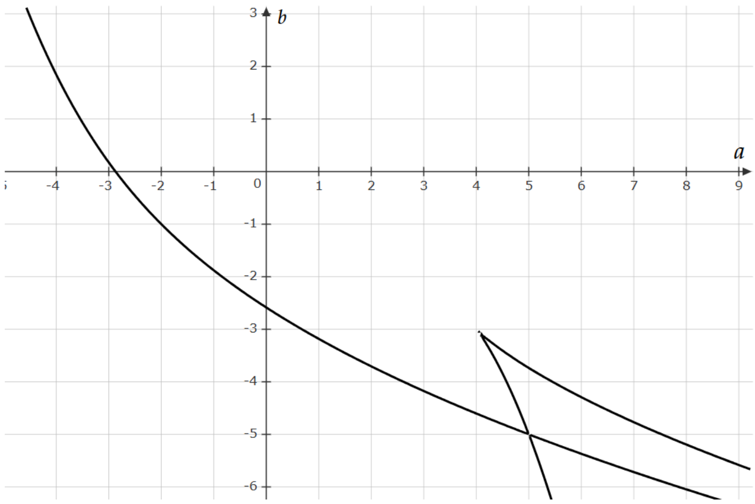


Figure 5.21: Real picture of Y.

5.4.3 Fundamental Group

The action of $\mathbb{Z}_5$ in Y has a fixed point at the origin and via the splitting of the short exact sequence in Proposition 4.31 we have

$$\pi_1(FIPS,0) = \pi_1(Y,0) \rtimes \mathbb{Z}_5. \tag{5.22}$$

Hence we start by computing $\pi_1(Y,0)$. We use the Zariski-Van-Kampen method, starting by projecting onto the a-coordinate $\rho : Y \to \mathbb{C}_a$. This projection has generic fibre $\rho^{-1}(a_0)$ isomorphic to $\mathbb{C}$ with 5 points missing since $f(a_0,b)$ is order 5 in b. We have 10 critical points given by

$$a_1 = \frac{5 \cdot 3^{4/5}}{2 \cdot 2^{3/5}} \approx 3.97,$$
$$a_2 = 5,$$

along with their orbits under the $\mathbb{Z}_5$ action. We denote the $\mathbb{C}_a$ without these points by B.

Take as our generic fibre

$$F = \rho^{-1}(0).$$

We pick loop generators in the base $\sigma_i, \tau_j \in \pi_1(B,0)$ and in the fibre over $g_k \in \pi_1(F,0)$ for $i,j,k \in \{1,2,3,4,5\}$ as per Figure 5.22 and using Theorem 4.25 we have fundamental group presentation

$$\pi_1(Y,0) = \langle g_1,g_2,g_3,g_4,g_5 | g_i^{\sigma_j} = g_i, g_i^{\tau_j} = g_i \rangle.$$

Note that we have not yet shown that the choice of zero section satisfies the conditions described in §4.1.3 that ensures that the lifts of the base loops σ_i, τ_i are meridians of the critical fibres and hence trivial, but indeed we will see that this is the case when we analyse the monodromy. Because of the action of $\mathbb{Z}_5$ on $\pi_1(Y,0)$

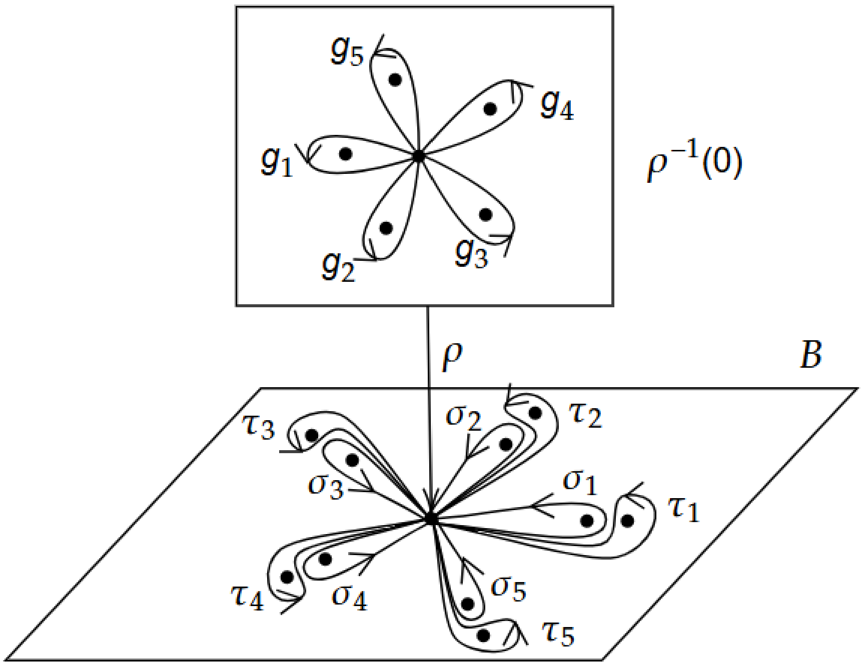


Figure 5.22: A picture of ρ and our fundamental group generators. For $i,j,k \in \{1,2,3,4,5\}$, $\sigma_i, \tau_j \in \pi_1(B,0)$ are loops in the base and $g_k \in \pi_1(F,0)$ are loops in the fibre over 0.

