UNIVERSITY COLLEGE LONDON

2025-26 Main Summer Assessment Period

MATH0004: Analysis 2

Duration: **2 hours** Additional Materials: **No** Calculators Allowed: **No**
Assessment Pattern: **MATH00044AUE, MATH00044AUF**

CANDIDATE NUMBER	
SEAT NUMBER	

INSTRUCTIONS: (READ CAREFULLY!)

- **Do not turn this page over until the invigilators instruct you to do so. No aids are permitted.**
- This exam booklet contains 34 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **If you need extra space for a question, you may use pages 31 - 34 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Only work written on this booklet will be marked.** Work written on additional pages or additional booklets will not be read by the examiners and **messy work** might receive little credit.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.

Question 1. **(25 marks)**

(a) *(3 marks)* What is a partition P of an interval $[a, b]$ and what is its mesh $\|P\|$?

Solution: (3 marks, BOOKWORK) A *partition* P of the interval $[a, b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The set of all partitions of a given interval $[a, b]$ will be denoted by $\mathcal{P}[a, b]$.

The *mesh (norm, width)* of the partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

(b) Let $f : [a, b] \to \mathbb{R}$ be bounded.

i. *(3 marks)* Given a partition P of the interval $[a, b]$, define the upper Darboux sum $U(f, P)$.

Solution: (3 marks, BOOKWORK) The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$.

ii. *(3 marks)* Define the upper Riemann integral $\overline{\int_a^b} f(x)\,dx$.

Solution: (3 marks, BOOKWORK) The *upper Riemann integral* of f over $[a,b]$ is defined as

$$\overline{\int_a^b} f(x)\,dx := \inf_{P\in\mathcal{P}[a,b]} U(f,P)\,.$$

(c) Consider the function $f : [0,1] \to \mathbb{R}$, $f(x) = x$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0,1]$.

i. *(9 marks)* Prove that for all $i = 1, 2, \ldots, n$ we have

$$\frac{1}{2}(x_i^2 - x_{i-1}^2) < \left(\sup_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) \le \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1})\|P\|\,.$$

Solution: (9 marks, UNSEEN) We have $\sup\limits_{x\in[x_{i-1},x_i]} f(x) = x_i$, so

$$\left(\sup_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) = x_i(x_i - x_{i-1}) = \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1})^2. \quad (1)$$

Formula (1) implies

$$\frac{1}{2}(x_i^2 - x_{i-1}^2) < \left(\sup_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) \le \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1})\|P\|\,. \quad (2)$$

Use this page if you need more space for Question 1(c)(i).

ii. *(3 marks)* Prove that $\frac{1}{2} < U(f, P) \leq \frac{1}{2}(1 + \|P\|)$.

Solution: (3 marks, UNSEEN) Summing up the inequalities (2) over $i = 1, 2, \ldots, n$, we get

$$\frac{1}{2} < U(f, P) \leq \frac{1}{2}(1 + \|P\|). \tag{3}$$

iii. *(4 marks)* Using the definition from 1(b)(ii) and the result from 1(c)(ii), prove that

$$\overline{\int_0^1} x\,dx = \frac{1}{2}\,.$$

Solution: (4 marks, UNSEEN) The left inequality (3) tells is that $\frac{1}{2}$ is a lower bound for upper Darboux sums, whereas the right inequality (3) tells us that $\frac{1}{2}$ is the greatest lower bound for upper Darboux sums (because the mesh can be chosen to be arbitrarily small). Hence, by the definition from (b)(ii) $\overline{\int_0^1} x\,dx = \frac{1}{2}$.

Question 2. **(25 marks)**

(a) Let r be a natural number and let $f : [r, +\infty) \to [0, +\infty)$ be a decreasing function. Put

$$T_n := \sum_{k=r}^{n} f(k) - \int_r^n f(x)\,dx\,, \qquad n \in \mathbb{N}, \quad n \geq r.$$

i. *(2 marks)* Explain why the integral in the above formula makes sense, i.e. explain why the function f is Riemann integrable on $[r, n]$.

Solution: (2 marks, BOOKWORK) The integral makes sense because the function is monotonic.

ii. *(4 marks)* Prove that the sequence $\{T_n\}$ is decreasing.

Solution: (4 marks, BOOKWORK) We have

$$T_n - T_{n+1} = \int_n^{n+1} f(x)\,dx - f(n+1) = \int_n^{n+1} f(x)\,dx - \int_n^{n+1} f(n+1)\,dx$$
$$= \int_n^{n+1} (f(x) - f(n+1))\,dx \geq 0,$$

so the sequence $\{T_n\}$ is decreasing.

iii. *(8 marks)* Prove that $T_n \geq f(n)$.

