MATH0004 Analysis 2

Solutions to the main examination paper for the academic year 2021/2022

Dmitri Vassiliev

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Solution to question 1 (total marks 25)

(a)(i)

Definition 1 *We say that* $L = \lim\limits_{n\to\infty} x_n$ *if* $\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N}$ *such that*

$$n > N \qquad \Longrightarrow \qquad |x_n - L| < \varepsilon.$$

[3 marks; bookwork]

(a)(ii)

Definition 2 *A sequence of real numbers* $\{x_n\}$ *is said to be a* Cauchy sequence *if* $\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N}$ *such that*

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \varepsilon.$$

[3 marks; bookwork]

(a)(iii)

Theorem 1 *A sequence of real numbers* $\{x_n\}$ *converges iff it is a Cauchy sequence.*

[3 marks; bookwork]

(b) Let ε be an arbitrary positive number. Definition 2 tells us that there exist $N', N'' \in \mathbb{N}$ such that

$$m, n > N' \qquad \Longrightarrow \qquad |x_m - x_n| < \frac{\varepsilon}{2}\,,$$

$$m, n > N'' \qquad \Longrightarrow \qquad |y_m - y_n| < \frac{\varepsilon}{2}\,.$$

Put $N := \max\{N', N''\}$. Then for $m, n > N$ we have

$$|z_m - z_n| = |(x_m + y_m) - (x_n + y_n)| = |(x_m - x_n) + (y_m - y_n)| \le |x_m - x_n| + |y_m - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so by Definition 2 $\{z_n\}$ is a Cauchy sequence. **[5 marks; unseen exercise]**

(c) For $\alpha = 0$ the statement is obviously true, so further on we assume that $\alpha \neq 0$.

Let ε be an arbitrary positive number. Definition 2 tells us that there exists an $N \in \mathbb{N}$ such that

$$m, n > N \qquad \Longrightarrow \qquad |x_m - x_n| < \frac{\varepsilon}{|\alpha|}\,.$$

Then for $m, n > N$ we have

$$|z_m - z_n| = |\alpha x_m - \alpha x_n| = |\alpha(x_m - x_n)| = |\alpha|\,|x_m - x_n)| \le |\alpha|\,\frac{\varepsilon}{|\alpha|} = \varepsilon,$$

so by Definition 2 $\{z_n\}$ is a Cauchy sequence. **[5 marks; unseen exercise]**

(d) Let ε be an arbitrary positive number. Choose an $N \in \mathbb{N}$ such that $N > 2/\varepsilon$. Then for $m, n > N$ we have

$$|x_m - x_n| \le \frac{1}{mn} < \frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so by Definition 2 $\{x_n\}$ is a Cauchy sequence. By Theorem 1 it converges. **[6 marks; unseen exercise]**

Solution to question 2 (total marks 25)

(a)

Definition 3 *Let $f : [a,b] \to \mathbb{R}$. A* primitive *of f is a function $F : [a,b] \to \mathbb{R}$ which is continuous on $[a,b]$, differentiable on (a,b) and satisfies $F' = f$ on (a,b).*

[3 marks; bookwork]

(b) No: one can always add a constant to F. **[2 marks; bookwork]**

(c)

Theorem 2 *Let $f \in \mathcal{R}[a,b]$ and let F be a primitive of f. Then*

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a) \;=\; F(x)\big|_a^b\,. \tag{1}$$

[3 marks; bookwork]

Note: $F(x)\big|_a^b$ is simply a short way of writing $F(b) - F(a)$.

