*Answer **all** questions.*

Section A

1. (a) Let $\{x_n\}$ be a sequence of real numbers.

(i) Give the mathematical definition of the statement "the sequence $\{x_n\}$ converges to a limit L".

(ii) Give the mathematical definition of the statement "$\{x_n\}$ is a Cauchy sequence".

(iii) State Cauchy's General Principle of Convergence for $\{x_n\}$.

(b) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences. Put $z_n := x_n + y_n$. Prove that the sequence $\{z_n\}$ is a Cauchy sequence.

(c) Let $\{x_n\}$ be a Cauchy sequence and let α be a real number. Put $z_n := \alpha x_n$. Prove that the sequence $\{z_n\}$ is a Cauchy sequence.

(d) Suppose that $|x_m - x_n| \leq \dfrac{1}{mn}$ for all $m, n \in \mathbb{N}$, $\mathbb{N} = \{1, 2, \dots\}$. Prove that the sequence $\{x_n\}$ converges.

(25 marks)

2. (a) Give the mathematical definition of the statement "the function $F : [a, b] \to \mathbb{R}$ is a primitive of the function $f : [a, b] \to \mathbb{R}$".

(b) Suppose that the function $f : [a, b] \to \mathbb{R}$ has a primitive $F : [a, b] \to \mathbb{R}$. Is this primitive unique?

(c) State the Fundamental Theorem of Calculus.

(d) Consider the functions $f_+, F_+ : [0, 1] \to \mathbb{R}$ defined by

$$f_+(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad 0 < x \leq 1, \end{cases} \qquad F_+(x) = x.$$

(i) Prove that the function F_+ is a primitive of the function f_+.

(ii) Prove that the function f_+ is Riemann integrable on the interval $[0, 1]$.

(iii) Evaluate $\displaystyle\int_0^1 f_+(x)\, dx$.

(e) Consider the function $\operatorname{sgn} : [-1, 1] \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if} \quad -1 \leq x < 0, \\ 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Explain why the function sgn is Riemann integrable on the interval $[-1, 1]$ and evaluate $\displaystyle\int_{-1}^1 \operatorname{sgn}(x)\, dx$. You may find it helpful to refer to the domain splitting property from the properties of the Riemann integral theorem.

(25 marks)

Section B

3. (a) Define the notion of the radius of convergence of a real power series.

(b) Let k be a nonnegative integer. Consider two real power series,

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n x^{n-k}\,.$$

How are their radii of convergence related? No proof required here.

(c) State the theorem about the differentiability of a real power series. Explain how the result from (b) is used in the statement of this theorem.

(d) State and prove the corollary about the infinite differentiability of a real power series.

(e) Consider the real power series

$$\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m}\,(m!)^2}\,. \tag{1}$$

(i) Prove that the power series (1) has infinite radius of convergence.

(ii) Consider the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x)$ is given by formula (1). Prove that this function satisfies the differential equation

$$xf'' + f' + xf = 0.$$

(25 marks)

4. (a) Give the mathematical definition of the statement "the function $f : (-\infty, -1] \to \mathbb{R}$ is locally Riemann integrable".

(b) Let $f : (-\infty, -1] \to \mathbb{R}$ be locally Riemann integrable. Give the mathematical definition of the statement "the function f is integrable over $(-\infty, -1]$ in the improper sense". Define the improper integral $\displaystyle\int_{-\infty}^{-1} f(x)\, dx\,.$

(c) Use the definition from (b) to prove the existence of the improper integral $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2}\, dx$ and to evaluate this integral.

(d) State the Comparison Theorem for Improper Integrals.

(e) State the theorem relating the integrability, in the improper sense, of the functions f and $|f|$.

(f) Prove the existence of the improper integral $\displaystyle\int_{-\infty}^{-1} \frac{\sin x}{x^2}\, dx\,.$

[*Hint:* you may find it helpful to use the results from (c), (d) and (e).]

(g) Prove the existence of the improper integral $\displaystyle\int_{-\infty}^{-1} \frac{\cos x}{x}\, dx\,.$

[*Hint:* you may find it helpful to integrate by parts.]

(h) How are the numerical values of the improper integrals from (f) and (g) related?

(25 marks)