**UNIVERSITY COLLEGE LONDON**
2024-25 Main Summer Assessment Period

# MATH0004

Module Code: **MATH0004**
Module Name: **Analysis 2**
Assessment Pattern: **MATH00044AUE, MATH00044AUF**
Duration: **2 hours**
Calculators Allowed: **No**
Additional Materials: **No**

**DO NOT PHOTOCOPY THIS BOOKLET**

FIRST NAME ..........................................................................

SURNAME ..........................................................................

STUDENT ID NUMBER ..........................................................................

CANDIDATE NUMBER ..........................................................................

SEAT NUMBER ..........................................................................

**INSTRUCTIONS:** (READ CAREFULLY!)

- **Do not turn this page over until the invigilators instruct you to do so.**
- This exam booklet contains 26 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **If you need extra space for a question, you may use pages 23 - 26 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Only work written on this booklet will be marked.** Work written on additional pages or additional booklets will not be read by the examiners.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **Organise your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, mobile phones, or any other electronic device.

**Question 1.** **(25 marks)**

(a) *(3 marks)* Give, in terms of $\varepsilon$ and $\delta$, the mathematical definition of the statement "the function $f : (0, +\infty) \to \mathbb{R}$ is uniformly continuous".

(b) *(6 marks)* Prove that the function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = e^{-x}$, is uniformly continuous.

[*Hint:* you may find it helpful to use the Mean Value Theorem.]

Use this page if you need more space for Question 1(b).

(c) *(3 marks)* Give the mathematical definition of the statement "the sequence of real numbers $\{x_n\}$ is a Cauchy sequence".

(d) *(3 marks)* State Cauchy's General Principle of Convergence.

(e) *(10 marks)* Let $g : (0, +\infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that the sequence $\left\{g\left(\frac{1}{n}\right)\right\}$ converges.

[*Hint:* you may find it helpful to prove that $\left\{g\left(\frac{1}{n}\right)\right\}$ is a Cauchy sequence.]

Use this page if you need more space for Question 1(e).

**Question 2.** **(25 marks)**

(a) *(5 marks)* State Riemann's Criterion for Integrability.

(b) *(2 marks)* Give the mathematical definition of the statement "the function $f : [a, b] \to \mathbb{R}$ is decreasing".

(c) *(2 marks)* Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is bounded.

(d) *(11 marks)* Prove that a decreasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. Here you may use without proof Riemann's Criterion for Integrability.

Use this page if you need more space for Question 2(d).

(e) *(5 marks)* Give an example of a bounded function $f : [-1, 1] \to \mathbb{R}$ which is not continuous but is Riemann integrable on $[-1, 1]$. Justify your answer.

**Question 3.** **(25 marks)**

(a) *(5 marks)* State Cauchy's Generalisation of the Mean Value Theorem.

(b) *(12 marks)* Let $n$ be a nonnegative integer and let $f : (-1, +\infty) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) := f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) := f(x) - P_n(x).$$

Given an $x > 0$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0, x)$.

Use this page if you need more space for Question 3(b).

Use this page if you need more space for Question 3(b).

(c) *(8 marks)* Using the result from 3(b), prove that $\dfrac{7}{12} < \ln 2 < \dfrac{5}{6}$.

Use this page if you need more space for Question 3(c).

**Question 4.** **(25 marks)**

(a) *(2 marks)* Give the mathematical definition of the statement "the function $f : [0, +\infty) \to \mathbb{R}$ is locally Riemann integrable".

(b) *(3 marks)* Let $f : [0, +\infty) \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function $f$ is integrable over $[0, +\infty)$ in the improper sense". Define the improper integral $\displaystyle\int_0^{+\infty} f(x)\, dx$ .

(c) *(3 marks)* Use the definition from 4(b) to prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx$ and to evaluate this integral.

(d) *(3 marks)* State the Comparison Theorem for Improper Integrals.

(e) *(3 marks)* State the theorem relating the integrability, in the improper sense, of the functions $f$ and $|f|$.

(f) *(6 marks)* Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+x^2}\, dx$.

[*Hint:* you may find it helpful to use the results from 4(c), 4(d) and 4(e).]

Use this page if you need more space for Question 4(f).

(g) *(5 marks)* Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \left(\frac{\pi}{2} - \arctan x\right) \cos x \, dx$.

[*Hint:* you may find it helpful to integrate by parts.]

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