UNIVERSITY COLLEGE LONDON

2024-25 Main Summer Assessment Period

MATH0004

Module Code: **MATH0004**
Module Name: **Analysis 2**
Assessment Pattern: **MATH00044AUE, MATH00044AUF**
Duration: **2 hours**
Calculators Allowed: **No**
Additional Materials: **No**

DO NOT PHOTOCOPY THIS BOOKLET

FIRST NAME ....

SURNAME ....

STUDENT ID NUMBER ....

CANDIDATE NUMBER ....

SEAT NUMBER ....

INSTRUCTIONS: (READ CAREFULLY!)

- **Do not turn this page over until the invigilators instruct you to do so.**
- This exam booklet contains 22 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **If you need extra space for a question, you may use pages 20 - 22 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Only work written on this booklet will be marked.** Work written on additional pages or additional booklets will not be read by the examiners.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **Organise your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, mobile phones, or any other electronic device.

Question 1. **(25 marks)**

(a) *(3 marks)* Give, in terms of ε and δ, the mathematical definition of the statement "the function $f:(0,+\infty)\to\mathbb{R}$ is uniformly continuous".

Solution: (3 marks, BOOKWORK) We say that the function $f:(0,+\infty)\to\mathbb{R}$ is *uniformly continuous* if $\forall\varepsilon>0\ \ \exists\delta>0$ such that

$$|x-y|<\delta \quad \& \quad x,y\in(0,+\infty) \qquad \Longrightarrow \qquad |f(x)-f(y)|<\varepsilon.$$

(b) *(6 marks)* Prove that the function $f:(0,+\infty)\to\mathbb{R}$, $f(x)=e^{-x}$, is uniformly continuous.

[*Hint:* you may find it helpful to use the Mean Value Theorem.]

Solution: (6 marks, UNSEEN) Let ε be an arbitrary positive number. Take arbitrary x,y such that

$$|x-y|<\varepsilon \quad \& \quad x,y\in(0,+\infty). \tag{1}$$

I claim that (1) implies

$$|e^{-x}-e^{-y}|<\varepsilon. \tag{2}$$

Indeed, if $x=y$ then (2) is, obviously, true. If $x\neq y$ then the Mean Value Theorem tells us that

$$|e^{-x}-e^{-y}|=e^{-\xi}|x-y| \tag{3}$$

for some ξ strictly between x and y. But $\xi>0$, so $e^{-\xi}<1$ and, consequently, (3) and (1) imply (2). Thus, the definition of uniform continuity works with $\delta=\varepsilon$.

Use this page if you need more space for Question 1(b).

(c) *(3 marks)* Give the mathematical definition of the statement "the sequence of real numbers $\{x_n\}$ is a Cauchy sequence".

Solution: (3 marks, BOOKWORK) A sequence of real numbers $\{x_n\}$ is said to be a *Cauchy sequence* if $\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

(d) *(3 marks)* State Cauchy's General Principle of Convergence.

Solution: (3 marks, BOOKWORK) A sequence of real numbers $\{x_n\}$ converges iff it is a Cauchy sequence.

(e) *(10 marks)* Let $g : (0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that the sequence $\left\{g\left(\frac{1}{n}\right)\right\}$ converges.

[*Hint:* you may find it helpful to prove that $\left\{g\left(\frac{1}{n}\right)\right\}$ is a Cauchy sequence.]

Solution: (10 marks, UNSEEN) Let ε be an arbitrary positive number. The definition of uniform continuity (part (a)) tells us that there exists a $\delta > 0$ such that

$$|x - y| < \delta \quad \& \quad x, y \in (0, +\infty) \qquad \Longrightarrow \qquad |g(x) - g(y)| < \varepsilon. \tag{4}$$

The sequence $\left\{\frac{1}{n}\right\}$ converges to 0, hence, by Cauchy's General Principle of Convergence (part (d)), it is a Cauchy sequence. The definition of a Cauchy sequence (part (c)) tells us that there exists an $N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad \left|\frac{1}{m} - \frac{1}{n}\right| < \delta, \tag{5}$$

where the δ is the one from formula (4). Combining formulae (4) and (5), we get

$$m, n > N \qquad \Longrightarrow \qquad \left|g\left(\frac{1}{m}\right) - g\left(\frac{1}{n}\right)\right| < \varepsilon,$$

which in view of the definition of a Cauchy sequence (part (c)) means that $\left\{g\left(\frac{1}{n}\right)\right\}$ is a Cauchy sequence. Cauchy's General Principle of Convergence (part (d)) now tells us that the sequence $\left\{g\left(\frac{1}{n}\right)\right\}$ converges.