(d)(i) The function F_+ is continuous on $[0,1]$ and differentiable on $(0,1)$. Differentiating F_+ on $(0,1)$, we get $F'(x) = 1 = f_+(x)$, so by Definition 3 the function F_+ is a primitive of the function f_+. **[3 marks; unseen exercise]**

(d)(ii) The function f_+ is Riemann integrable on the interval $[0,1]$ because it is increasing. **[3 marks; unseen exercise]**

(d)(iii) Theorem 2 gives us

$$\int_0^1 f_+(x)\,dx = \; F_+(x)\big|_0^1 = 1 - 0 = 1. \tag{2}$$

[3 marks; unseen exercise]

(e) Repeating the arguments from (d) with $f_-, F_- : [-1, 0] \to \mathbb{R}$ defined by

$$f_-(x) = \begin{cases} -1 & \text{if } \quad -1 \le x < 0, \\ 0 & \text{if } \quad x = 0, \end{cases} \qquad F_-(x) = -x,$$

we get

$$\int_{-1}^{0} f_-(x)\,dx = -1. \tag{3}$$

Formulae (3) and (2) can be rewritten as

$$\int_{-1}^{0} \operatorname{sgn}(x)\,dx = -1, \tag{4}$$

$$\int_{0}^{1} \operatorname{sgn}(x)\,dx = 1, \tag{5}$$

respectively. Formulae (4) and (5) and the domain splitting property from the properties of the Riemann integral theorem tell us that the function sgn is Riemann integrable on the interval $[-1, 1]$ and that $\int_{-1}^{1} \operatorname{sgn}(x)\,dx = -1 + 1 = 0$. **[8 marks; unseen exercise]**

Solution to question 3 (total marks 25)

(a) Let us introduce the set

$$S := \left\{ r \ge 0 \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \right\}. \tag{6}$$

Obviously, $\{0\} \subset S \subset [0, +\infty)$.

Definition 4 *The* radius of convergence *of a power series is the extended real number*

$$R := \sup S$$

where S is the set (6).

[2 marks; bookwork]

(b) The two power series,

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n x^{n-k}.$$

have the same radii of convergence. **[1 marks; bookwork]**

(c)

Theorem 3 *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f : (-R, R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.*

[3 marks; bookwork]

Note that the result from (b) plays an important role in the statement of the above theorem: it guarantees that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ appearing in the statement of this theorem converges. **[1 mark; bookwork]**

(d)

Corollary 1 *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$ *and let* $f : (-R, R) \to \mathbb{R}$ *be the sum of this series. Then* f *is infinitely differentiable on the interval* $(-R, R)$ *and* $f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{a_n n!}{(n-k)!} x^{n-k}$.

[2 marks; bookwork]

Proof Repeated application of Theorem 3 and the result from (b). □ **[1 mark; bookwork]**

(e)(i) Fix an $x \neq 0$ and apply the ratio test: we have

$$\left| \frac{\frac{(-1)^{m+1}\, x^{2m+2}}{2^{2m+2}\,((m+1)!)^2}}{\frac{(-1)^m\, x^{2m}}{2^{2m}\,(m!)^2}} \right| = \frac{|x|^2}{4(m+1)^2} \to 0$$

as $m \to \infty$. Hence, the power series converges absolutely for any x, so it has infinite radius of convergence. **[4 marks; unseen exercise]**

(e)(ii) The results from (d) and (e)(i) tell us that our function $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and that its derivatives are obtained by differentiating the power series for f term-by-term.

Recall that $f(x)$ is given by the formula

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m}\,(m!)^2}\ . \tag{7}$$

Differentiating (7):

$$f'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m\, x^{2m-1}}{2^{2m-1}\,(m-1)!\,(m!)}\ .$$

Introducing a new summation index $p = m - 1$, we get

$$f'(x) = -\sum_{p=0}^{\infty} \frac{(-1)^p\, x^{2p+1}}{2^{2p+1}\, p!\,(p+1)!}\ .$$

Renaming p to m, we get

$$f'(x) = -\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m+1}}{2^{2m+1}\, m!\,(m+1)!}\ . \tag{8}$$

Differentiating (8):

$$f''(x) = -\sum_{m=0}^{\infty} \frac{(-1)^m\,(2m+1)\, x^{2m}}{2^{2m+1}\, m!\,(m+1)!}\ . \tag{9}$$

Formulae (7)–(9) imply

$$xf'' + f' + xf = \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m+1}}{2^{2m}\,(m!)^2} \left[-\frac{2m+1}{2(m+1)} - \frac{1}{2(m+1)} + 1 \right] = 0.$$

[11 marks; unseen exercise]

Solution to question 4 (total marks 25)

(a)

Definition 5 *A function* $f : (-\infty, -1] \to \mathbb{R}$ *is said to be* locally Riemann integrable *over* $(-\infty, -1]$ *if* $\forall a, b \in (-\infty, -1]$ *we have* $f \in \mathcal{R}[a, b]$. *The set of all locally Riemann integrable functions over the interval* $(-\infty, -1]$ *will be denoted by* $\mathcal{R}_{loc}(-\infty, -1]$.