from equation (5.22), we only need to consider the monodromy around σ_1 and τ_1, which we shall denote σ and τ from now on. Using (5.22) we have

$$\pi_1(FIPS,0) = \langle \omega, g_1, g_2, g_3, g_4, g_5 | g_i^\sigma = g_i, g_i^\tau = g_i, \omega g_i \omega^{-1} = g_{i+3}, \omega^5 = 1 \rangle, \tag{5.23}$$

for $i \in \{1,2,3,4,5\}$, where we note that you read the indices of g_i modulo 5.

We now need to understand ${g_i}^\sigma$ and ${g_i}^\tau$. Following the globalising programme from §4.1.3 we break up each loop as in Figure 5.23

$$\sigma = \lambda^{-1}\tilde{\sigma}\lambda,$$
$$\tau = \eta^{-1}\tilde{\tau}\eta,$$

where

- λ is a path that takes $0 \in B$ to a point $a_1' \in B$ arbitrarily close to a_1 where $0 < a_1' < a_1$,

- η is a path that takes 0 to a point $a_2' \in B$ arbitrarily close to a_2 where $a_1 < a_2' < a_2$,
- $\tilde{\sigma}$ and $\tilde{\tau}$ are small loops around a_1 and a_2 (small enough so that the Puiseux series in the critical fibres are convergent, see §4.1.2).

We also have path breakdown

$$\eta = \lambda' \theta \lambda,$$

where

$$\theta(t) = \tilde{\sigma}(t/2),\ t \in [0,1],$$

is the path that traverses half of $\tilde{\sigma}$ and λ' is a path on the real line connecting $\theta(1)$ to a_2'. Let $F_1 = \rho^{-1}(a_1')$ and $F_2 = \rho^{-1}(a_2')$.

The small loops $\tilde{\sigma}$ and $\tilde{\tau}$ determine the local monodromy. To globalise rigorously we also need analyse the monodromy action $\pi_1(F) \to \pi_1(F_i)$ of λ and η. We start

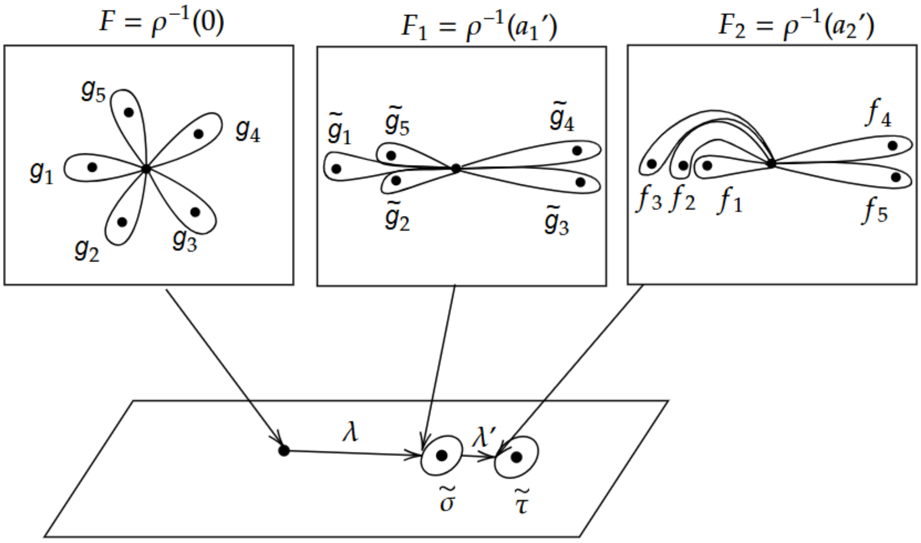


Figure 5.23: σ and τ paths broken up into parts $\lambda, \lambda', \tilde{\sigma}$, and $\tilde{\tau}$. We have $g_i \in \pi_1(\rho^{-1}(0))$, $\tilde{g}_j \in \pi_1(F_1)$, and $f_k \in \pi_1(F_2)$ for $i,j,k \in \{1,2,3,4,5\}$.

by looking at the monodromy action of λ. We need to use computation tools to understand this. Using Mathematica (see Appendix C for the code) we plot the movement of the missing points in the fibres as we traverse a path from 0 to a_1, see Figure 5.24. We see that the monodromy action is trivial.

