**UNIVERSITY COLLEGE LONDON**

2025-26 Main Summer Assessment Period

# MATH0004: Analysis 2

Duration: **2 hours** Additional Materials: **No** Calculators Allowed: **No**

Assessment Pattern: **MATH00044AUE, MATH00044AUF**

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| CANDIDATE NUMBER | |
| SEAT NUMBER | |

**INSTRUCTIONS:** (READ CAREFULLY!)

- **Do not turn this page over until the invigilators instruct you to do so. No aids are permitted.**
- This exam booklet contains 34 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **If you need extra space for a question, you may use pages 31 - 34 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Only work written on this booklet will be marked.** Work written on additional pages or additional booklets will not be read by the examiners and **messy work** might receive little credit.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.

**Question 1.** **(25 marks)**

(a) *(3 marks)* What is a partition $P$ of an interval $[a, b]$ and what is its mesh $\|P\|$ ?

(b) Let $f : [a, b] \to \mathbb{R}$ be bounded.

i. *(3 marks)* Given a partition $P$ of the interval $[a, b]$, define the upper Darboux sum $U(f, P)$.

ii. *(3 marks)* Define the upper Riemann integral $\overline{\int_a^b} f(x)\, dx$.

(c) Consider the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0, 1]$.

i. *(9 marks)* Prove that for all $i = 1, 2, \ldots, n$ we have

$$\frac{1}{2}(x_i^2 - x_{i-1}^2) < \left( \sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \leq \frac{1}{2}(x_i^2 - x_{i-1}^2) + \frac{1}{2}(x_i - x_{i-1}) \|P\| .$$

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ii. *(3 marks)* Prove that $\frac{1}{2} < U(f, P) \leq \frac{1}{2}(1 + \|P\|)$.

iii. *(4 marks)* Using the definition from 1(b)(ii) and the result from 1(c)(ii), prove that

$$\overline{\int_0^1} x \, dx = \frac{1}{2} \, .$$

**Question 2.** **(25 marks)**

(a) Let $r$ be a natural number and let $f : [r, +\infty) \to [0, +\infty)$ be a decreasing function. Put

$$T_n := \sum_{k=r}^{n} f(k) - \int_r^n f(x)\, dx\,, \qquad n \in \mathbb{N}, \quad n \geq r.$$

i. *(2 marks)* Explain why the integral in the above formula makes sense, i.e. explain why the function $f$ is Riemann integrable on $[r, n]$.

ii. *(4 marks)* Prove that the sequence $\{T_n\}$ is decreasing.

iii. *(8 marks)* Prove that $T_n \geq f(n)$.

iv. *(1 mark)* Prove that $T_n \geq 0$.

v. *(2 marks)* Prove that the sequence $\{T_n\}$ converges.

vi. *(3 marks)* Prove that the series $\sum_{k=r}^{\infty} f(k)$ converges if and only if $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists.

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(b) *(5 marks)* Does the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converge or diverge? Justify your answer.

**Question 3.** **(25 marks)**

(a) *(8 marks)* Suppose that the functions $f, g : (-1, 1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1, 1)$. Prove that $f(x) = g(x) + c$ for all $x \in (-1, 1)$, where $c$ is a constant.

[*Hint:* you may find it helpful to use the Mean Value Theorem.]

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(b) *(3 marks)* Define the notion of the radius of convergence of a real power series.

(c) *(4 marks)* State the theorem about the differentiability of a real power series.

(d) *(4 marks)* Find the radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1}$.

(e) *(4 marks)* Consider the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \, x^{2k+1}$. Prove that

$$g'(x) = \frac{1}{1+x^2} \, .$$

(f) *(2 marks)* Prove that $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1}$ for all $x \in (-1,1)$.

[*Hint:* you may find it helpful to use the results from 3(a) and 3(e).]

**Question 4.** **(25 marks)**

(a) *(12 marks)* Let $n$ be a nonnegative integer and let $f : (-1, +\infty) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) := f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) := f(x) - P_n(x).$$

Given an $x \in (-1, 0)$, prove that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}\,(x-\xi)^n x$$

for some $\xi \in (x, 0)$.

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Use this page if you need more space for Question 4(a).

(b) Consider the function $f : (-1, +\infty) \to \mathbb{R}$, $f(x) = \ln(1 + x)$.

i. *(3 marks)* Write down explicitly Maclaurin's polynomial $P_n(x)$ defined in 4(a).

ii. *(4 marks)* Write down explicitly Cauchy's remainder $R_n(x)$ defined in 4(a).

iii. *(6 marks)* Using the results from 4(a), 4(b)(i) and 4(b)(ii), prove that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} .$$

for all $x \in (-1, 0)$.

$\Big[$ *Hint:* you may find it helpful to evaluate $\max\limits_{\xi \in [x,0]} \dfrac{\xi - x}{1+\xi}$ . $\Big]$

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