Use this page if you need more space for Question 1(e).

Question 2. **(25 marks)**

(a) *(5 marks)* State Riemann's Criterion for Integrability.

Solution: (5 marks, BOOKWORK) Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ iff $\forall \varepsilon > 0$ $\exists P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon. \tag{6}$$

(b) *(2 marks)* Give the mathematical definition of the statement "the function $f : [a,b] \to \mathbb{R}$ is decreasing".

Solution: (2 marks, BOOKWORK) A function $f : [a,b] \to \mathbb{R}$ is said to be *decreasing* on $[a,b]$ if

$$x_1 < x_2 \quad \& \quad x_1, x_2 \in [a,b] \qquad \Longrightarrow \qquad f(x_1) \geq f(x_2).$$

(c) *(2 marks)* Prove that a decreasing function $f : [a,b] \to \mathbb{R}$ is bounded.

Solution: (2 marks, for the case of an increasing function this argument was done in the lectures) We have $f(a) \geq f(x) \geq f(b)$, so our decreasing function is bounded.

(d) *(11 marks)* Prove that a decreasing function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. Here you may use without proof Riemann's Criterion for Integrability.

Solution: (11 marks, for the case of an increasing function this proof was done in the lectures) Consider a partition P_n into n subintervals of equal length: $P_n = \{x_0, x_1, \ldots, x_n\}$,

$$x_i = a + \frac{b-a}{n} i, \qquad i = 0, 1, \ldots, n.$$

Then

$$L(f, P_n) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i),$$

$$U(f, P_n) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i),$$

$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n}(f(x_0) - f(x_n)) = \frac{b-a}{n}(f(a) - f(b)).$$

Now, let ε be an arbitrary positive number. Choose n so large that

$$\frac{b-a}{n}(f(a) - f(b)) < \varepsilon.$$

Then

$$U(f, P_n) - L(f, P_n) < \varepsilon$$

and Riemann's Criterion for Integrability (part (a)) tells us that $f \in \mathcal{R}[a,b]$.

Use this page if you need more space for Question 2(d).

(e) *(5 marks)* Give an example of a bounded function $f : [-1, 1] \to \mathbb{R}$ which is not continuous but is Riemann integrable on $[-1, 1]$. Justify your answer.

Solution: (5 marks, UNSEEN) The function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x > 0, \end{cases}$$

is not continuous. However, it is decreasing, so by part (d) it is integrable on $[-1, 1]$.

Question 3. **(25 marks)**

(a) *(5 marks)* State Cauchy's Generalisation of the Mean Value Theorem.

Solution: (5 marks, BOOKWORK) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\xi \in (a, b)$ such that

$$(g(b) - g(a))f'(\xi) = (f(b) - f(a))g'(\xi)\,.$$

(b) *(12 marks)* Let n be a nonnegative integer and let $f : (-1, +\infty) \to \mathbb{R}$ be $n + 1$ times differentiable. Put

$$P_n(x) := f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$
$$R_n(x) := f(x) - P_n(x).$$

Given an $x > 0$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0, x)$.

Solution: (12 marks, BOOKWORK) Fix $x > 0$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - \dots$$
$$- \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x-t)^n\,, \quad (7)$$

where $t \in (-1, +\infty)$ is the independent variable. Since $f : (-1, +\infty) \to \mathbb{R}$ is $n+1$ times differentiable, $g : (-1, +\infty) \to \mathbb{R}$ is differentiable. Differentiating (7) we get

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \dots$$
$$\dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$
$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem (part (a)) to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[0, x]$ (which we can because g is continuous on $[0, x] \subset (-1, +\infty)$ and differentiable on $(0, x) \subset (-1, +\infty)$) we get

$$\frac{g(x) - g(0)}{h(x) - h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (0, x)$. But

$$g(0) = R_n(x), \quad g(x) = 0, \quad h(0) = x^{n+1}, \quad h(x) = 0,$$

so the above formula can be rewritten as

$$\frac{-R_n(x)}{-x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}$.

Use this page if you need more space for Question 3(b).

Use this page if you need more space for Question 3(b).

(c) *(8 marks)* Using the result from 3(b), prove that $\dfrac{7}{12} < \ln 2 < \dfrac{5}{6}$.

Solution: (8 marks, UNSEEN) Let us consider the function $f : (-1, +\infty) \to \mathbb{R}$, $f(x) = \ln(1 + x)$, and apply the result from (b) with $n = 3$. We get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(1+\xi)^4}\,. \tag{8}$$

Here x is an arbitrary positive number and ξ is some number on the interval $(0, x)$. Formula (8) implies

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}\,, \qquad \forall x > 0. \tag{9}$$

Setting $x = 1$ in (9) we arrive at the required result.

