UNIVERSITY COLLEGE LONDON
Department of Mathematics

MATH0004 - Analysis 2
2023-24 Main Summer Assessment Period

Duration: 2 hours

CANDIDATE NUMBER ....

SEAT NUMBER ....

INSTRUCTIONS: (READ CAREFULLY!)

- This exam booklet contains 18 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **Only work written on this booklet will be marked.** Work written on additional pages or additional booklets will not be read by the examiners.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **If you need extra space for a question, you may use pages 15 - 18 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Organise your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, mobile phones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

Good luck!

Question 1. **(25 marks)**

(a) *(3 marks)* Give the mathematical definition of the statement "the sequence of real numbers $\{x_n\}$ is a Cauchy sequence".

Solution: (3 marks, BOOKWORK) A sequence of real numbers $\{x_n\}$ is said to be a *Cauchy sequence* if $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

(b) *(7 marks)* Let $\{x_n\}$ be a sequence of real numbers such that $|x_{n+1} - x_n| \le \dfrac{1}{2^n}$. Using the definition from (a), prove that it is a Cauchy sequence.

Solution: (7 marks, A VERSION OF THIS QUESTION APPEARED IN AN ONLINE QUIZ) Observe that for $m, n \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned} |x_m - x_n| &\le |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \cdots + |x_m - x_{m-1}| \\ &\le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-1}}. \end{aligned} \tag{1}$$

Now, given an arbitrary $\varepsilon > 0$ choose an $N \in \mathbb{N}$ so large that

$$\frac{1}{2^N} < \varepsilon. \tag{2}$$

Formulae (1) and (2) imply that the definition from (a) is satisfied.

(c) *(5 marks)* Using the definition from (a), prove that the sequence $\{x_n\}$, $x_n = n$, is not a Cauchy sequence.

Solution: (5 marks, UNSEEN) Suppose the sequence is Cauchy. Applying the definition from (a) with $\varepsilon = \frac{1}{2}$ we conclude that there exists an $N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \frac{1}{2}. \tag{3}$$

Put $m = N + 2$, $n = N + 1$. Then

$$m, n > N$$

but

$$|x_m - x_n| = 1,$$

contradicting (3).

(d) *(3 marks)* Give the mathematical definition of the statement "the sequence of real numbers $\{x_n\}$ converges to a limit L".

Solution: (3 marks, BOOKWORK) We say that $L = \lim_{n\to\infty} x_n$ if $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N}$ such that

$$n > N \qquad \Longrightarrow \qquad |x_n - L| < \varepsilon.$$

(e) *(7 marks)* Suppose that a sequence of real numbers $\{x_n\}$ converges to a limit L. Prove that it is a Cauchy sequence.

Solution: (3 marks, BOOKWORK) Let ε be an arbitrary positive number. Then by definition from (d) $\exists N \in \mathbb{N}$ such that

$$n > N \qquad \Longrightarrow \qquad |x_n - L| < \frac{\varepsilon}{2},$$

where L is the limit. Now take arbitrary m and n such that $m, n > N$. We have

$$|x_m - x_n| = |(x_m - L) + (L - x_n)| \leq |x_m - L| + |L - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Question 2. **(25 marks)**

(a) *(4 marks)* Give the mathematical definition of the statement "the function $F : [a, b] \to \mathbb{R}$ is a primitive of the function $f : [a, b] \to \mathbb{R}$".

Solution: (4 marks, BOOKWORK; note that in the lectures this definition was given for an arbitrary interval I and not specifically for the closed bounded interval $[a, b]$) A *primitive* of f is a function $F : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$, differentiable on (a, b) and satisfies $F' = f$ on (a, b).

(b) *(1 mark)* Suppose that the function $f : [a, b] \to \mathbb{R}$ has a primitive $F : [a, b] \to \mathbb{R}$. Is this primitive unique?

Solution: (1 marks, BOOKWORK) No: one can always add a constant to F.

(c) *(12 marks)* State and prove the Fundamental Theorem of Calculus.

Solution: (12 marks, BOOKWORK)

Fundamental Theorem of Calculus Let $f \in \mathcal{R}[a, b]$ and let F be a primitive of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)|_a^b\,. \tag{4}$$

Note: $F(x)|_a^b$ is simply a short way of writing $F(b) - F(a)$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Applying the Mean Value Theorem to the function F on the subinterval $[x_{i-1}, x_i]$ we get

$$F(x_i) - F(x_{i-1}) = f(\xi_i)(x_i - x_{i-1})$$

for some $\xi_i \in (x_{i-1}, x_i)$. [Note: here we can apply the Mean Value Theorem because F is continuous on $[x_{i-1}, x_i] \subseteq [a, b]$ and differentiable on $(x_{i-1}, x_i) \subseteq (a, b)$.] But

$$m_i \le f(\xi_i) \le M_i$$

where I use the standard notation $m_i := \inf_{x\in[x_{i-1},x_i]} f(x)$, $M_i := \sup_{x\in[x_{i-1},x_i]} f(x)$. Hence,

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

Summing up over i from 1 to n we get

$$L(f, P) \le \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \le U(f, P).$$

