UNIVERSITY COLLEGE LONDON
Department of Mathematics

MATH0004- Analysis 2
2023 Main Summer Assessment Period

Duration: 2 hours

CANDIDATE NUMBER ....

INSTRUCTIONS: (READ CAREFULLY!)

- This exam booklet contains 20 pages including this one. It consists of **4** questions, for a total of **100** marks.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **If you need extra space for a question, you may use pages 17 - 20 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Organize your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, mobile phones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

Good luck!

Question 1. **(25 marks)**

(a) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the point $x_0 \in \mathbb{R}$".

(b) Give the mathematical definition of the statement "the function $f : \mathbb{R} \to \mathbb{R}$ is continuous".

(c) Give, in terms of ε and δ, the mathematical definition of the statement "the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous".

(d) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{1+x^2}$, is uniformly continuous.

(e) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is not uniformly continuous.

Question 2. **(25 marks)**

(a) What is a partition P of an interval $[a,b]$ and what is its mesh $\|P\|$?

(b) Let $f : [a,b] \to \mathbb{R}$ be bounded.

i. Given a partition P of the interval $[a,b]$, define the lower Darboux sum $L(f,P)$ and the upper Darboux sum $U(f,P)$.

ii. Define the lower Riemann integral $\underline{\int}_a^b f(x)\,dx$ and the upper Riemann integral $\overline{\int}_a^b f(x)\,dx$.

iii. Prove that $\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx$. Here you may use without proof the fact that $L(f,P) \le U(f,Q)$ for any pair of partitions

iv. Give the mathematical definition of the statement "the function f is Riemann integrable on $[a,b]$". Define the Riemann integral $\int_a^b f(x)\,dx$.

(c) Consider the function $f : [0,1] \to \mathbb{R}$, $f(x) = e^x$, and let $P_n = \{x_0, x_1, \dots, x_n\}$ be the partition of the interval $[0,1]$ into n subintervals of equal length, that is, $x_i = \frac{i}{n}$, $i = 0, 1, \dots, n$.

i. Evaluate the Darboux sums $L(f, P_n)$ and $U(f, P_n)$.

ii. Use the definitions from (b)(ii) and the results from (c)(i) to prove that $\underline{\int}_0^1 e^x\,dx \geq e-1$ and $\overline{\int}_0^1 e^x\,dx \leq e-1$.

iii. Use the definition from (b)(iv) and the results from (b)(iii) and (c)(ii) to prove that $\int_0^1 e^x\,dx = e - 1$.

Question 3. **(25 marks)**

(a) State Cauchy's Generalisation of the Mean Value Theorem.

(b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a,b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Given an $x \in (0,b)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0,x)$.

(c) Let $x \in (0,1)$. Prove that $\arcsin x = x + \dfrac{1+2\xi^2}{6(1-\xi^2)^{5/2}}\, x^3$ for some $\xi \in (0,x)$.

(d) Prove that $3 + \dfrac{1}{8} < \pi < 3 + \dfrac{2\sqrt{3}}{9}$. [*Hint:* apply the result from (c) with a particular x.]

Question 4. **(25 marks)**

(a) Let $I \subset \mathbb{R}$ be an interval, bounded or unbounded at either end, open or closed at either end. Give the mathematical definition of the statement "the function $f : I \to \mathbb{R}$ is locally Riemann integrable".

(b) Let $f : (0, 1] \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $(0, 1]$ in the improper sense". Define the improper integral $\displaystyle\int_0^1 f(x)\,dx$.

(c) Let $f : [1, +\infty) \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $[1, +\infty)$ in the improper sense". Define the improper integral $\displaystyle\int_1^\infty f(x)\,dx$.

(d) Let α be a positive real number. Prove the existence of the following two improper integrals. Here you may use, without proof, all the results from the lecture course. You may also use the fact that for any $\beta, \gamma > 0$ we have $\lim\limits_{x \to +\infty} x^\beta e^{-\gamma x} = 0$.

i. $\displaystyle\int_0^1 x^{\alpha-1}\, e^{-x}\, dx\,.$

ii. $\displaystyle\int_1^{+\infty} x^{\alpha-1}\, e^{-x}\, dx\,.$

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