

UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics M11B: Analysis 2

COURSE CODE : MATHM11B

UNIT VALUE : 0.50

DATE : 05-MAY-98

TIME : 14.30

TIME ALLOWED : 2 hours

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All questions may be attempted but only marks obtained on the best **five** solutions will count. The use of an electronic calculator is **not** permitted during this examination.

1. Let f be a real-valued function defined on some open interval (a, b) . If $c \in (a, b)$, explain what is meant by the statement " f is differentiable at c ".

By using this definition show that, if $f(x) = x^n$, n a non-negative integer, then f is differentiable at c with derivative nc^{n-1} .

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R .

Show that, for $|x| < R$, g is continuous at x .

2. State and prove each of Rolle's Theorem, the Cauchy Mean Value Theorem and L'Hôpital's Rule.

Evaluate

(i) $\lim_{x \rightarrow 0} \frac{2x + x^2 + 2 \log(1 - x)}{x^3},$

(ii) $\lim_{x \rightarrow 0^+} x (\log x)^2,$

(iii) $\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right)^2 \tan^2 x.$

3. (i) Without using any general results, show that, if $a_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \log n$, the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded below.

Hence show that $a_n \rightarrow a$ as $n \rightarrow \infty$, where $0 \leq a \leq 1$.

- (ii) Given that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and that a power series can be differentiated term by term within its radius of convergence, show that

$$\cos(x + y) = \cos x \cos y - \sin x \sin y, \forall x, y \in \mathbb{R}.$$

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4. (i) Let f be a bounded real-valued function on $[a, b]$ and suppose that, for each $\varepsilon > 0$, there exists a partition P such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Prove that f is Riemann integrable on $[a, b]$.

- (ii) If f is a continuous function on $[a, b]$ show that f is uniformly continuous on $[a, b]$. Hence prove that f is Riemann integrable on $[a, b]$.

5. State and prove Newton's iterative method for finding roots. If

$$g(x) = x^3 - x^2 + x - 1, \quad x_1 = 1.01$$

and

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \quad n = 1, 2, \dots,$$

show that $x_n \rightarrow 1$ as $n \rightarrow \infty$.

6. Indicate how the approximation to $\int_a^b f(x) dx$, called Simpson's Rule, is obtained. If the fourth derivative $f^{(4)}(x)$ exists and is continuous on $[a, b]$, give an overestimate for the error incurred.

In estimating $\int_0^1 e^{-x^2} dx$, show that a subdivision of $[0, 1]$ into 10 equal parts will be enough to produce an error of at most 10^{-4} .

CONTINUED

7. (i) Suppose F and G are differentiable functions on $[a, b]$ and $F' = f$, $G' = g$ where f and g are Riemann integrable on $[a, b]$. Show that

$$\int_a^b F(x) g(x) dx = F(b) G(b) - F(a) G(a) - \int_a^b f(x) G(x) dx.$$

Evaluate $\int_0^{\frac{\pi}{2}} x \cos^2 x dx$.

- (ii) Suppose that $\int_a^b f(x) dx$ exists and $x = g(t)$ where

(a) $a = g(c)$, $b = g(d)$,

(b) $g'(t)$ exists and is continuous on $[c, d]$,

(c) $f(x)$ is continuous on $[a, b]$.

Show that

$$\int_a^b f(x) dx = \int_c^d f(g(t)) g'(t) dt.$$

Evaluate $\int_2^e \frac{1}{x (\log x)^2} dx$; where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

END OF PAPER