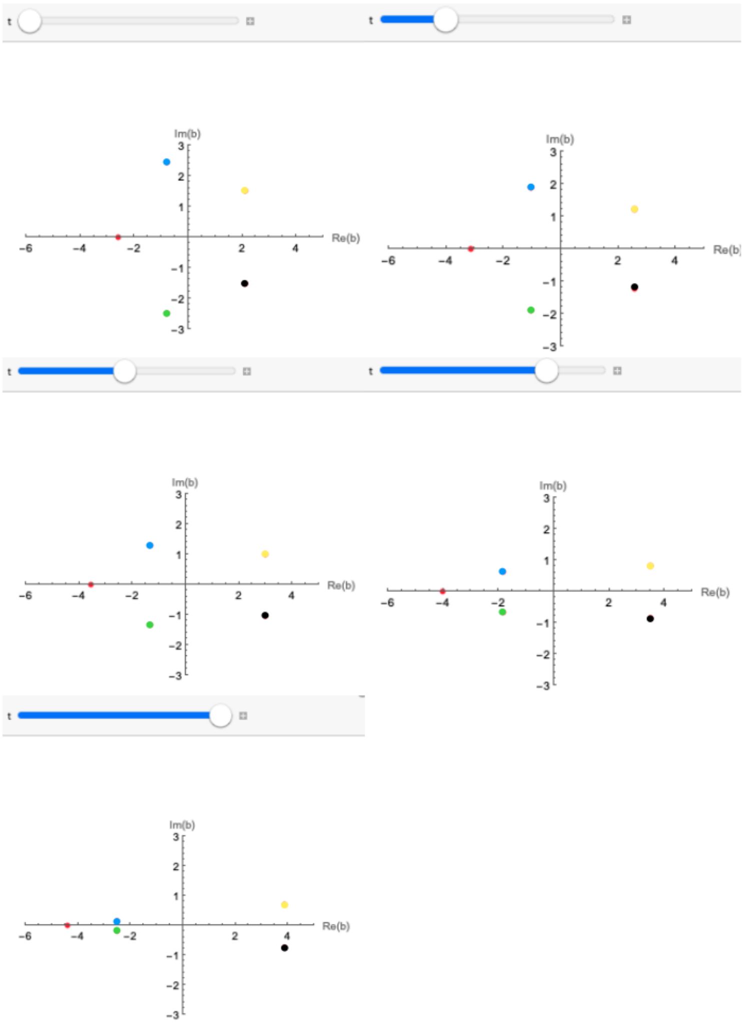


Figure 5.24: Monodromy of λ. The sliders for t parametrises the path, and below each one is the snapshot of the missing points in the fibre (the coloured dots).

So the monodromy around σ is determined by the local monodromy around $\tilde{\sigma}$. So we want understand local braid for $\tilde{\sigma}_3$ from fibre F_1. Over $a_1 \in \mathbb{C}_a$, we can use Mathematica (see Appendix B) to deduce that we have 4 missing points. We can moreover use Mathematica to find approximate Puiseux expansions[7]

$$b(a) = -4.59 + 0.41(a-a_1) + 0.02(a-a_1)^2 + O((a-a_1)^3),$$

$$b(a) = -2.97 - 1.12(a-a_1) + 0.53(a-a_1)^{3/2} + \sum_{i=4}^{\infty} c_i (a-a_1)^{i/2},$$

$$b(a) = 4.11 + 0.67i + (0.45 - 0.10i)(a-a_1) + O((a-a_1)^2),$$

$$b(a) = 4.11 - 0.67i + (0.45 + 0.10i)(a-a_1) + O((a-a_1)^2),$$

[7]We had to use Mathematica functions that only found approximate numerical solutions for the coefficients of the Puiseux expansions. Fortunately this isn't a problem for us because we are concerned with the powers in the expansion more than the coefficients, see §4.1.2.

where $c_i \in \mathbb{C}$ are some coefficients. We see therefore that at approximately $(a_1, -2.97)$ there is a cuspidal singularity (Example 4.21). Using our understanding of the λ-monodromy we see that as a tends to a_1, it is the two missing points around g_2 and g_5 that tend to this singular point. By Example 4.15, we know that as we traverse $\tilde{\sigma}$ we see these two missing points in the fibre rotate 3 times around each other anti-clockwise. Hence, we find that the relations $g_i^{\sigma} = g_i$ can be summarised by the the braid relation $g_2g_5g_2 = g_5g_2g_5$.

Now we look at the monodromy action of η. Using computational tools again we find that the braid from λ' is trivial, so that g_i^{η} is determined by the monodromy of θ. We understand the monodromy of θ as it's just traversing half of $\tilde{\sigma}$. We find

$$\begin{aligned}
f_1^{\eta^{-1}} &= g_2^{-1}g_5g_2,\\
f_2^{\eta^{-1}} &= g_2,\\
f_3^{\eta^{-1}} &= g_5g_1g_5^{-1},\\
f_4^{\eta^{-1}} &= g_4,\\
f_5^{\eta^{-1}} &= g_3.
\end{aligned}$$

Lastly, we need to compute the monodromy around $\tilde{\tau}$. Again, over $a_2 = 5 \in \mathbb{C}_a$, we can use Mathematica (see Appendix B) to deduce that we have 4 missing points and find approximate Puiseux expansions:

$$\begin{aligned}
b(a) &= -5 - 2.62(a-5) - 0.61(a-5)^2 + O((a-5)^3),\\
b(a) &= -5 - 0.38(a-5) + 0.013(a-5)^2 + O((a-5)^3),\\
b(a) &= -3.74 - 0.61(a-5) + 0.75(a-5)^2 + O((a-5)^3),\\
b(a) &= (4.55+0.57i) + (0.42-0.08i)(a-5) + O((a-5)^2),\\
b(a) &= (4.55-0.57i) + (0.42+0.08i)(a-5) + O((a-5)^2)
\end{aligned}$$