[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I **and not specifically for the interval** $(-\infty, -1]$**]**

(b) We say that f is integrable on $(-\infty, -1]$ in the improper sense if $\lim_{a \to -\infty} \int_a^{-1} f(x)\, dx$ exists. We will denote this limit by $\int_{-\infty}^{-1} f(x)\, dx$.

[2 marks; bookwork; note that in the lectures this definition was given for an arbitrary interval I **and not specifically for the interval** $(-\infty, -1]$**]**

(c) Note that the integrand is continuous on $(-\infty, -1]$, hence locally Riemann integrable on $(-\infty, -1]$. So we have to check only integrability at $-\infty$. We have

$$\int_a^{-1} \frac{1}{x^2}\, dx = 1 + \frac{1}{a} \to 1 \qquad \text{as} \qquad a \to -\infty,$$

so $\int_{-\infty}^{-1} \frac{1}{x^2}\, dx = 1$. **[3 marks; unseen exercise]**

(d)

Theorem 4 *Let* $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$, *and suppose that*

- $0 \le f(x) \le g(x)$ *for all* $x \in I$;
- $\int_c^d g(x)\, dx$ *exists.*

Then $\int_c^d f(x)\, dx$ *exists.*

[2 marks; bookwork]

(e)

Theorem 5 *Suppose that* $f \in \mathcal{R}_{loc}(I)$ *and* $|f|$ *is integrable over* I. *Then* f *is integrable over* I.

[2 marks; bookwork]

(f) Let us prove the existence of the improper integral

$$\int_{-\infty}^{-1} \frac{\sin x}{x^2}\, dx. \tag{10}$$

Note that the integrand is continuous on $(-\infty, -1]$, hence locally Riemann integrable on $(-\infty, -1]$. So we have to check only integrability at $+\infty$.

By Theorem 5, in order to prove the existence of the integral (10) it is sufficient to prove the existence of the integral

$$\int_{-\infty}^{-1} \frac{|\sin x|}{x^2}\, dx. \tag{11}$$

Compare now the functions $\frac{|\sin x|}{x^2}$ and $\frac{1}{x^2}$ on the interval $(-\infty, -1]$. We have

$$0 \le \frac{|\sin x|}{x^2} \le \frac{1}{x^2}\,,$$

so, by Theorem 4, in order to prove the existence of the integral (11) it is sufficient to prove the existence of the integral $\int_{-\infty}^{-1} \frac{1}{x^2}\,dx$. But this has already been done in (c). **[6 marks; unseen exercise]**

(g) Note that the integrand is continuous on $(-\infty, -1]$, hence locally Riemann integrable on $(-\infty, -1]$. So we have to check only integrability at $-\infty$.

Integrating by parts we get

$$\int_a^{-1} \frac{\cos x}{x}\,dx = \sin 1 - \frac{\sin a}{a} + \int_a^{-1} \frac{\sin x}{x^2}\,dx\,.$$

We have $\frac{\sin a}{a} \to 0$ as $a \to -\infty$, so in order to prove the existence of the original integral we have to prove the existence of the integral $\int_{-\infty}^{-1} \frac{\sin x}{x^2}\,dx$. But this has already been done in (f). **[6 marks; unseen exercise]**

(h)

$$\int_{-\infty}^{-1} \frac{\cos x}{x}\,dx = \sin 1 + \int_{-\infty}^{-1} \frac{\sin x}{x^2}\,dx\,.$$

[2 marks; unseen exercise]