Solution: (8 marks, BOOKWORK) We have

$$T_n = \sum_{k=r}^{n-1} \int_k^{k+1} f(k)\, dx + f(n) - \sum_{k=r}^{n-1} \int_k^{k+1} f(x)\, dx$$
$$= \sum_{k=r}^{n-1} \int_k^{k+1} (f(k) - f(x))\, dx + f(n) \geq f(n).$$

iv. *(1 mark)* Prove that $T_n \geq 0$.

Solution: (1 marks, BOOKWORK) The result from (a)(iii) and the fact that f is non-negative imply $T_n \geq 0$.

v. *(2 marks)* Prove that the sequence $\{T_n\}$ converges.

Solution: (2 marks, BOOKWORK) We are looking at a decreasing sequence $\{T_n\}$ which is bounded from below by zero. Such a sequence has to converge.

vi. *(3 marks)* Prove that the series $\sum_{k=r}^{\infty} f(k)$ converges if and only if $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists.

Solution: (3 marks, BOOKWORK) Suppose that the series $\sum_{k=r}^{\infty} f(k)$ converges. The formula for T_n can be rewritten as

$$\int_r^n f(x)\,dx = \sum_{k=r}^{n} f(k) - T_n\,.$$

The fact that the sequence $\{T_n\}$ converges and the algebra of limits tell us that $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists.

Conversely, suppose that $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists. The formula for T_n can be rewritten as

$$\sum_{k=r}^{n} f(k) = \int_r^n f(x)\,dx + T_n\,.$$

The fact that the sequence $\{T_n\}$ converges and the algebra of limits tell us that the series $\sum_{k=r}^{\infty} f(k)$ converges.

Use this page if you need more space for Question 2(a)(vi).

(b) *(5 marks)* Does the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converge or diverge? Justify your answer.

Solution: (5 marks, UNSEEN) Let us introduce the function $f : [2, +\infty) \to [0, +\infty)$, $f(x) := \frac{1}{x(\ln x)^2}$. This function is decreasing and takes nonnegative values, so by the Integral Test for the Convergence of Series (result from (a)(vi)) our series converges if and only if

$$\lim_{n\to\infty} \int_2^n \frac{dx}{x(\ln x)^2}$$

exists. But

$$\int_2^n \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\bigg|_{x=2}^{x=n} = \frac{1}{\ln 2} - \frac{1}{\ln n} \to \frac{1}{\ln 2} \qquad \text{as} \qquad n \to \infty.$$

Hence, the series converges.

Question 3. **(25 marks)**

(a) *(8 marks)* Suppose that the functions $f, g : (-1,1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1,1)$. Prove that $f(x) = g(x) + c$ for all $x \in (-1,1)$, where c is a constant.

[*Hint:* you may find it helpful to use the Mean Value Theorem.]

Solution: (8 marks, UNSEEN) Put $h(x) := f(x) - g(x)$. Then $h'(x) = 0$ for all $x \in (-1,1)$. I claim that

$$h(x) = h(0), \qquad \forall x \in (-1,1). \tag{4}$$

Case 1: $x = 0$. Obviously, (4) is true.

Case 2: $x \in (0,1)$. Applying the Mean Value Theorem to the function h on the interval $[0,x]$ we get

$$\frac{h(x) - h(0)}{x - 0} = h'(\xi) \tag{5}$$

for some $\xi \in (0,x)$. But $h'(\xi) = 0$, so (5) implies (4).

Case 3: $x \in (-1,0)$. Applying the Mean Value Theorem to the function h on the interval $[x,0]$ we get

$$\frac{h(0) - h(x)}{0 - x} = h'(\xi) \tag{6}$$

for some $\xi \in (x,0)$. But $h'(\xi) = 0$, so (6) implies (4).

Use this page if you need more space for Question 3(a).

(b) *(3 marks)* Define the notion of the radius of convergence of a real power series.

Solution: (3 marks, BOOKWORK) In the lectures the radius of convergence was defined as follows. Introduce the set

$$S := \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{7}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$. The *radius of convergence* of a power series is the extended real number

$$R := \sup S$$

where S is the set (7).

Another possible definition is that the radius of convergence of a power series is the extended real number R such that for $|x| < R$ the series converges absolutely and for $|x| > R$ the series diverges. I will accept this definition even though it is not *a priori* obvious that an R with these properties exists.

(c) *(4 marks)* State the theorem about the differentiability of a real power series.

Solution: (4 marks, BOOKWORK) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

(d) *(4 marks)* Find the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$.

Solution: (4 marks, UNSEEN) Fix an $x \neq 0$ and apply the ratio test:

$$\left| \frac{\frac{(-1)^{k+1}}{2k+3} x^{2k+3}}{\frac{(-1)^k}{2k+1} x^{2k+1}} \right| = \frac{2k+1}{2k+3} x^2 = \frac{1+\frac{1}{2k}}{1+\frac{3}{2k}} x^2 \to x^2$$

as $k \to \infty$. Hence, the power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is 1.