But

$$\sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a),$$

so the above inequality can be rewritten as

$$L(f,P) \le F(b) - F(a) \le U(f,P). \tag{5}$$

The left inequality (5) is true for any partition P, so formula (5) and the definitions of the lower Riemann integral and the Riemann integral imply

$$\int_a^b f(x)\,dx \le F(b) - F(a) \le U(f,P). \tag{6}$$

The right inequality (6) is true for any partition P, so formula (6) and the definitions of the upper Riemann integral and the Riemann integral imply

$$\int_a^b f(x)\,dx \le F(b) - F(a) \le \int_a^b f(x)\,dx\,. \tag{7}$$

Formula (7) implies (4). □

(d) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad 0 < x < 1, \\ 2 & \text{if} \quad x = 1. \end{cases}$$

i. *(2 marks)* Prove that the function f is Riemann integrable on the interval $[0, 1]$. Here you are allowed to use all known facts from the lectures.

Solution: (2 marks, UNSEEN) The function f is Riemann integrable on the interval $[0, 1]$ because it is increasing.

ii. *(3 marks)* Find a primitive $F : [0, 1] \to \mathbb{R}$ for the function f.

Solution: (3 marks, UNSEEN) $F(x) = x$.

iii. *(3 marks)* Using the Fundamental Theorem of Calculus, evaluate $\int_0^1 f(x)\, dx$.

Solution: (3 marks, UNSEEN) $\int_0^1 f(x)\, dx = x|_0^1 = 1$.

Question 3. **(25 marks)**

(a) *(4 marks)* Define the notion of the radius of convergence of a real power series.

Solution: (4 marks, BOOKWORK) Let us introduce the set

$$S := \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{8}$$

Obviously, $\{0\} \subseteq S \subseteq [0, +\infty)$.

The *radius of convergence* of a power series is the extended real number

$$R := \sup S$$

where S is the set (8).

(b) *(9 marks)* Let R be the radius of convergence of a real power series. Prove that if $|x| < R$ then the power series converges absolutely and if $|x| > R$ then the power series diverges.

Solution: (9 marks, BOOKWORK) The fact that the power series diverges for $|x| > R$ follows from the definition of the radius of convergence R: indeed, the fact that $|x| > R$ and the definition of the radius of convergence imply that $|x| \notin S$, S being the set (8). Note that here we made use of the *or* in formula (8). The argument wouldn't work if we'd had an *and* in formula (8).

Thus, we only need to prove absolute convergence for $|x| < R$.

If $R = 0$ there is nothing to prove, hence further on we assume that $R > 0$.

Suppose $|x| < R$. Then, by the definition of R, there exists a y such that $|x| < |y| \leq R$ and $\sum\limits_{n=0}^{\infty} a_n y^n$ converges. But convergence of the series $\sum\limits_{n=0}^{\infty} a_n y^n$ implies that $\lim\limits_{n\to\infty} a_n y^n = 0$, which in turn implies that the sequence $\{a_n y^n\}$ is bounded (a convergent sequence is bounded). Thus, there exists an $M \geq 0$ such that $|a_n y^n| \leq M$, $\forall n \in \mathbb{N} \cup \{0\}$. Hence,

$$|a_n x^n| = |a_n y^n| \left|\frac{x}{y}\right|^n \leq M \left|\frac{x}{y}\right|^n.$$

But

$$\sum_{n=0}^{\infty} M \left|\frac{x}{y}\right|^n < \infty$$

because this is a geometric series with ratio $\left|\frac{x}{y}\right| < 1$, so $\sum\limits_{n=0}^{\infty} |a_n x^n|$ converges by the comparison test. $\square$

(c) *(1 mark)* Consider two real power series, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$. How are their radii of convergence related? No proof required here.

Solution: (1 marks, BOOKWORK) The two power series have the same radii of convergence.

(d) *(4 marks)* State the theorem about the differentiability of a real power series.

Solution: (4 marks, BOOKWORK) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

(e) Consider the real power series $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = \frac{(-1)^{n+1}}{n}$.

i. *(3 marks)* Prove that this power series has radius of convergence 1.

Solution: (3 marks, UNSEEN) Fix an $x \neq 0$ and apply the ratio test:

$$\left| \frac{\frac{(-1)^{n+2}}{n+1} x^{n+1}}{\frac{(-1)^{n+1}}{n} x^n} \right| = \frac{n}{n+1} |x| \to |x|$$

as $n \to \infty$. Hence, our power series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. The result from (b) tells us that $R = 1$.

ii. *(4 marks)* Prove that the function $f : (-1, 1) \to \mathbb{R}$, $f(x) = \sum_{n=1}^{\infty} a_n x^n$, satisfies the differential equation $(1 + x)f'(x) = 1$.

