UNIVERSITY COLLEGE LONDON
Department of Mathematics

MATH0004- Analysis 2
2023 Main Summer Assessment Period

Duration: 2 hours

CANDIDATE NUMBER ....

INSTRUCTIONS: (READ CAREFULLY!)

- This exam booklet contains 20 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **If you need extra space for a question, you may use pages 17 - 20 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Organize your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, mobile phones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

Good luck!

Question 1. **(25 marks)**

(a) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the point $x_0 \in \mathbb{R}$".

Solution: (4 marks, BOOKWORK) We say that the function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at the point* $x_0 \in \mathbb{R}$ if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

(b) Give the mathematical definition of the statement "the function $f : \mathbb{R} \to \mathbb{R}$ is continuous".

Solution: (2 marks, BOOKWORK) We say that the function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if it is continuous at every point $x_0 \in \mathbb{R}$.

(c) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous".

Solution: (4 marks, BOOKWORK) We say that the function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

(d) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{1+x^2}$, is uniformly continuous.

Solution: (6 marks, UNSEEN) I claim that

$$\left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| \le |x-y|, \qquad \forall x, y \in \mathbb{R}. \tag{1}$$

If $x = y$ the inequality (1) is certainly true, so further on we consider the case $x \neq y$. Without loss of generality we also assume that $y < x$.

Our function is differentiable (and, therefore, continuous) on the whole real line, hence we can apply the Mean Value Theorem to our function on the interval $[y, x]$ to get

$$\frac{d(1+\xi^2)^{-1}}{d\xi} = -\frac{2\xi}{(1+\xi^2)^2} = \frac{(1+x^2)^{-1} - (1+y^2)^{-1}}{x-y},$$

$$\left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| = \frac{2|\xi|}{(1+\xi^2)^2} |x-y|, \tag{2}$$

for some $\xi \in (y, x)$. Clearly,

$$2|\xi| \le 1 + \xi^2 \qquad \text{for all} \qquad \xi \in \mathbb{R},$$

so that

$$\frac{2|\xi|}{(1+\xi^2)^2} \le \frac{1}{1+\xi^2} \le 1 \qquad \text{for all} \qquad \xi \in \mathbb{R}. \tag{3}$$

Formulae (2) and (3) imply (1).

The inequality (1) tells us that the definition of uniform continuity works for our function with $\delta = \varepsilon$.

(e) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is not uniformly continuous.

Solution: (9 marks, UNSEEN) Suppose that our function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is uniformly continuous. Applying the definition of uniform continuity with $\varepsilon = 1$ we get a $\delta > 0$ such that

$$|x - y| < \delta \qquad \Longrightarrow \qquad |x^2 - y^2| < 1. \tag{4}$$

Put $y = n$ and $x = n + \frac{1}{n}$, where $n \in \mathbb{N}$. We have

$$|x - y| = x - y = \frac{1}{n},$$

so if we choose $n > \frac{1}{\delta}$ we are guaranteed to have $|x - y| < \delta$. But

$$|x^2 - y^2| = x^2 - y^2 = n^2 + 2 + n^{-2} - n^2 = 2 + n^{-2} > 2,$$

contradicting (4).

Question 2. **(25 marks)**

(a) What is a partition P of an interval $[a, b]$ and what is its mesh $\|P\|$?

Solution: (2 marks, BOOKWORK) A *partition* P of the interval $[a, b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The set of all partitions of a given interval $[a, b]$ will be denoted by $\mathcal{P}[a, b]$.
The *mesh (norm, width)* of the partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number

$$\|P\| := \max_{i=1,\ldots,n} (x_i - x_{i-1}).$$

(b) Let $f : [a, b] \to \mathbb{R}$ be bounded.

i. Given a partition P of the interval $[a, b]$, define the lower Darboux sum $L(f, P)$ and the upper Darboux sum $U(f, P)$.

Solution: (2 marks, BOOKWORK) The *lower Darboux sum* of f with respect to the partition $P = \{x_0, x_1, \ldots, x_n\}$ is defined as

$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$. The *upper Darboux sum* of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$.

ii. Define the lower Riemann integral $\underline{\int_a^b} f(x)\, dx$ and the upper Riemann integral $\overline{\int_a^b} f(x)\, dx$.

