

UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics M11B: Analysis 2

COURSE CODE : MATHM11B

UNIT VALUE : 0.50

DATE : 04-MAY-99

TIME : 14.30

TIME ALLOWED : 2 hours

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All questions may be attempted but only marks obtained on the best five solutions will count. The use of an electronic calculator is not permitted in this examination.

1. (i) Let f be defined on an open interval containing the point a . Define what it means for f to be differentiable at a .

(ii) If $f(x) = x^n$, n a positive integer, show that $f'(a) = na^{n-1}$.

(iii) Let

$$f(x) = \begin{cases} x^3 + 8, & x \geq 2, \\ 12x - 8, & x \leq 2. \end{cases}$$

Show that f is differentiable at 2 with derivative 12.

2. State and prove

(i) Rolle's Theorem,

(ii) the Mean Value Theorem.

Suppose that f is defined and differentiable for every $x > 0$ and put $g(x) = f(x+1) - f(x)$. If $f'(x) \rightarrow 0$ as $x \rightarrow \infty$, prove that $g(x) \rightarrow 0$ as $x \rightarrow \infty$.

If $f'(x) \rightarrow 1$ as $x \rightarrow \infty$, show that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$.

3. (i) Use L'Hôpital's Rule to evaluate the following limits.

$$\lim_{x \rightarrow 0} \frac{\tan^3 x - \sin^3 x}{x^5},$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2},$$

$$\lim_{x \rightarrow 0} \frac{\log(1-x) + \log(1+x)}{x^2}.$$

- (ii) Suppose that f is differentiable in (a, b) , and that $a < x < b$. Suppose also that $\alpha_n \rightarrow x$ and $\beta_n \rightarrow x$ as $n \rightarrow \infty$, where $x < \alpha_n < \beta_n < b$ for $n = 1, 2, 3, \dots$.

Show that the quotients

$$\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

need not converge to $f'(x)$ as $n \rightarrow \infty$.

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4. Suppose that f is defined on $[a, b]$ and that the first $(n + 1)$ derivatives $f_{(x)}^{(1)}, \dots, f_{(x)}^{(n+1)}$ exist for all $x \in [a, b]$. Let the remainder term $R_{n,a}(x)$ be defined by

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + R_{n,a}(x),$$

for $a \leq x \leq b$.

Show that

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

for some $\xi \in (a, x)$.

If $f(x) = \cos x \sin x$, show that $R_{n,0}(x) \rightarrow 0$ as $n \rightarrow \infty$ and hence deduce that

$$\sin x \cos x = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{x^{2n+1}}{(2n+1)!}.$$

5. Let f be a bounded function on $[a, b]$, $a < b$.

Define the upper Riemann integral $\overline{\int}_a^b f(x) dx$ and the lower Riemann integral

$\underline{\int}_a^b f(x) dx$, and show that

$$\underline{\int}_a^b f(x) dx \leq \overline{\int}_a^b f(x) dx.$$

The function $f(x)$ is defined for x in $[0, 1]$ by setting $f(x) = 0$ if x is irrational and $f\left(\frac{p}{q}\right) = \frac{1}{q}$ if $p \geq 0$, $q > 0$ are integers with no common factors. Show that f is

Riemann integrable on $[0, 1]$ and determine $\int_0^1 f(x) dx$.

6. (i) Suppose f is Riemann integrable on $[a, b]$ and $m \leq f(x) \leq M$, $\forall x \in [a, b]$. Suppose g is continuous on $[m, M]$ and write $h(x) = g(f(x))$, $\forall x \in [a, b]$. Show that h is Riemann integrable on $[a, b]$.
- (ii) Show that the function $\cos\left[\frac{1}{x}\right]$, where $[x]$ denotes the integer part of x , is Riemann integrable on $[a, b]$, where $0 < a < b$.

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7. (i) Show that if f is a continuous function on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.
- (ii) Are the following functions uniformly continuous over the indicated interval? Give justifications.
- (a) $\cos\left(\frac{1}{x}\right)$ over $\left(0, \frac{\pi}{2}\right)$,
- (b) $\exp(-|x|)$ over $(-\infty, \infty)$,
- (c) x^3 over $(0, 1)$.

END OF PAPER