By Example 4.22, we see that $(5,-5)$ is a nodal singuularity and that as a tends to a_2, the missing points around f_3 and f_2 are tending to the nodal singularity. Hence using Examples 4.16 we can summarise the monodromy relation $f_i = f_i^{\tilde{\tau}}$ as

$$[f_2, f_3] = 1.$$

We can figure out what this means in terms of g_i by using the η^{-1} monodromy action. It follows that there is a fundamental group presentation

$$\begin{aligned}
\pi_1(FIPS) = \langle \omega, g_1, g_2, g_3, g_4, g_5 \mid g_2g_5g_2 = g_5g_2g_5,\ [g_5^{-1}g_2g_5, g_1] = 1,\\
\omega g_i \omega^{-1} = g_{i+3},\ \omega^5 = 1\rangle,
\end{aligned} \tag{5.24}$$

with $i \in \{1,2,3,4,5\}$ and indices taken modulo 5.

5.4.4 Relations between Functors

The mirror symmetry monodromy conjecture predicts $\pi_1(FIPS)$ acts on the bounded derived category of the GIT quotient

$$X = \left[\mathbb{A}^3 \big/_{1,2,2} \mathbb{Z}_5\right].$$

More specifically, by Conjecture 3.17 we want that the meridians of the principal discriminant component g_i are represented by a spherical twists around $\mathscr{O}_0 \otimes \mathscr{L}$ for $\mathscr{L} \in \mathrm{Pic}\, X \cong \mathbb{Z}_5$ and that ω corresponds to a generator of $\mathrm{Pic}\, X \cong \mathbb{Z}_5$. We choose the matchings:

$$g_1 = T_{\mathscr{O}_0(2)},\ g_2 = T_{\mathscr{O}_0},\ g_3 = T_{\mathscr{O}_0(3)},\ g_4 = T_{\mathscr{O}_0(1)},\ g_5 = T_{\mathscr{O}_0(4)},\ \omega = \otimes\mathscr{O}(4).$$

To prove that this representation holds, we need to demonstrate the the autoequivalences satisfy the relations in the group presentation (5.24).

Theorem 5.7. Conjecture 3.17 is true for GIT quotient X in Example (d).

Proof. We first note that $\omega^5 = 1$ is satisfied for $\omega = \otimes\mathscr{O}(4)$ as $\mathrm{Pic}X \cong \mathbb{Z}_5$. Additionally, we can verify that $\omega g_i \omega^{-1} = g_{i+3}$ are satisfied for our matching of functors by directly applying Lemma 2.24.

Let S_i denote the spherical object corresponding to the spherical twist g_i. By Proposition 2.18, if $\dim \mathrm{Hom}_{D^b(X)}(S_2, S_5) = \dim \mathrm{Hom}_{D^b(X)}(\mathscr{O}_0, \mathscr{O}_0(4)) = 1$, then g_2 and g_5 satisfy the braid relation. We have Koszul resolution of $\mathscr{O}_0$:

$$\cdots \to 0 \to \mathscr{O} \to \mathscr{O}(-3)^{\oplus 2} \oplus \mathscr{O}(-4) \to \mathscr{O}(-2)^{\oplus 2} \oplus \mathscr{O}(-1) \to \mathscr{O} \to 0 \to \dots. \tag{5.25}$$

and apply $\mathrm{Hom}_{D^b(X)}(\ \ , \mathscr{O}_0(4))$ to obtain

$$\dots \leftarrow 0 \leftarrow \mathscr{O}_0(4) \leftarrow \mathscr{O}_0(2)^{\oplus 2} \oplus \mathscr{O}_0(3) \leftarrow \mathscr{O}_0(1)^{\oplus 2} \oplus \mathscr{O}_0 \leftarrow \mathscr{O}_0(4) \to 0 \leftarrow \dots$$

Now we take global sections. This amounts to taking $\mathbb{Z}_5$ invariants, and we find $\mathrm{Hom}_{D^b(X)}(\mathscr{O}_0, \mathscr{O}_0(4)) = \mathbb{C}[-1]$. Hence the Hom space is 1-dimensional, and g_2 and g_5 satisfy the braid relation.

