

UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics M11B: Analysis 2

COURSE CODE : **MATHM11B**

UNIT VALUE : **0.50**

DATE : **09-MAY-01**

TIME : **14.30**

TIME ALLOWED : **2 hours**

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TURN OVER

All questions may be attempted but only marks obtained on the best five solutions will count. The use of an electronic calculator is **not** permitted in this examination.

1. (i) Let f, g be functions differentiable at the point a . If $h(x) = f(x)g(x)$, show that $h'(a) = f'(a)g(a) + f(a)g'(a)$.
- (ii) Let f be a bounded differentiable function defined for all $x \in \mathbb{R}$. If $f(a) = \min f(x)$, show that $f'(a) = 0$.
- (iii) Let
$$f(x) = \begin{cases} x^3 & x \leq 1 \\ x^2 + x - 1 & x \geq 1 \end{cases}$$

Show that $f'(1)$ exists but $f''(1)$ does not exist.

2. (i) Let f be a continuous strictly increasing function on a closed interval $[a, b]$. Show that f has a continuous strictly increasing inverse f^{-1} defined on $[f(a), f(b)]$. Show further that if $y \in (f(a), f(b))$, $y = f(x)$ say, with $f'(x)$ existing, $f'(x) \neq 0$, then f^{-1} is differentiable at y and

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

- (ii) Let $f(x) = 1 - \cos x$, $x \in [0, \frac{\pi}{2}]$. Show that f has a continuous strictly increasing inverse f^{-1} defined on $[0, 1]$ and find $(f^{-1})'(y)$, where $0 < y < 1$.

3. (i) Define π and show that $2\sqrt{2} < \pi < 2\sqrt{3}$.
- (ii) Show that $\sin x > 0$, $0 < x \leq 2$.
- (iii) Show that $\sin \frac{\pi}{2} = 1$.

(Definitions based on geometric considerations will not be accepted).

4. (i) State and prove Rolle's theorem.
 (ii) State and prove the Mean Value Theorem.
- (iii) a) Find $\lim_{x \rightarrow 0} \frac{e^x - \cos x}{\tan x}$, b) $\lim_{x \rightarrow 0} \frac{(e^x - 1) \sin x}{\cos x - \cos^2 x}$,
 c) $\lim_{x \rightarrow +\infty} x \log \frac{x+1}{x-1}$.
5. (i) Let f be a twice differentiable function on $\mathbb{R}$ whose second derivative f'' is continuous on $\mathbb{R}$. If $f'(a) = 0$ and $f''(a) > 0$ show that f has a local minimum at a .
 (ii) Show that $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, $-1 < x \leq 1$.
6. (i) Let f be an increasing function defined on $[0, 1]$. Show that f is Riemann integrable on $[0, 1]$.
 (ii) If f and g are Riemann integrable functions on $[0, 1]$, show that, if $h(x) = f(x)g(x)$, $x \in [0, 1]$, then h is Riemann integrable on $[0, 1]$.
 (iii) If f is Riemann integrable on $[0, 1]$ and there exists $m > 0$ such that $f(x) \geq m$, $\forall x \in [0, 1]$, show that the function $1/f$ is Riemann integrable on $[0, 1]$.
7. (i) State and prove the Fundamental Theorem of Calculus.
 (ii) Evaluate $\int_0^{\pi/4} \frac{dx}{3 \cos^2 x + \sin^2 x}$.
 (iii) Find $\lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 \cos \frac{k\pi}{2n}$.