(e) *(4 marks)* Consider the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1}$. Prove that

$$g'(x) = \frac{1}{1+x^2}\,.$$

Solution: (4 marks, UNSEEN) The results from (c) and (d) tell us that the function g is differentiable on $(-1, 1)$ and that its derivative is obtained by differentiating the power series for g term-by-term:

$$g'(x) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1} \right)' = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1}\, x^{2k+1} \right)' = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1+x^2}$$

because we are looking at a geometric series with ratio $-x^2$.

(f) *(2 marks)* Prove that $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ for all $x \in (-1,1)$.

[*Hint:* you may find it helpful to use the results from 3(a) and 3(e).]

Solution: (2 marks, UNSEEN) Application of the result from (a) with $f(x) = \arctan x$ and $g(x)$ as in (e) gives

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + c\,, \qquad \forall x \in (-1,1).$$

Setting $x = 0$ we conclude that $c = 0$.

Question 4. **(25 marks)**

(a) *(12 marks)* Let n be a nonnegative integer and let $f:(-1,+\infty)\to\mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) := f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$
$$R_n(x) := f(x) - P_n(x).$$

Given an $x\in(-1,0)$, prove that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}\,(x-\xi)^n x$$

for some $\xi\in(x,0)$.

Solution: (12 marks, PARTIALLY SEEN. In the lectures the proof was done for Lagrange's form of the remainder and only for $x>0$. I mentioned the proof for Cauchy's form of the remainder only briefly, saying that at the last step one has to use the regular Mean Value Theorem as opposed to Cauchy's Generalisation of the Mean Value Theorem.) Fix $x\in(-1,0)$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots \\ - \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,, \quad (8)$$

where $t\in(-1,+\infty)$ is the independent variable. Since $f:(-1,+\infty)\to\mathbb{R}$ is $n+1$ times differentiable, $g:(-1,+\infty)\to\mathbb{R}$ is differentiable. Differentiating (8) we get

$$g'(t) = -f'(t) + \big[f'(t) - f''(t)(x-t)\big] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots \\ \dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

(all the terms cancelled out, apart from the last one). Applying the Mean Value Theorem to the function $g(t)$ on the interval $[x,0]$ (which we can because g is continuous on $[x,0]\subset(-1,+\infty)$ and differentiable on $(x,0)\subset(-1,+\infty)$), we get

$$\frac{g(0)-g(x)}{0-x} = g'(\xi) = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

for some $\xi\in(x,0)$. But

$$g(0) = R_n(x), \quad g(x) = 0,$$

so the previous formula can be rewritten as

$$\frac{R_n(x)}{-x} = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{n!}\,(x-\xi)^n x$.

Use this page if you need more space for Question 4(a).

Use this page if you need more space for Question 4(a).

(b) Consider the function $f : (-1, +\infty) \to \mathbb{R}$, $f(x) = \ln(1 + x)$.

i. *(3 marks)* Write down explicitly Maclaurin's polynomial $P_n(x)$ defined in 4(a).

Solution: (3 marks, SEEN) Straightforward calculations give

$$P_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} \,.$$

ii. *(4 marks)* Write down explicitly Cauchy's remainder $R_n(x)$ defined in 4(a).

Solution: (4 marks, PARTIALLY SEEN. This appeared as question 3 in online quiz 4. I do not class this as fully seen for three reasons. Firstly, I do not publish solutions to online quizzes. Secondly, even if a student did the online quiz successfully, reproducing the argument in examination conditions isn't easy. Thirdly, the online quiz had more steps and hints.) Straightforward calculations give

$$R_n(x) = \frac{(\xi - x)^n}{(1+\xi)^{n+1}}\, x\,.$$

iii. *(6 marks)* Using the results from 4(a), 4(b)(i) and 4(b)(ii), prove that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

for all $x \in (-1, 0)$.

$\left[\right.$ *Hint:* you may find it helpful to evaluate $\max_{\xi \in [x,0]} \dfrac{\xi - x}{1+\xi}$. $\left. \right]$

Solution: (6 marks, PARTIALLY SEEN. See comment to 4(b)(ii) and note that 4(b)(iii) is the most tricky bit of the argument.) Consider the function $h : [x, 0] \to \mathbb{R}$, $h(\xi) = \dfrac{\xi - x}{1+\xi}$. Observe that $h(x) = 0$, $h(0) = |x|$, and that the function h has no critical points on $(x, 0)$. Hence,

$$\max_{\xi \in [x,0]} \frac{\xi - x}{1+\xi} = h(0) = |x|.$$

This implies

$$|R_n(x)| \leq \frac{|x|^{n+1}}{1+\xi} < \frac{|x|^{n+1}}{1-|x|} \to 0$$

as $n \to \infty$.

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