Lastly, we need to show that $\left[g_5^{-1} g_2 g_5, g_1\right] = \left[T_{(T_{\mathscr{O}_0(4)})^{-1}\mathscr{O}_0}, T_{\mathscr{O}_0(2)}\right] = 1$. This will hold if

$$\mathrm{Hom}\left(\mathscr{O}_0(2), (T_{\mathscr{O}_0(4)})^{-1}\mathscr{O}_0\right) = 0. \tag{5.26}$$

By definition

$$\begin{aligned}(T_{\mathscr{O}_0(4)})^{-1}\mathscr{O}_0 &= C(\mathscr{O}_0 \to \mathrm{Hom}(\mathscr{O}_0, \mathscr{O}_0(4))^{\vee} \otimes \mathscr{O}_0(4)) \\ &= C(\mathscr{O}_0 \to \mathscr{O}_0(4)[1]),\ \text{by the Koszul Resolution (5.25).}\end{aligned}$$

Notice we have a short exact sequence:

$$0 \to \mathscr{O}_0(4) \to \frac{\mathbb{C}[c,d,e]}{\langle c^2,d,e\rangle} \to \mathscr{O}_0 \to 0.$$

Let $R = \mathbb{C}[c,d,e]/\langle c^2,d,e\rangle$. R is the unique non-split $\mathrm{Ext}^1(\mathscr{O}_0, \mathscr{O}_0(4))$ so it follows that $(T_{\mathscr{O}_0(4)})^{-1}\mathscr{O}_0 = R[1]$. Using the Koszul complex again it's easy to show that (5.26) holds, completing the verification of Conjecture 3.17 for Example (d).

□

We can generalise this example a bit. Consider a primary polygon that is a 2D lattice triangle (the vertices are lattice points) with no lattice points on the edges, so that there are no minimal faces other than the entire triangle itself. By exploiting the $\mathbb{Z}^2 \rtimes GL_2(\mathbb{Z})$ action we find that all such primary polytopes have a representative with vertices (0,0), (1,0), and (n,m), where $n,m \in \mathbb{Z}_{\geq 0}$, $1 < n \leq \frac{m+1}{2}$, $\gcd(n,m) = \gcd(n-1,m) = 1$.

One of the GIT quotients is the toric stack with fan the cone on the triangle, and is it isomorphic to the smooth DM stack $[\mathbb{A}^3/_{n-1,-n,1}\mathbb{Z}_m]$. By analogy with Example (d), we find some relations between spherical twists of $D^b([\mathbb{A}^3/_{n-1,-n,1}\mathbb{Z}_m])$.

Proposition 5.8. Consider the smooth DM stack

$$X = [\mathbb{A}^3/_{n-1,-n,1}\mathbb{Z}_m],$$

where

$$n,m \in \mathbb{Z}_{\geq 0},\ 2 < n \leq \frac{m+1}{2},\ \gcd(n,m) = \gcd(n-1,m) = 1.$$

The spherical twists about the spherical objects $\mathscr{O}_0(l)$ and $\mathscr{O}_0(l+1)$ satisfy the braid relation. Additionally, for $k = -n+1 \bmod m$ and $k = n \bmod m$ we have

$$\left[T_{\mathscr{O}_0(k)}, T_{T_{\mathscr{O}_0(-1)}^{-1}\mathscr{O}_0}\right] = 1.$$

Proof. We have Koszul resolution of $\mathscr{O}_0$:

$$[\mathscr{O} \to \mathscr{O}(1-m) \oplus \mathscr{O}(-n) \oplus \mathscr{O}(-m+n-1) \to \mathscr{O}(-(n-1)) \oplus \mathscr{O}(-(m-n)) \oplus \mathscr{O}(-1) \to \mathscr{O}].$$

Therefore for $n \neq 2$, $\mathscr{O}_0$ and $\mathscr{O}_0(1)$ braid.

The proof of the second relation follows line by line in the same way as we proved it in Example (d) which coincides, for example, with $n = 3$ and $m = 5$. □

An interesting feature of Proposition 5.8 is that it inverts the approach taken in our preceding examples. Rather than analysing the fundamental group of *FIPS* in order to deduce relations among autoequivalences, we instead established relations between autoequivalences which, in turn, are expected to impose corresponding relations in the more intractable fundamental group of a hypersurface complement.

Notation

Abbreviations:
Geometric invariant theory (GIT), Fayet-Iliopoulos Parameter Space ($FIPS$), Deligne-Mumford stack (DM stack), Calabi-Yau (CY), semi-orthogonal decomposition (SOD), homological mirror symmetry (HMS), Fourier-Mukai (FM), Landau-Ginzburg model (LG model)

Conventions:

- We read composition of functors, functions, and objects in fundamental groups from right to left.
- By triangulations of polytopes, we only mean coherent triangulations, see the definition in [Gel+94, p.218].
- We often omit Tot when referring to total spaces of line bundles.
- We say orbifold and DM stack basically interchangeably, see Remark 2.3.

Notations:

1. For X an algebraic variety and G a group acting, $[X/G]$ denotes a quotient stack. For G finite, $BG := [\mathrm{Spec}\,\mathbb{C}/G]$ denotes a gerbe or point in a DM stack.
2. By $\mathbb{A}^n$ we denote standard n-dimensional affine space over the complex numbers. If we write it with subscripts, $\mathbb{A}^n_{x_1,\ldots,x_n}$, we mean that we take coordinates $x_1,\ldots,x_n$. Similarly, by $\mathbb{P}^{n-1}$ we denote standard $(n-1)$-dimensional projective space over the complex numbers, and if we write $\mathbb{P}^{n-1}_{x_1,\ldots,x_n}$ we mean that we take projective coordinates $[x_1,\ldots,x_n]$. Additionally, by $\mathbb{P}^{n-1}_{x_1,\ldots,x_n}(r_1,\ldots,r_n)$ we denote that weighted projective stack given by $\mathbb{C}^*$ acting on $\mathbb{C}^n_{x_1,\ldots,x_n}\setminus 0$

with weights $r_1, \dots, r_n$:

$$\mathbb{P}^{n-1}(r_1, \dots, r_n) = [\,(\mathbb{C}^n \setminus 0)\,/_{r_1,\dots,r_n}\,\mathbb{C}^*\,].$$

Note that when we add subscripts to $/$ in a quotient stack $[X/G]$ then those are the weights of the action of G on X.