Solution: (4 marks, UNSEEN) The results from (d) and (e)(i) tell us that our power series defines a differentiable function $f(x)$ on the interval $(-1, 1)$ and that this power series can be differentiated term-by-term. We have

$$f'(x) = \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1} \overset{\text{by introducing new summation index } m=n-1}{=} \sum_{m=0}^{\infty}(-1)^m x^m,$$

$$xf'(x) = \sum_{n=1}^{\infty}(-1)^{n+1}x^{n} \overset{\text{by renaming } n \text{ to } m}{=} \sum_{m=1}^{\infty}(-1)^{m+1}x^m = -\sum_{m=1}^{\infty}(-1)^m x^m,$$

so

$$(1+x)f'(x) = \sum_{m=0}^{\infty}(-1)^m x^m - \sum_{m=1}^{\infty}(-1)^m x^m = 1.$$

Alternatively, just observe that

$$\sum_{m=0}^{\infty}(-1)^m x^m$$

is a geometric series with ratio $-x$, so that

$$\sum_{m=0}^{\infty}(-1)^m x^m = \frac{1}{1-(-x)} = \frac{1}{1+x}\,.$$

Question 4. **(25 marks)**

(a) *(3 marks)* Let $g : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$ be such that $g(c) = 0$ and $g'(x) \neq 0$ for $x \neq c$. Prove that $g(x) \neq 0$ for $x \neq c$.

Solution: (3 marks, BOOKWORK) Suppose $g(x) = 0$ for some $x \in (a, b)$, $x \neq c$.

Case 1: $x > c$. Apply Rolle's Theorem to the function g on the interval $[c, x]$: Rolle's Theorem tells us that there exists a $\xi \in (c, x)$ such that $g'(\xi) = 0$, which is a contradiction.

Case 2: $x < c$. Apply Rolle's Theorem to the function g on the interval $[x, c]$: Rolle's Theorem tells us that there exists a $\xi \in (x, c)$ such that $g'(\xi) = 0$, which is a contradiction.

(b) *(11 marks)* State and prove l'Hôpital's Rule for finding $\lim_{x \to c} \frac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$.

Solution: (11 marks, BOOKWORK) **L'Hôpital's Rule** Let $f, g : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$ be such that $f(c) = g(c) = 0$ and $g'(x) \neq 0$ for $x \neq c$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{9}$$

provided the latter limit exists.

Proof Suppose $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists and equals L. Then

$$\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L \tag{10}$$

and

$$\lim_{x \to c^-} \frac{f'(x)}{g'(x)} = L. \tag{11}$$

In order to prove (9) we need to show that

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = L \tag{12}$$

and

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = L. \tag{13}$$

Let us prove (12).

Take an arbitrary sequence $\{x_n\} \subset (c, b)$ such that

$$\lim_{n \to \infty} x_n = c. \tag{14}$$

Applying Cauchy's Generalisation of the Mean Value Theorem on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ such that

$$c < \xi_n < x_n \tag{15}$$

and

$$(g(x_n) - g(c))f'(\xi_n) = (f(x_n) - f(c))g'(\xi_n)$$

which can be equivalently rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi_n)}{g'(\xi_n)} \,. \tag{16}$$

Formulae (14), (15) and the Sandwich Principle imply

$$\lim_{n\to\infty} \xi_n = c. \tag{17}$$

Formulae (10), (17) and the sequential definition of a limit of a function, one-sided version, imply

$$\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L. \tag{18}$$

Formulae (16) and (18) imply

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L. \tag{19}$$

Formula (19) and the sequential definition of a limit of a function, one-sided version, imply formula (12).

Formula (13) is proved similarly. □

(c) Evaluate the following limits.

i. *(3 marks)* $\lim_{x\to 1} \frac{\cos\frac{\pi}{2}x}{x^2-1}$.

Solution: (3 marks, UNSEEN)

$$\lim_{x\to 1} \frac{\cos\frac{\pi}{2}x}{x^2-1} = -\lim_{x\to 1} \frac{\frac{\pi}{2}\sin\frac{\pi}{2}x}{2x} = -\lim_{x\to 1} \frac{\frac{\pi}{2}}{2} = -\frac{\pi}{4}.$$

ii. *(8 marks)* $\lim_{x\to 0} \frac{\tan^2 x - \arctan^2 x}{x^4}$.

Solution: (8 marks, UNSEEN) We have

$$\frac{\tan^2 x - \arctan^2 x}{x^4} = \frac{\tan x - \arctan x}{x^3}\,\frac{\tan x + \arctan x}{x}\,. \tag{20}$$

Applying in each case L'Hôpital's Rule once we get

$$\lim_{x\to 0} \frac{\tan x + \arctan x}{x} = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} + \frac{1}{1+x^2}}{1} = 2, \tag{21}$$

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - \arctan x}{x^3} &= \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - \frac{1}{1+x^2}}{3x^2} \\ &= \lim_{x\to 0} \frac{1}{(1+x^2)\cos^2 x} \lim_{x\to 0} \frac{1+x^2-\cos^2 x}{3x^2} \\ &= \lim_{x\to 0} \frac{x^2+\sin^2 x}{3x^2} = \frac{1}{3} + \frac{1}{3}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = \frac{2}{3}\,. \end{aligned} \tag{22}$$

Note that in (22) we used the following standard fact (proved by applying L'Hôpital's Rule once):

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Combining formulae (20)–(22) we get

$$\lim_{x\to 0} \frac{\tan^2 x - \arctan^2 x}{x^4} = \frac{4}{3}\,.$$

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