UNIVERSITY COLLEGE LONDON
Department of Mathematics

MATH0004 - Analysis 2
2023-24 Main Summer Assessment Period

Duration: 2 hours

CANDIDATE NUMBER ....

SEAT NUMBER ....

INSTRUCTIONS: (READ CAREFULLY!)

- This exam booklet contains 24 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **Only work written on this booklet will be marked.** Work written on additional pages or additional booklets will not be read by the examiners.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **If you need extra space for a question, you may use pages 22 - 24 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Organise your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, mobile phones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

Good luck!

Question 1. **(25 marks)**

(a) *(3 marks)* Give the mathematical definition of the statement "the sequence of real numbers $\{x_n\}$ is a Cauchy sequence".

(b) *(7 marks)* Let $\{x_n\}$ be a sequence of real numbers such that $|x_{n+1} - x_n| \leq \dfrac{1}{2^n}$. Using the definition from (a), prove that it is a Cauchy sequence.

Use this page if you need more space for Question 1(b).

(c) *(5 marks)* Using the definition from (a), prove that the sequence $\{x_n\}$, $x_n = n$, is not a Cauchy sequence.

(d) *(3 marks)* Give the mathematical definition of the statement "the sequence of real numbers $\{x_n\}$ converges to a limit L".

(e) *(7 marks)* Suppose that a sequence of real numbers $\{x_n\}$ converges to a limit L. Prove that it is a Cauchy sequence.

Use this page if you need more space for Question 1(e).

Question 2. **(25 marks)**

(a) *(4 marks)* Give the mathematical definition of the statement "the function $F : [a, b] \to \mathbb{R}$ is a primitive of the function $f : [a, b] \to \mathbb{R}$".

(b) *(1 mark)* Suppose that the function $f : [a, b] \to \mathbb{R}$ has a primitive $F : [a, b] \to \mathbb{R}$. Is this primitive unique?

(c) *(12 marks)* State and prove the Fundamental Theorem of Calculus.

Use this page if you need more space for Question 2(c).

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(d) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad 0 < x < 1, \\ 2 & \text{if} \quad x = 1. \end{cases}$$

i. *(2 marks)* Prove that the function f is Riemann integrable on the interval $[0,1]$. Here you are allowed to use all known facts from the lectures.

ii. *(3 marks)* Find a primitive $F : [0,1] \to \mathbb{R}$ for the function f.

iii. *(3 marks)* Using the Fundamental Theorem of Calculus, evaluate $\displaystyle\int_0^1 f(x)\, dx$.

Question 3. **(25 marks)**

(a) *(4 marks)* Define the notion of the radius of convergence of a real power series.

(b) *(9 marks)* Let R be the radius of convergence of a real power series. Prove that if $|x| < R$ then the power series converges absolutely and if $|x| > R$ then the power series diverges.

Use this page if you need more space for Question 3(b).

(c) *(1 mark)* Consider two real power series, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$. How are their radii of convergence related? No proof required here.

(d) *(4 marks)* State the theorem about the differentiability of a real power series.

(e) Consider the real power series $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = \frac{(-1)^{n+1}}{n}$.

i. *(3 marks)* Prove that this power series has radius of convergence 1.

ii. *(4 marks)* Prove that the function $f : (-1, 1) \to \mathbb{R}$, $f(x) = \sum_{n=1}^{\infty} a_n x^n$, satisfies the differential equation $(1 + x)f'(x) = 1$.

Question 4. **(25 marks)**

(a) *(3 marks)* Let $g : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$ be such that $g(c) = 0$ and $g'(x) \neq 0$ for $x \neq c$. Prove that $g(x) \neq 0$ for $x \neq c$.

(b) *(11 marks)* State and prove l'Hôpital's Rule for finding $\lim_{x \to c} \frac{f(x)}{g(x)}$ when $f(c) = g(c) = 0$.

Use this page if you need more space for Question 4(b).

(c) Evaluate the following limits.

i. *(3 marks)* $\lim\limits_{x \to 1} \dfrac{\cos \frac{\pi}{2} x}{x^2 - 1}$.

ii. *(8 marks)* $\lim_{x \to 0} \dfrac{\tan^2 x - \arctan^2 x}{x^4}$.

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