3. $D^b(X)$ denotes the bounded derived category of coherent sheaves on X a smooth DM stack.

4. We write $\mathscr{T} = \langle \mathscr{T}_1, \dots, \mathscr{T}_k \rangle$ to denote SODs of triangulated categories.

5. For a smooth DM stack X, Pic X denotes the group of line bundles on X under tensor product. We use the notation $\mathscr{L} \in \operatorname{Pic} X$ to denote a general choice of line bundle.

6. $\pi_1(X,x)$ denotes the orbifold fundamental group of a smooth DM stack X based at x. For finite subset $B \subseteq X$, $\pi_1(X,B)$ denotes the fundamental groupoid of paths between points in B.

7. ω_X is the canonical sheaf of X a smooth DM stack.

8. $\mathbb{Z}_{>0}$ are the strictly positive integers, and $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{>0} \cup \{0\}$.

9. For a subset $I \subseteq J$, $I^c = J \setminus I$ denotes the complement.

10. For a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, $V(f) \subseteq \mathbb{C}^n$ denotes the vanishing locus.

11. For an abelian group G, $G_{\mathbb{R}} := G \otimes_{\mathbb{Z}} \mathbb{R}$.

12. For a subset of vectors $I \subseteq V$ of a vector space over a field k, we denote by $\langle I \rangle_k \subseteq V$ the subspace linearly spanned by the vectors in A.

13. By a rank-r, dimension-n linear toric GIT problem, we mean that we have an action of a r-dimensional algebraic torus $T \cong (\mathbb{C}^*)^r$ on a n-dimensional vector space $V \cong (\mathbb{C})^n$, see §2.1 for full details. This action defines a weight map $Q : \mathbb{Z}^n \to L^\vee$, where $L^\vee = \operatorname{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^r$ is called the *character lattice*.

14. $A^\vee : \mathbb{Z}^n \to N$ is defined to be the cokernel of the dual weight map $Q^\vee : L \to (\mathbb{Z}^n)^\vee$ (where we identify $(L^\vee)^\vee \cong L$). We note that N is a finitely generated group which may have torsion. We package this information into the short exact sequence
$$0 \to L \xrightarrow{Q^\vee} (\mathbb{Z}^n)^\vee \xrightarrow{A^\vee} N \to 0.$$
We can see $A := (A^\vee)^\vee : M \to \mathbb{Z}^n$ as a $n \times (n-r)$ dimensional matrix by picking a basis for the rank $n-r$ lattice $M = N^\vee$.

15. $e_i \in \mathbb{Z}^n$ are standard basis vectors, and $(e_i)^\vee \in (\mathbb{Z}^n)^\vee$ are dual basis vectors. $q_i = Q(e_i)$ are called weight vectors, and $a_i = A^\vee((e_i)^\vee) \otimes \mathbb{R} \in N_\mathbb{R} \cong \mathbb{R}^{n-r}$ are the rays of our GIT problem, which are integer vectors. The character is $\det V = \sum_{i=1}^n q_i$.

16. Suppose you a liner toric GIT problem given by T acting on V. $X_\theta := [X_\theta^{ss}/T]$ is the GIT quotient for stability condition $\theta \in L^\vee$, where $X_\theta^{ss} \subseteq V$ denotes the semi-stable locus, see §2.1 for the details. The stacky fan for the toric DM stack X_θ is given by $\boldsymbol{\Sigma}_\theta$, see equation (2.9).

17. Our GIT quotients X are toric DM stacks and $\mathrm{Pic}\, X \cong \mathbb{Z}^k \otimes \mathrm{Tor}(X)$ will be a finitely generated abelian group for some $k \in \mathbb{Z}_{\geq 0}$, where $\mathrm{Tor}(X)$ is the torsion subgroup. When we write $\mathscr{O}(n_1, \ldots, n_k, g) \in \mathrm{Pic}\, X$ for $g \in \mathrm{Tor}(X)$ we mean the line bundle corresponding to $(n_1, \ldots, n_k, g) \in \mathbb{Z}^k \otimes \mathrm{Tor}(X)$.

18. W denotes a wall in the secondary fan (or GKZ fan) Σ_{GKZ}, a codimension 1 cone. See §2.1. λ_W is a primitive one-parameter subgroup that is orthogonal to W.

19. $P \subseteq N_\mathbb{R}$ denotes the primary polytope, see Definition 2.7.