Solution: (2 marks, BOOKWORK) The *lower Riemann integral* of f over $[a, b]$ is defined as

$$\underline{\int_a^b} f(x)\, dx := \sup_{P \in \mathcal{P}[a,b]} L(f, P).$$

The *upper Riemann integral* of f over $[a, b]$ is defined as

$$\overline{\int_a^b} f(x)\, dx := \inf_{P \in \mathcal{P}[a,b]} U(f, P).$$

iii. Prove that $\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx$. Here you may use without proof the fact that $L(f,P) \le U(f,Q)$ for any pair of partitions

Solution: (3 marks, BOOKWORK) For any pair of partitions P and Q we have $L(f,P) \le U(f,Q)$. Taking the supremum over $P \in \mathcal{P}[a,b]$ we get

$$\underline{\int}_a^b f(x)\,dx \le U(f,Q). \tag{5}$$

Note that formula (5) holds for any partition Q. Taking the infimum over $Q \in \mathcal{P}[a,b]$ we get $\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx$.

iv. Give the mathematical definition of the statement "the function f is Riemann integrable on $[a,b]$". Define the Riemann integral $\int_a^b f(x)\,dx$.

Solution: (2 marks, BOOKWORK) The function f is said to be *Riemann integrable* on $[a,b]$ if

$$\underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

$$\int_a^b f(x)\,dx$$

and is called the *Riemann integral* of f over $[a,b]$.

(c) Consider the function $f : [0,1] \to \mathbb{R}$, $f(x) = e^x$, and let $P_n = \{x_0, x_1, \ldots, x_n\}$ be the partition of the interval $[0,1]$ into n subintervals of equal length, that is, $x_i = \frac{i}{n}$, $i = 0, 1, \ldots, n$.

i. Evaluate the Darboux sums $L(f, P_n)$ and $U(f, P_n)$.

Solution: (5 marks, UNSEEN) We have

$$L(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} e^{(i-1)/n} = \frac{1}{n} \sum_{i=1}^{n} (e^{1/n})^{i-1} = \frac{1}{n} \frac{1-e}{1-e^{1/n}} = \frac{\frac{1}{n}}{e^{1/n}-1}(e-1),$$

$$U(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} e^{i/n} = e^{1/n} L(f, P_n) = e^{1/n} \frac{\frac{1}{n}}{e^{1/n}-1}(e-1).$$

ii. Use the definitions from (b)(ii) and the results from (c)(i) to prove that $\underline{\int}_0^1 e^x\,dx \geq e-1$ and $\overline{\int}_0^1 e^x\,dx \leq e-1$.

Solution: (7 marks, UNSEEN) We have

$$\underline{\int_0^1} e^x\,dx := \sup_{P\in\mathcal{P}[0,1]} L(f,P) \geq \sup_{n\in\mathbb{N}} L(f,P_n) \geq \lim_{n\to\infty} L(f,P_n),$$

$$\overline{\int_0^1} e^x\,dx := \inf_{P\in\mathcal{P}[0,1]} U(f,P) \leq \inf_{n\in\mathbb{N}} U(f,P_n) \leq \lim_{n\to\infty} U(f,P_n).$$

Hence, in order to establish the required results it suffices to show that

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{e^{1/n}-1} = 1, \tag{6}$$

$$\lim_{n\to\infty} e^{1/n} = 1. \tag{7}$$

Formula (7) is obvious, whereas in order to prove (6) it suffices to show that

$$\lim_{x\to 0} \frac{x}{e^x-1} = 1.$$

But the latter is established by an application of L'Hôpital's Rule.

iii. Use the definition from (b)(iv) and the results from (b)(iii) and (c)(ii) to prove that $\int_0^1 e^x\,dx = e - 1$.

Solution: (2 marks, UNSEEN) Combining the results from (b)(iii) and (c)(ii) we get

$$e - 1 \leq \underline{\int_0^1} e^x\,dx \leq \overline{\int_0^1} e^x\,dx \leq e - 1$$

which implies

$$\underline{\int_0^1} e^x\,dx = \overline{\int_0^1} e^x\,dx = e - 1\,.$$

According to the definition from (b)(iv) this means that $\int_0^1 e^x\,dx = e - 1$.

Question 3. **(25 marks)**

(a) State Cauchy's Generalisation of the Mean Value Theorem.