Use this page if you need more space for Question 3(c).

Question 4. **(25 marks)**

(a) *(2 marks)* Give the mathematical definition of the statement "the function $f : [0, +\infty) \to \mathbb{R}$ is locally Riemann integrable".

Solution: (2 marks, BOOKWORK, although in the lectures the definition was given for an arbitrary interval I and not specifically for the interval $[0, +\infty)$) A function $f : [0, +\infty) \to \mathbb{R}$ is said to be *locally Riemann integrable* over $[0, +\infty)$ if $\forall a, b \in [0, +\infty)$ we have $f \in \mathcal{R}[a, b]$.

(b) *(3 marks)* Let $f : [0, +\infty) \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $[0, +\infty)$ in the improper sense". Define the improper integral $\displaystyle\int_0^{+\infty} f(x)\,dx$.

Solution: (3 marks, BOOKWORK, although in the lectures the definition was given for an arbitrary interval I and not specifically for the interval $[0, +\infty)$) We say that f is integrable on $[0, +\infty)$ in the improper sense if $\displaystyle\lim_{b \to +\infty} \int_0^b f(x)\,dx$ exists. We will denote this limit by $\displaystyle\int_0^{+\infty} f(x)\,dx$.

(c) *(3 marks)* Use the definition from 4(b) to prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx$ and to evaluate this integral.

Solution: (3 marks, UNSEEN) Note that the integrand is continuous on $[0, +\infty)$, hence locally integrable on $[0, +\infty)$. Therefore, we have to check only integrability at $+\infty$. We have

$$\int_0^b \frac{1}{1+x^2}\,dx = \arctan b \to \frac{\pi}{2} \qquad \text{as} \qquad b \to +\infty,$$

so $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

(d) *(3 marks)* State the Comparison Theorem for Improper Integrals.

Solution: (3 marks, BOOKWORK) Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$, and suppose that

- $0 \le f(x) \le g(x)$ for all $x \in I$;
- $\int_c^d g(x)\, dx$ exists.

Then $\int_c^d f(x)\, dx$ exists.

(e) *(3 marks)* State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

Solution: (3 marks, BOOKWORK) Suppose that $f \in \mathcal{R}_{loc}(I)$ and $|f|$ is integrable over I. Then f is integrable over I.

(f) *(6 marks)* Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+x^2}\, dx$.

[*Hint:* you may find it helpful to use the results from 4(c), 4(d) and 4(e).]

Solution: (6 marks, UNSEEN) Let us prove the existence of the improper integral

$$\int_0^{+\infty} \frac{\sin x}{1+x^2}\, dx\,. \tag{10}$$

Note that the integrand is continuous on $[0, +\infty)$, hence locally integrable on $[0, +\infty)$. So we have to check only integrability at $+\infty$.

By part (e), in order to prove the existence of the integral (10) it is sufficient to prove the existence of the integral

$$\int_0^{+\infty} \frac{|\sin x|}{1+x^2}\, dx\,. \tag{11}$$

Compare now the functions $\dfrac{|\sin x|}{1+x^2}$ and $\dfrac{1}{1+x^2}$ on the interval $[0, +\infty)$. We have

$$0 \le \frac{|\sin x|}{1+x^2} \le \frac{1}{1+x^2}\,,$$

so, by part (d), in order to prove the existence of the integral (11) it is sufficient to prove the existence of the integral $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\, dx$. But this has already been done in part (c).

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(g) *(5 marks)* Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \left(\frac{\pi}{2} - \arctan x\right) \cos x \, dx\,.$

[*Hint:* you may find it helpful to integrate by parts.]

Solution: (5 marks, UNSEEN) Note that the integrand is continuous on $[0, +\infty)$, hence locally integrable on $[0, +\infty)$. So we have to check only integrability at $+\infty$.

Integrating by parts we get

$$\int_0^b \left(\frac{\pi}{2} - \arctan x\right) \cos x \, dx = \left(\frac{\pi}{2} - \arctan b\right) \sin b + \int_0^b \frac{\sin x}{1+x^2} \, dx\,.$$

We have $\left(\frac{\pi}{2} - \arctan b\right) \sin b \to 0$ as $b \to +\infty$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+x^2} \, dx\,.$ But this has already been done in part (f). In fact, the numerical values of the improper integrals in parts (f) and (g) are the same.

Use this page if you need more space for Question 4(g).

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