20. $\mathfrak{F}$ denotes the secondary toric stack, defined by stacky fan $\boldsymbol{\Sigma_{GKZ}}$ as given in Equation (2.6).

21. Associated to a wall W in the secondary fan, there is a Higgs GIT problem (Definition 2.9), and Z_W is a GIT quotient of this problem, with stability condition induced by W (Definition 2.10).

22. $F : \mathscr{A} \to \mathscr{B}$ denotes a spherical functor between triangulated categories $\mathscr{A}$ and $\mathscr{B}$. $T_F : \mathscr{B} \to \mathscr{B}$ denotes its spherical twist, see equation (2.15).

23. Given a wall W in the secondary fan Σ_{GKZ}, there are two adjacent chambers C_+ and $C-$, with GIT quotients X_+ and X_- respectively. There is a special spherical functor $F_W : D^b(Z_W) \to D^b(X_+)$ coming from the wall W, whose detailed definition can be found in Theorem 2.25 and the preceeding discussion.

24. For any morphism of smooth DM stacks $f : X \to Y$, the right adjoint to (derived) f_* is denoted $f^! : D^b(Y) \to D^b(X)$.

25. E_A denotes the principal determinant of A, where A is now the set rays of our GIT problem (Definition 3.3). $|A| = n$, with each element a $(n-r)$-dimensional integer vector. $\nabla = V(E_A) \subseteq \mathbb{C}^n$ is called the discriminant locus.

For subsets $\Gamma \subseteq A$, we have components $\Delta_\Gamma \in \mathbb{C}[a_1, \dots, a_n]$ and ∇_Γ (equation (3.6)).

26. $FIPS \subseteq \mathfrak{F}$ is the parameter space over which our Hori-Vafa mirrors live defined in Definition 3.6.

27. We write $L_W, L_X \subseteq FIPS$ to denote large radius regions associated to walls W or GIT quotients X. We write $c_X \in \mathfrak{F}$ for the large radius limit associated to GIT quotient X (Definition 3.10).

28. B_d denotes the braid group on d-strands, with $\mathfrak{b}_1, \dots, \mathfrak{b}_{d-1}$ as the standard braid group generators. For $d = 2$, we also denote the standard generator of B_2 by $\mathfrak{b}$.

29. By $\mathscr{C} \subseteq \mathbb{C}^2$ we denote plane algebraic curves, and by $\rho : \mathbb{C}^2 \backslash \mathscr{C} \to \mathbb{C}$ we denote a linear projection.

30. Consider a linear projection $\rho : \mathbb{C}^2 \backslash \mathscr{C} \to \mathbb{C}$ where $\mathscr{C}$ is an algebraic curve. Given a path γ in the base outside the critical set, $\mathfrak{b}^\gamma$ denotes the braid on d-strands traced out by the missing points in the fibres over γ as described in §4.1.1.

31. Consider a linear projection $\rho : \mathbb{C}^2 \backslash \mathscr{C} \to \mathbb{C}$ where $\mathscr{C}$ is an algebraic curve. $S \subseteq \mathbb{C}$ is the finite set of critical points, and we denote the critical fibres by $\mathscr{H} = \rho^{-1}(S)$. If there is only one critical fibre we denote it by H. $\rho|_{\mathbb{C}^2 \backslash (\mathscr{C} \cup \mathscr{H})}$ denotes the restriction of ρ
$$\rho|_{\mathbb{C}^2 \backslash (\mathscr{C} \cup \mathscr{H})} : \mathbb{C}^2 \backslash (\mathscr{C} \cup \mathscr{H}) \to \mathbb{C} \setminus S,$$
which is a locally trivial fibration, see Theorem 4.5.

32. In the examples in Chapter 5, we denote the weights by $\alpha, \beta, \gamma, \delta, \varepsilon \in L^\vee$ instead of the usual q_i notation. By abuse of notation we also identify the weight symbols with coordinates on the vector space V. Lastly we denote the rays by $a, b, c, d, e \in N$ instead of by a_i.

Appendix

A Computing Principal A-Determinants

We used Macaulay2 to compute E_A (Definition 3.3) for Example (c) (§5.3.2) and Example (d) (§5.4.2). To install Macaulay 2, follow https://github.com/Macaulay2/M2/wiki. For the documentation for the "HolonomicSystems" package we used, see: https://macaulay2.com/doc/Macaulay2/share/doc/Macaulay2/HolonomicSystems/html/_gkz.html. The code that we used in the macOS Terminal can be found below.