Solution: (4 marks, BOOKWORK) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)\,.$$

(b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n + 1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Given an $x \in (0, b)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0, x)$.

Solution: (11 marks, BOOKWORK) Fix $x > 0$ and consider the function

$$g(t) := f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2!}(x - t)^2 - \frac{f'''(t)}{3!}(x - t)^3 - \dots$$
$$- \frac{f^{(n-1)}(t)}{(n-1)!}(x - t)^{n-1} - \frac{f^{(n)}(t)}{n!}(x - t)^n\,, \quad (8)$$

where $t \in (a, b)$ is the independent variable. Since $f : (a, b) \to \mathbb{R}$ is $n + 1$ times differentiable, $g : (a, b) \to \mathbb{R}$ is differentiable. Differentiating (8) we get

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x - t)] + \left[f''(t)(x - t) - \frac{f'''(t)}{2!}(x - t)^2\right] + \dots$$
$$\dots + \left[\frac{f^{(n-1)}(t)}{(n-2)!}(x - t)^{n-2} - \frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n-1}\right] + \left[\frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x - t)^n\right]$$
$$= -\frac{f^{(n+1)}(t)}{n!}(x - t)^n$$

(all the terms cancelled out, apart from the last one). Applying Cauchy's Generalisation of the Mean Value Theorem to the functions $g(t)$ and $h(t) := (x - t)^{n+1}$ on the interval $[0, x]$ (which we can because g is continuous on $[0, x] \subset (a, b)$ and differentiable on $(0, x) \subset (a, b)$) we get

$$\frac{g(x) - g(0)}{h(x) - h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x - \xi)^n}{-(n+1)(x - \xi)^n} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (0, x)$. But

$$g(0) = R_n(x), \quad g(x) = 0, \quad h(0) = x^{n+1}, \quad h(x) = 0,$$

so the above formula can be rewritten as

$$\frac{-R_n(x)}{-x^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,,$$

or, equivalently, as $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$.

(c) Let $x \in (0,1)$. Prove that $\arcsin x = x + \dfrac{1+2\xi^2}{6(1-\xi^2)^{5/2}}\, x^3$ for some $\xi \in (0,x)$.

Solution: (5 marks, UNSEEN) We have

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}\,, \qquad \arcsin'' x = \frac{x}{(1-x^2)^{3/2}}\,, \qquad \arcsin''' x = \frac{1+2x^2}{(1-x^2)^{5/2}}\,,$$

so application of the result from (b) to the function $\arcsin : (-1,1) \to \mathbb{R}$ with $n=2$ gives the required result.

(d) Prove that $3+\dfrac{1}{8}<\pi<3+\dfrac{2\sqrt{3}}{9}$. [*Hint:* apply the result from (c) with a particular x.]

Solution: (5 marks, UNSEEN) Take $x=\frac{1}{2}$. Then the formula from (c) reads

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1+2\xi^2}{6(1-\xi^2)^{5/2}}\,\frac{1}{8}$$

for some $\xi\in(0,\frac{1}{2})$. The function $\xi\mapsto\frac{1+2\xi^2}{(1-\xi^2)^{5/2}}$ is strictly increasing on the interval $[0,\frac{1}{2}]$ (the numerator increases as ξ increases, whereas the denominator decreases as ξ increases), so

$$\frac{1}{2}+\frac{1}{48}<\frac{\pi}{6}<\frac{1}{2}+\frac{1+\frac{1}{2}}{6\left(1-\frac{1}{4}\right)^{5/2}}\,\frac{1}{8}\,,$$

or, equivalently, $3+\dfrac{1}{8}<\pi<3+\dfrac{2\sqrt{3}}{9}$.

Question 4. **(25 marks)**

(a) Let $I \subset \mathbb{R}$ be an interval, bounded or unbounded at either end, open or closed at either end. Give the mathematical definition of the statement "the function $f : I \to \mathbb{R}$ is locally Riemann integrable".

Solution: (3 marks, BOOKWORK) A function $f : I \to \mathbb{R}$ is said to be *locally Riemann integrable* over I if $\forall a, b \in I$ we have $f \in \mathcal{R}[a, b]$.

(b) Let $f : (0, 1] \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $(0, 1]$ in the improper sense". Define the improper integral $\displaystyle\int_0^1 f(x)\, dx$.