Example (c)

```
username@computer % M2

Macaulay2, version 1.25.06

i1 : loadPackage "HolonomicSystems"

i2 : A = matrix{{1,1,1,1,1},{-1,0,0,1,0},{-1,-1,0,0,1}}

i3 : H = gkz(A,{0,0,0})

i4 : factor (singLocus H)_0
```

o4 = $(x_1)(x_2)(x_4)(x_5)(-x_2 x_3^4 + 8 x_2^2 x_3^2 x_5 - 36 x_1 x_2 x_3 x_4 x_5 + 27 x_1^2 x_4^2 x_5 - 16 x_2^3 x_5^2)$

Example (d)

```
username@computer % M2

Macaulay2, version 1.25.06
```

```
i1 : loadPackage "HolonomicSystems"

i2 : A = matrix{{1,1,1,1,1},{0,0,0,1,-1},{0,1,2,0,-1}}

i3 : H = gkz(A,{0,0,0})

i4 : factor (singLocus H)_0
```

$$o4 = (x_5)(x_4)(x_3)(x_1^3x_2^4 - 8x_1^4x_2^2x_3 + 16x_1^5x_3^2 + 27x_2^5x_4x_5 - 225x_1x_2^3x_3x_4x_5 + 500x_1^2x_2x_3^2x_4x_5 - 3125x_3^3x_4^2x_5^2)$$

B Computing Critical Points and Puiseux Series

We used Mathematica v. 14.2. to compute the critical points and Puiseux expansions in Example (c) (§5.3.3) and Example (d) (§5.4.3). In this appendix we include the code that was used. Note that we used different functions in the two examples as Example (d) could not be symbolically solved, and numerical approximations were required for the coefficients of the series.

Example (c)

```
(*Define the polynomial*)
g[b_, e_] := b*e*(-b + 1 + 8 b^2 e - 36 b*e + 27 e - 16 b^3
   e^2);

(*Find Critical Values)

(*Define the critical values*)criticalBs = {0, 1, 9/8};

(*For each critical value, \
expand the implicit solution e(b) using Solve+Series*)
seriesSolutions =
  Table[Module[{sols, seriesList},(*Solve g(b,e)==0 for e
      near b0*)
     sols = Solve[g[b, e] == 0, e];
     (*Series expand each branch of e(b) around b0*)
     seriesList = Quiet@Series[#, {b, b0, 4}] & /@ (e /. sols
        );
```

```
    {b0, seriesList}], {b0, criticalBs}];

(*Display the results*)
TableForm[
 Flatten[Table[
   Table[{"b" -> seriesSolutions[[i, 1]],
     "e(b)" -> seriesSolutions[[i, 2, j]]}, {j,
     Length[seriesSolutions[[i, 2]]]}], {i,
    Length[seriesSolutions]}], 1],
 TableHeadings -> {None, {"b", "Series for e(b)"}}]
```

Example (d)

```
(*Define the polynomial*)
f[a_, b_] :=
  a^3 b^4 - 8 a^4 b^2 + 16 a^5 + 27 b^5 - 225 a b^3 + 500 a
     ^2 b + 3125;

(*Compute partial derivatives*)fa = D[f[a, b], a];
fb = D[f[a, b], b];

(*Use Reduce over complexes*)
criticalConditions = Reduce[{fa == 0, fb == 0}, {a, b},
   Complexes]

criticalAs = {5, (5 3^(4/5))/(2 2^(3/5)) };

(*Compute Puiseux-style expansions for b(a) near each
   critical a0*)
seriesSolutions =
  Table[Module[{a0 = N[aCrit], sol},
    sol = AsymptoticSolve[f[a, b] == 0, b, {a, a0, 4}];
    {a0, sol}], {aCrit, criticalAs}];

(*Display results neatly*)
TableForm[seriesSolutions,
 TableHeadings -> {None, {"a0", "Puiseux-style series for b(
    a)"}}]
```

C Computing Global Monodromy

We used Mathematica v. 14.2. to understand the global monodromy of λ in Example (d) (§5.4.3), see Figure 5.24. Here is the code.

```
(* Define the polynomial *)
f[a_, b_] :=
  a^3 b^4 - 8 a^4 b^2 + 16 a^5 + 27 b^5 - 225 a b^3 + 500 a
     ^2 b + 3125;

DynamicModule[{previousValues, colors, matchRoots},

  (* Function to match roots from one frame to the next *)
  matchRoots[old_, new_] :=
   Module[{perm},
    perm =
     FindPermutation[
       Nearest[new -> Automatic] /@ old /. Nearest ->
        (Nearest[#1 -> "Index"] &)]; (* Find best matching
           order *)
    Permute[new, perm]
    ];

  Manipulate[

   Module[{sol, values},
    sol = NSolve[f[t, b] == 0, b];
    values = b /. sol;

    If[! ValueQ[previousValues],
     (* First frame initialization *)
     previousValues = values;
     colors = {Red, Blue, Green, Purple, Orange};
     ,
     (* Match roots to maintain consistent ordering *)
     values = matchRoots[previousValues, values];
     previousValues = values;
     ];

    Graphics[
     Table[
```

```
    {colors[[i]], PointSize[Large],
     Point[{Re[values[[i]]], Im[values[[i]]]}]},
    {i, Length[values]}
    ],
   Axes -> True,
   AxesLabel -> {"Re(b)", "Im(b)"},
   PlotRange -> {{-10, 10}, {-10, 10}},
   ImageSize -> 400
   ]
  ],

 {{t, 0.01, "Parameter a"}, 0.01, 3.9, 0.05, Appearance ->
     "Labeled"},
 TrackedSymbols :> {t}
 ]
]
```

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