Solution: (3 marks, BOOKWORK) We say that f is integrable on $(0, 1]$ in the improper sense if $\displaystyle\lim_{a\to 0^+} \int_a^1 f(x)\, dx$ exists. We will denote this limit by $\displaystyle\int_0^1 f(x)\, dx$.

(c) Let $f : [1, +\infty) \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $[1, +\infty)$ in the improper sense". Define the improper integral $\displaystyle\int_1^\infty f(x)\, dx$.

Solution: (3 marks, BOOKWORK) We say that f is integrable on $[1, +\infty)$ in the improper sense if $\displaystyle\lim_{b\to +\infty} \int_1^b f(x)\, dx$ exists. We will denote this limit by $\displaystyle\int_1^{+\infty} f(x)\, dx$.

(d) Let α be a positive real number. Prove the existence of the following two improper integrals. Here you may use, without proof, all the results from the lecture course. You may also use the fact that for any $\beta, \gamma > 0$ we have $\displaystyle\lim_{x\to +\infty} x^\beta e^{-\gamma x} = 0$.

i. $\displaystyle\int_0^1 x^{\alpha-1}\, e^{-x}\, dx\,.$

Solution: (6 marks, UNSEEN) The two functions $x^{\alpha-1}\, e^{-x}$ and $x^{\alpha-1}$ are continuous on $(0,1]$, hence locally Riemann integrable on $(0,1]$. Also, these two functions satisfy the inequalities

$$0 < x^{\alpha-1}\, e^{-x} < x^{\alpha-1}$$

for $x \in (0,1]$. The Comparison Theorem for Improper Integrals tells us that in order to prove the existence of the improper integral $\displaystyle\int_0^1 x^{\alpha-1}\, e^{-x}\, dx$ it is sufficient to prove the existence of the improper integral $\displaystyle\int_0^1 x^{\alpha-1}\, dx\,.$ But the existence of the latter for $\alpha > 0$ is a standard fact.

ii. $\displaystyle\int_1^{+\infty} x^{\alpha-1}\, e^{-x}\, dx\,.$

Solution: (10 marks, UNSEEN) Firstly, observe that for any $\mu > 0$ the improper integral

$$\int_1^{+\infty} e^{-\mu x}\, dx$$

exists. Indeed, we have

$$\int_1^{b} e^{-\mu x}\, dx = \frac{e^{-\mu} - e^{-\mu b}}{\mu} \to \frac{e^{-\mu}}{\mu} \qquad \text{as} \qquad b \to +\infty. \tag{9}$$

We now consider two cases.

Case 1: $0 < \alpha \le 1$. The two functions $x^{\alpha-1}\, e^{-x}$ and e^{-x} are continuous on $[1, +\infty)$, hence locally Riemann integrable on $[1, +\infty)$. Also, these two functions satisfy the inequalities

$$0 < x^{\alpha-1}\, e^{-x} \le e^{-x}$$

for $x \in [1, +\infty)$. The Comparison Theorem for Improper Integrals tells us that in order to prove the existence of the improper integral $\displaystyle\int_1^{+\infty} x^{\alpha-1}\, e^{-x}\, dx$ it is sufficient to prove the existence of the improper integral $\displaystyle\int_1^{+\infty} e^{-x}\, dx\,.$ But the existence of the latter was established above, see formula (9).

Case 2: $\alpha > 1$. We know that $\lim\limits_{x\to+\infty} x^{\alpha-1} e^{-x/2} = 0$, hence

$$0 < x^{\alpha-1} e^{-x/2} \le M, \qquad \forall x \in [1, +\infty),$$

for some positive constant M. The two functions $x^{\alpha-1}\, e^{-x}$ and $e^{-x/2}$ are continuous on $[1, +\infty)$, hence locally Riemann integrable on $[1, +\infty)$. Also, these two functions satisfy the inequalities

$$0 < x^{\alpha-1}\, e^{-x} \le M e^{-x/2}$$

for $x \in [1, +\infty)$. The Comparison Theorem for Improper Integrals tells us that in order to prove the existence of the improper integral $\displaystyle\int_1^{+\infty} x^{\alpha-1}\, e^{-x}\, dx$ it is sufficient to prove the existence of the improper integral $\displaystyle\int_1^{+\infty} e^{-x/2}\, dx\,.$ But the existence of the latter was established above, see formula (9).

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